EECS 121: Introduction to Digital Communication Systems

Problem Set 6

Due Fri, March 14, 4pm at Therese's desk

1. In this exercise we study some properties of the $Q(\cdot)$ function defined as the following.

(a) Consider a Gaussian random variable n with mean 0 and variance 1. Then

$$Q(a) \stackrel{\text{def}}{=} \mathbb{P}(n > a) \quad (1)$$
$$= \int_a^\infty \frac{1}{2\pi} \exp\left(-\frac{a^2}{2}\right) \quad da. \quad (2)$$

Using a table from standard probability and statistics books or using MATLAB, write down the values of a at which $Q(a)$ is equal to $10^{-k}, k = 1, 2, \ldots, 10$.

(b) Now consider a Gaussian random variable n with mean μ and variance σ^2. Write $\mathbb{P}(n > a)$ in terms of the $Q(\cdot)$ function.

2. Consider the binary detection problem discussed in class, where a bit b is sent by transmitting either $x = +\sqrt{E}$ or $x = -\sqrt{E}$, and the received signal is $y = x + w$, where $w \sim N(0, \sigma^2)$. Derive the detection rule that minimizes the error probability and give an expression of that minimum error probability in terms of the Q-function.

3. Consider detecting whether voltages A_1 or A_2 was sent over a channel with independent Gaussian noise. The Gaussian noise has mean A_3 volts and variance σ^2. The voltages A_1 and A_2 are equally likely (i.e., they occur with probability 0.5 each).

(a) What decision rule would you use to minimize the average error probability? You can use any result derived in class or in the lecture notes but it needs to be stated precisely and connected to your derivation.

(b) What is the average error probability for the decision rule you derived in the previous part? Use the Q function notation.

(c) If the transmit voltage is A_1 then the average transmit energy is proportional to A_1^2. If the voltage is held constant for 1 second, then the energy is exactly A_1^2. Calculate the *average* transmit energy (the transmit energy is random, because the transmit voltage can be either A_1 or A_2) in this set up.

(d) Now suppose you have a constraint that the average transmit energy cannot be more than E. How would you pick the voltage levels A_1 and A_2 to meet the average energy constraint and still arrive at the smallest error probability using the decision rule derived earlier?

4. In this exercise we see the tradeoff between data rate and SNR while communicating reliably at certain desired level using the sequential communication scheme.

Suppose we want to communicate 8192 bits (1 KB) of information over an AWGN channel using n time instants. The data rate is denoted by

$$R \stackrel{\text{def}}{=} \frac{8192}{n} \quad \text{bits/unit time instant.} \tag{3}$$

Suppose n divides 8192. Then a sequential communication scheme transmits R of the bits every time instant using the energy E available to the transmitter. This is done by transmitting one of 2^R equally spaced voltages between $\pm\sqrt{E}$. Suppose we want to maintain an overall unreliability level of 10^{-5} (this is the chance that at least one of the 8096 bits is wrongly communicated). We would like to understand how much SNR (defined, as usual, as the ratio of E to σ^2) we need, as a function of n, to maintain this reliability level. Using MATLAB, plot the data rate R as a function of $10\log_{10}\mathsf{SNR}$ (where SNR is chosen such that the desired reliability level of communication is met). You can plot only for values of n that are powers of 2 and interpolate between these points to generate the graph. You might find the MATLAB functions `erfc` and `erfcinv` useful; they represent the $Q(\cdot)$ and the inverse-$Q(\cdot)$ functions.