EECS 121: Introduction to Digital Communication Systems

Problem Set 7

Due Fri, April 11

1. In class we showed that rate efficient reliable communication is possible with binary modulation using a linear code C as long as the rate is less than R^*.

(a) Prove that the same linear code (working at a data rate R very close to the limiting value of R^*) also affords energy efficient reliable communication (this means that the energy efficiency of the code C is *non-zero*).

(b) Derive an explicit expression for the energy efficiency of the code C. Energy efficiency is the simply the energy expended per reliable bit communicated.

(c) Show using the expression above that the ratio of energy efficiency to the noise variance σ^2 is solely a function of $\mathsf{SNR} \stackrel{\text{def}}{=} \frac{\mathsf{E}}{\sigma^2}$. What value of SNR maximizes this ratio?

(d) Plot the ratio above as a function of SNR with the SNR measured in a logarithmic scale. Specifically let us use the *decibel* (dB) notation: Any value of SNR is measured in decibel scale as $10 \log_{10} \mathsf{SNR}$ dB. So, you are asked to plot the ratio of energy efficiency to the noise variance σ^2 as a function of SNR measured in dB; choose SNR between -15dB and +10dB in your plot.

2. (*Random block codes over erasure channels*) Consider a random block code represented by a generator matrix $\mathbf{G}$ with T rows and RT columns, where $R \leq 1$. The RT^2 entries in $\mathbf{G}$ are independent random variables, taking on values either 0 or 1 with equal probability. This code is used to communicate over an erasure channel, where each coded symbol is erased with probability p. Let $\mathbf{G}'$ denote the generator matrix after the erasures have taken place, i.e. $\mathbf{G}'$ is equal to $\mathbf{G}$ with some rows removed, corresponding to the erased symbols.

(a) Suppose that no symbols are erased, so that $\mathbf{G}' = \mathbf{G}$. For the case of $T = 3$ and $R = 2/3$, compute the probability that $\mathbf{G}'$ is full rank.

Hint: Consider the probability that the second column is linearly independent of the first column, conditioned on the first column being linearly independent (i.e. not all zeros). Multiply this by the probability that the first column is full linearly independent, to get the answer.

(b) Suppose now that ϵT symbols are erased (assume ϵT is an integer). Find an exact expression for the probability that $\mathbf{G}'$ is full rank, in terms of R, T and ϵ.

Hint: Extend your technique from part (a) to a matrix with an arbitrary number of columns.

(c) Show that as $T \to \infty$ the probability that $\mathbf{G}'$ is *not* full rank goes to zero as $T \to \infty$ if the rate $R < 1 - \epsilon$.

Hint: you may find the inequality $1 - e^{-x} \leq x$ *for all* $x \in \mathbb{R}$*, useful, as well as the inequality* $-\log_e(1-x)/2 \leq x$ *for all* $x \in [0, 1/2]$.

(d) Give an exact expression for the probability of more than ϵT erasures occurring. If $p < \epsilon$, what do you think happens to this probability as $T \to \infty$? State a relevant probability law in your answer.

(e) Based on your answer to parts (c) and (d), for what set of rates can you say that reliable communication over the erasure channel is possible? Briefly explain your reasoning.

(f) Do you think it is possible to communicate reliably at rates higher than those in your answer to part (e)? Why/why not?

3. (*Random block codes for an M-PAM constellation*) Recall that in class it was shown that for a 2-PAM constellation, reliable communication was possible at all rates less than

$$R_2^* = 1 - \log_2\left(1 + e^{-\frac{E}{2\sigma^2}}\right).$$

In this problem we generalize this result for an M-PAM constellation. Consider the channel

$$\mathbf{y} = \mathbf{u} + \mathbf{w}$$

where $\mathbf{w}$ is a $T \times 1$ noise vector whose entries are i.i.d. Gaussian random variables with zero mean and variance σ^2. The vector $\mathbf{u}$ is a $T \times 1$ vector representing the codeword that is transmitted. There are 2^{RT} possible codewords to transmit which we label $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{2^{RT}}$. These codewords are generated as follows. Let $\mathbf{G}$ be a random generator matrix with T rows and $RT/\log_2 M$ columns. It has $RT^2/\log_2 M$ entries and these are i.i.d. random variables taking on values in the set $\{1, 2, \ldots, M\}$ with equal probability $1/M$. The data vector is denoted $\mathbf{x}$. It has dimension $RT/\log_2 M \times 1$. There are $M^{RT/\log_2 M} = 2^{RT}$ possible data vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2^{RT}}$, representing all possible combinations of elements from the set $\{0, 1, \ldots, M-1\}$. Thus $R \leq \log_2 M$. For example the first data vector is $\mathbf{x}_1 = [0, 0, \ldots, 0]^T$ and the last one is $\mathbf{x}_{2^{RT}} = [M-1, M-1, \ldots, M-1]^T$. The codeword $\mathbf{u}_i$ is given by

$$\mathbf{u}_i = \sqrt{E}\left(\frac{2}{M-1}\left(\mathbf{G}\mathbf{x}_i \mod M\right) - 1\right)$$

In other words, we map the M-ary valued entries of the data vector into an M-PAM constellation with maximum amplitude $\sqrt{E}$. For example if $M = 4$, and $\mathbf{G}\mathbf{x}_i = [0, 1, 2, 3]^T$, then we transmit

$$\mathbf{u_i} = \begin{pmatrix} -\sqrt{E} \\ -\sqrt{E}/3 \\ +\sqrt{E}/3 \\ +\sqrt{E} \end{pmatrix}.$$

(a) Using the union bound, show that the probability that the ML decoder makes an error satisfies

$$\Pr(\mathcal{E}) \leq \sum_{i=1}^{2^{RT}} \sum_{j=1, j\neq i}^{2^{RT}} \Pr(\mathbf{u}_i \to \mathbf{u}_j | \mathbf{u} = \mathbf{u}_i) \Pr(\mathbf{u} = \mathbf{u}_i)$$

where $\Pr(\mathbf{u}_i \to \mathbf{u}_j | \mathbf{u} = \mathbf{u}_i)$ denotes the probability that the received vector lies closer to codeword $\mathbf{u}_j$ than to codeword $\mathbf{u}_i$ conditioned on codeword $\mathbf{u}_i$ being transmitted. Carefully explain your reasoning for each step.

(b) Show that

$$\Pr(\mathbf{u}_i \to \mathbf{u}_j | \mathbf{u} = \mathbf{u}_i) \leq \sum_{l=0}^{T} \binom{T}{l} \left(\frac{1}{M}\right)^{T-l} \left(1 - \frac{1}{M}\right)^{l} Q\left(\sqrt{\frac{lE}{(M-1)^2\sigma^2}}\right)$$

Carefully explain your reasoning for each step.

(c) Show that if $R < R_M^*$ the probability of making a decoding error goes to zero as $T \to \infty$, where

$$R_M^* = \log_2\left(\frac{M}{1 + (M-1)e^{-\frac{\mathsf{SNR}}{2(M-1)^2}}}\right).$$

where $\mathtt{SNR} = E/\sigma^2$.

(d) Now suppose that we no longer use a random code, but instead we wish to use a single, "good" fixed code $\mathbf{G}$ to communicate. What does the answer to part (c) tell us about our ability to communicate reliably in this case?

(d) Use MATLAB to plot R_M^* as a function of SNR for $M = 2, 4, 8, 16$, using a logarithmic scale for the x-axis, with `SNR` ranging from 10^{-4} to 10^4. *Hint: this is most easily done by using the functions* `logspace` *to generate the* SNR *vector, and* `semilogx` *to do the plot.* At what value does R_M^* saturate at high `SNR`? Explain why it saturates at this value?

Explain why it is that for low values of `SNR` we can communicate reliably at higher rates by using a smaller constellation size, but at high values of `SNR` we have to use a larger constellation size in order to be able to communicate reliably at higher rates.