# EE 121 - Introduction to Digital Communications
# **Homework 6 Solutions**

March 15, 2008

**1.** (a) If you have access to the latest version of MATLAB you can use a build in function qfuncinv.m to evaluate the inverse Q-function. Else you can simply use the norminv.m function, via the command '-norminv(10.ˆ (-1:-1:-10))'. This yields values

1.2816 2.3263 3.0902 3.7190 4.2649 4.7534 5.1993 5.6120 5.9978 6.3613

(b)

$$\begin{aligned} P(n > a) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^\infty e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^\infty e^{-x^2/2} dx \\ &= Q\left(\frac{a-\mu}{\sigma}\right) \end{aligned}$$

**2.** (a) The detection rule that minimizes the error probability of error is the MAP detection rule,

$$\Pr(x = +\sqrt{E}|y) \underset{x=-\sqrt{E}}{\overset{x=+\sqrt{E}}{\gtrless}} \Pr(x = -\sqrt{E}|y).$$

Using Bayes rule we can write this as

$$\begin{aligned} &\frac{\Pr(y|x = +\sqrt{E})p_1}{\Pr(y)} \underset{x=-\sqrt{E}}{\overset{x=+\sqrt{E}}{\gtrless}} \frac{\Pr(y|x = -\sqrt{E})p_0}{\Pr(y)} \\ \Rightarrow &\frac{\Pr(y|x = +\sqrt{E})}{\Pr(y|x = -\sqrt{E})} \underset{x=-\sqrt{E}}{\overset{x=+\sqrt{E}}{\gtrless}} \frac{p_0}{p_1} \end{aligned}$$

where $p_0 \triangleq \Pr(x = -\sqrt{E})$ and $p_1 \triangleq \Pr(x = -\sqrt{E})$. Substituting for the distribution of the noise we have

$$e^{-(y-\sqrt{E})^2/2\sigma^2+(y+\sqrt{E})^2/2\sigma^2} \underset{x=-\sqrt{E}}{\overset{x=+\sqrt{E}}{\gtrless}} \frac{p_0}{p_1}$$

$$\Rightarrow -(y-\sqrt{E})^2/2\sigma^2 + (y+\sqrt{E})^2/2\sigma^2 \underset{x=-\sqrt{E}}{\overset{x=+\sqrt{E}}{\gtrless}} \log\frac{p_0}{p_1}$$

$$\Rightarrow y \underset{x=-\sqrt{E}}{\overset{x=+\sqrt{E}}{\gtrless}} \frac{\sigma^2}{2\sqrt{E}}\log\frac{p_0}{p_1}.$$

The probability of error is then

$$\begin{aligned}
\Pr(\text{error}) &= \Pr\left(y > \frac{\sigma^2}{2\sqrt{E}}\log\frac{p_0}{p_1}\middle| x = -\sqrt{E}\right)p_0 + \Pr\left(y < \frac{\sigma^2}{2\sqrt{E}}\log\frac{p_0}{p_1}\middle| x = +\sqrt{E}\right)p_1 \\
&= \Pr\left(w > \frac{\sigma^2}{2\sqrt{E}}\log\frac{p_0}{p_1} + \sqrt{E}\right)p_0 + \Pr\left(w < \frac{\sigma^2}{2\sqrt{E}}\log\frac{p_0}{p_1} - \sqrt{E}\right)p_1 \\
&= p_0 \cdot Q\left(\frac{\sqrt{E}}{\sigma} + \frac{\sigma}{2\sqrt{E}}\log\frac{p_0}{p_1}\right) + p_1 \cdot Q\left(\frac{\sqrt{E}}{\sigma} - \frac{\sigma}{2\sqrt{E}}\log\frac{p_0}{p_1}\right).
\end{aligned}$$

**3.** (a) Since both outputs are equiprobable, the MAP rule is the same as the ML rule. The likelihood ratio is:

$$LR(y) = \frac{f(y|b = A_1)}{f(y|b = A_2)}$$

and the ML rule is

$$\hat{b} = \begin{cases} A_1, & \text{if } LR(y) > 1 \\ A_2, & \text{if } LR(y) \le 1 \end{cases}$$

Now note that

$$\begin{aligned}
f(y|b = A_1) &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-A_1-A_3)^2/2\sigma^2} \\
f(y|b = A_2) &= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-A_2-A_3)^2/2\sigma^2}.
\end{aligned}$$

Therefore

$$LR(y) = e^{(A_1-A_2)(y-(A_1+A_2)/2-A_3)/\sigma^2}.$$

Without loss of generality we can assume $A_1 > A_2$ in which case the ML rule is

$$\hat{b} = \begin{cases} A_1, & \text{if } y > (A_1 + A_2)/2 + A_3 \\ A_2, & \text{if } y \le (A_1 + A_2)/2 + A_3 \end{cases}$$

(b)

$$\begin{aligned}\Pr(\text{error}) &= \Pr(\hat{b} = A_2 | b = A_1) \Pr(b = A_1) + \Pr(\hat{b} = A_1 | b = A_2) \Pr(b = A_2) \\ &= \Pr(y < (A_1 + A_2)/2 + A_3 | b = A_1) \Pr(b = A_1) \\ &\quad + \Pr(y > (A_1 + A_2)/2 + A_3 | b = A_2) \Pr(b = A_2) \\ &= \frac{1}{2}\left(1 - Q((A_2 - A_1)/2\sigma)\right) + \frac{1}{2} Q((A_1 - A_2)/2\sigma) \\ &= Q((A_1 - A_2)/2\sigma)\end{aligned}$$

(c) If the voltage is held constant for 1 second

$$\mathbb{E}[\mathrm{E}] = A_1^2 \Pr(b = A_1) + A_2^2 \Pr(b = A_2) = \frac{A_1^2 + A_2^2}{2}$$

(d) The problem is

$$\begin{aligned}&\min_{(A_1^2 + A_2^2)/2 \leq E} Q\left(\frac{A_1 - A_2}{2\sigma}\right) \\ &\qquad = \max_{(A_1^2 + A_2^2)/2 \leq E} A_1 - A_2 \\ &\qquad = \max_{(A_1^2 + A_2^2)/2 \leq E} (A_1 - A_2)^2\end{aligned}$$

Now note that

$$(A_1 - A_2)^2 \leq (|A_1| + |A_2|)^2 \leq 4E$$

and we have equality if and only if $A_1 = -A_2$. Therefore the optimal choice is $A_1 = \sqrt{E}$ and $A_2 = -\sqrt{E}$.

**4.** The overall reliability is

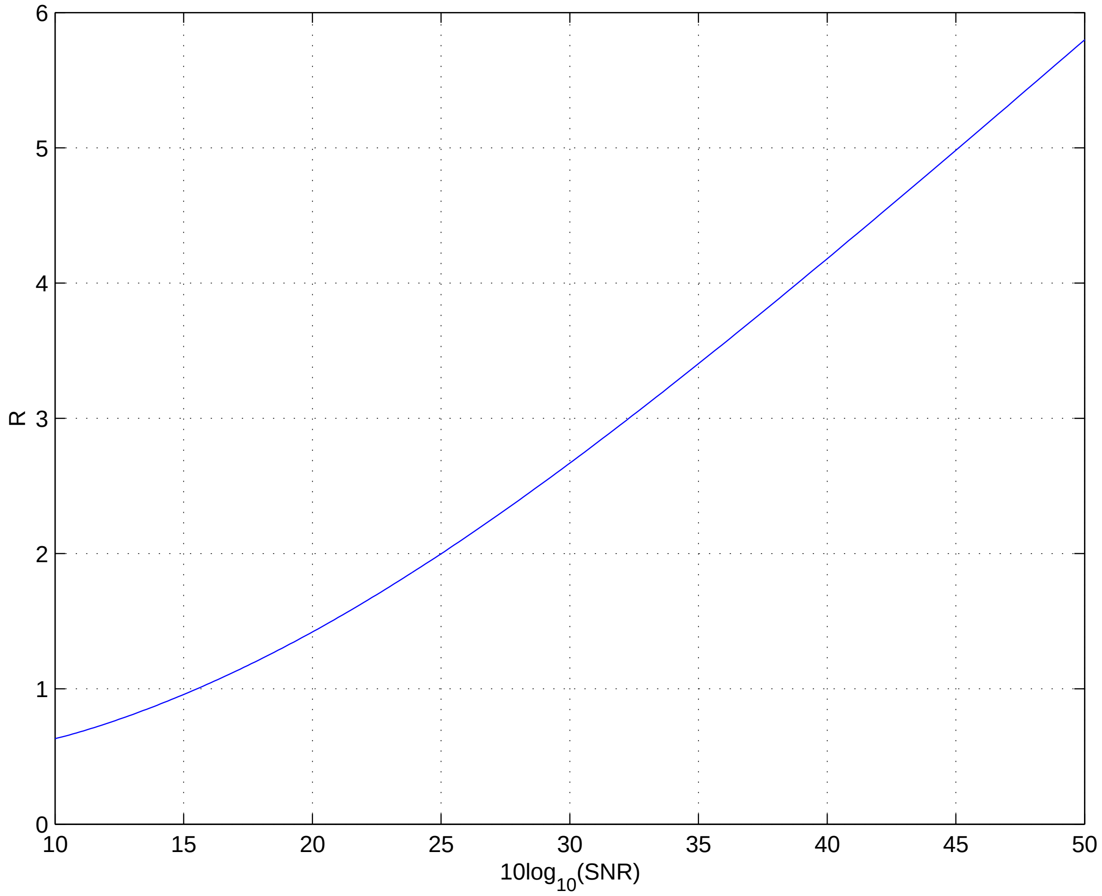
$$\begin{aligned}10^{-5} &= \Pr(\text{at least one time slot has error}) \\ &= 1 - (1 - P_e)^n\end{aligned}$$

where

$$P_e = \left(2 - \frac{1}{2^{R-1}}\right) Q\left(\frac{\sqrt{\mathrm{SNR}}}{2^R - 1}\right),$$

$n = 8192/R$ and $\mathrm{SNR} = E/\sigma^2$.

$$\Rightarrow 10 \log_{10} \mathrm{SNR} = 20 \log_{10}\left((2^R - 1) Q^{-1}\left(\frac{1}{2 - \frac{1}{2^{R-1}}}\left(1 - (1 - 10^{-5})^{\frac{R}{8192}}\right)\right)\right)$$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10log10(SNR)