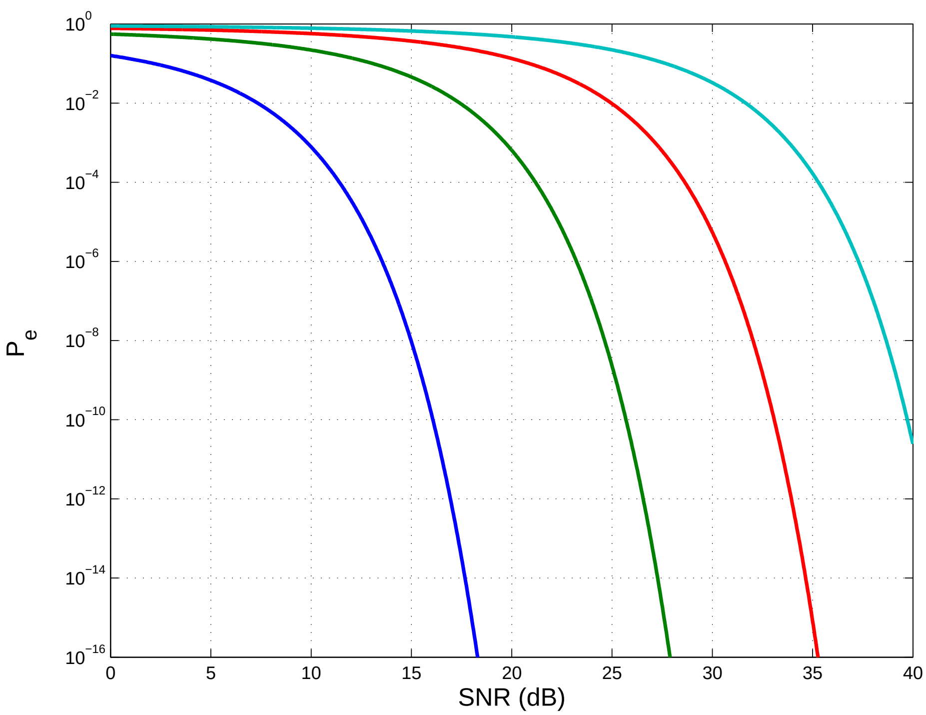
EE 121 - Introduction to Digital Communications
Homework 8 Solutions

May 7, 2008

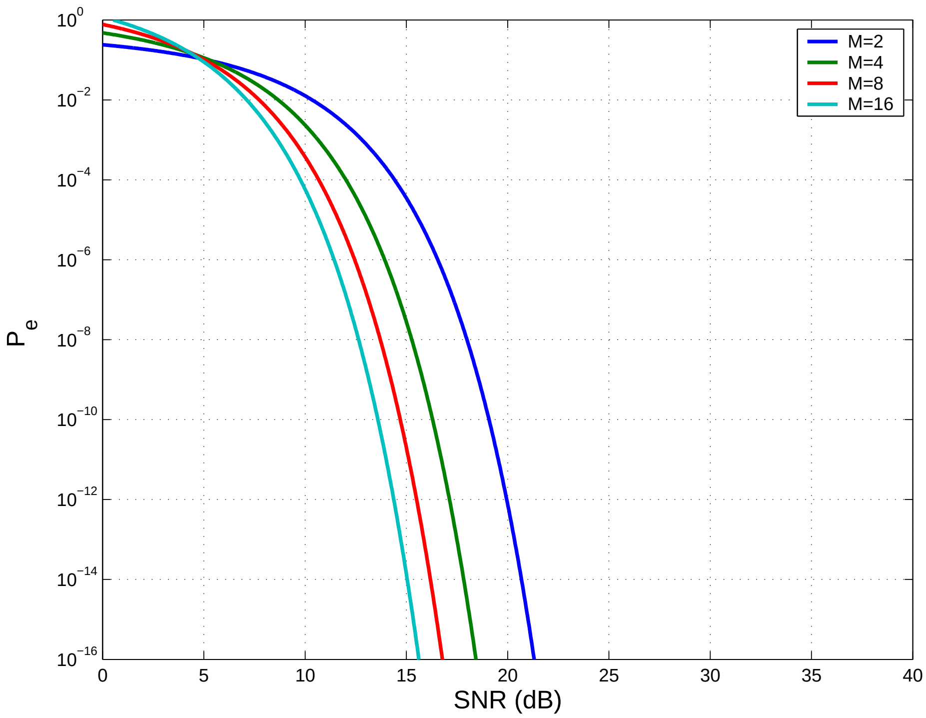
1. (a) The error probability is $P_e = Q(\sqrt{\mathsf{SNR}})$.



(b) $\mathsf{SNR}_{\min} = 2^{2C} - 1$. Thus to reliably communicate 1 bit per channel use, we need $\mathsf{SNR} > 3 =\approx 5$ dB. At an error probability of 10^{-4} 2-PAM requires roughly 12dB, an additional $\approx$7 dB beyond the minimum. At an error probability of 10^{-5} it requires $\approx$13 dB, an additional $\approx$8 dB beyond the minimum. As the target error probability decreases, the gap to the minimum SNR increases without bound (albeit very slowly). This is consistent with what was learnt in class, that coding is required to achieve arbitrarily low error probabilities with finite SNR.

(c) For M-PAM, $P_e = 2(1 - 1/M)Q(\sqrt{\mathsf{SNR}}/(M-1))$, see above figure.

(d) For M-PPM, define $\mathsf{SNR} = \mathcal{E}_b/\sigma^2$, then $P_e < (M-1)Q\left(\sqrt{\frac{\log_2 M}{2}\mathsf{SNR}}\right)$. As $M \to \infty$, the gap decreases.



2. (a) With 1Mhz of bandwidth and ideal sinc pulses we have 2Msamples/s. To achieve R = 4Mbits/s we need to send 2 bits per sample. This requires either 4-PAM with no repetition coding, 8-PAM with x1.5 repetition, or 16-PAM with x2 repetition. PPM can only achieve rates of less than 1 bit per sample so we rule this scheme out. From the graph plotted in Q1, we see that using 4-PAM to achieve an error probability of 10^{-4} requires an SNR of around 22 dB. For this scheme $E_b/N_0 \approx 22 + 10\log_{10}(1/2) = 19$ dB. As using M-PAM with repetition does not improve the error probability for the same amount of total energy expended, 8-PAM with 1.5x repetition and 16-PAM with 2x repetition will use the same energy per bit to achieve a 10^{-4} error probability. Thus either 4-PAM with no repetition, 8-PAM with 1.5x repetition, or 16-PAM with 2x repetition are all best choices in terms of minimizing energy use whilst meeting the required error probability and data rate.

(b) To achieve R = 150 kbits/sec we need to send 0.075 bits per sample (or 1 bit every 13.33 samples). This can be achieved using 2-PAM and x13 repetition. The required SNR to achieve the target error probability is ≈12 dB - $10\log_{10}(13) \approx 1$ dB. The corresponding $E_b/N_0 \approx 12 + 10\log_{10}(13) + 10\log_{10}(1) = 12$ dB. If we use M-PAM with $M > 2$, E_b/N_0 remains unchanged. If we use M-PPM we need $\log_2 M/M \geq 0.075$. From the plot in Q1 we see that at an error probability of 10^{-4} the larger the value of M the smaller the required SNR. Thus we choose $M = 64$, the maximum value in the proscribed range, in which case we can compute the required E_b/N_0 to be around 8 dB. So this is the best option.

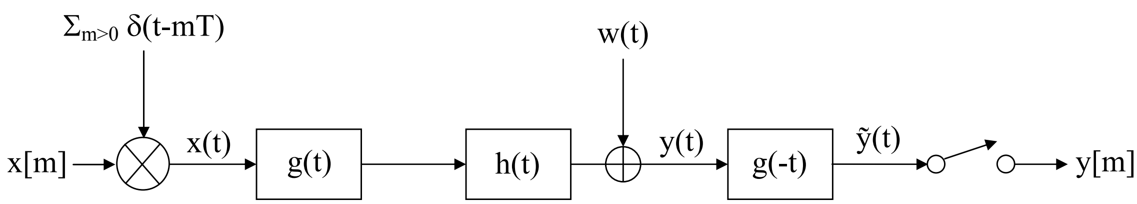
Another possibility is to use only a fraction of the allocated bandwidth. Say we use a bandwidth of $0.075W$, then the noise variance is reduced by a factor of $1/0.075$ and the SNR is boosted by $10 \log 10(1/0.075) \approx 11$ dB. With 2-PAM we would now be able to communicate at R $= 150$ kbits/sec using an SNR of $\approx 12 - 11 = 1$ dB and an $E_b/N_0 \approx 1$ dB. This is less energy per bit than 64-PPM, so it is a more desirable option if we are not restricted to using the full bandwidth allocated.

3. (a) A system diagram is shown below, where

$$g(t) \triangleq \text{sinc}(t/T).$$

We have

$$\begin{aligned}
y(t) &= \sum_{m>0} x(m) \cdot g(t - mT) * h(t) + w(t) \\
\tilde{y}(t) &= \sum_{m>0} x(m) \cdot g(t - mT) * g^*(-t) * h(t) + \tilde{w}(t) \\
y[n] &= \tilde{y}(nT) = \sum_{m>0} x(m) \cdot g(t - mT) * g^*(-t) * h(t)\Big|_{t=nT} + w[n]
\end{aligned}$$



(b) As the set of functions $\{\text{sinc}(t/T - m)\}_{m \in \mathbb{Z}}$ forms an orthonormal basis for the space L^2, which has the inner product defined by

$$< f(t), g(t) >= \int_{-\infty}^{\infty} f(t)g(t)dt,$$

we can write the filtered noise in terms of this basis as

$$\tilde{w}(t) = \sum_{m=-\infty}^{+\infty} \alpha_m \text{sinc}(t/T - m),$$

for some coefficients α_m. Solving for α_m we find $\alpha_m = w[m]$, which yields

$$\begin{aligned}
\lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \mathbb{E}[\tilde{w}(t)]^2 dt &= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \mathbb{E}\left[\sum_{m=-\infty}^{+\infty} w[m]\text{sinc}(t/T - m)\right]^2 dt \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \mathbb{E}\left[\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} w[m]w[n]\text{sinc}(t/T - m)\text{sinc}(t/T - n)\right] dt \\
&= \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathbb{E}\left[w[m]w[n]\right] \int_{-\tau/2}^{\tau/2} \text{sinc}(t/T - m)\text{sinc}(t/T - n)dt
\end{aligned}$$

Now if for any $\epsilon > 0$, $|m| > (1+\epsilon)\tau/2T$ or $|n| > (1+\epsilon)\tau/2T$, then

$$\begin{aligned}
\left|\int_{-\tau/2}^{\tau/2} \mathrm{sinc}(t/T-m)\mathrm{sinc}(t/T-n)dt\right| &\leq \int_{-\tau/2}^{\tau/2} \left|\mathrm{sinc}(t/T-m)\mathrm{sinc}(t/T-n)\right| dt \\
&< \frac{1}{\pi}\int_{-\tau/2}^{\tau/2} \frac{T}{|t-(1+\epsilon)\tau/2|}dt \\
&= \frac{1}{\pi}\int_{-\tau/2}^{\tau/2} \frac{T}{(1+\epsilon)\tau/2 - t}dt \\
&= \frac{T}{\pi}\left[\frac{1}{((2+\epsilon)\tau/2)^2} - \frac{1}{(\epsilon\tau/2)^2}\right] \\
&\leq \frac{T}{\pi(1+\epsilon/2)^2\tau^2} \\
&\to 0
\end{aligned}$$

as $\tau \to \infty$. Thus

$$\begin{aligned}
&\lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \mathbb{E}[\tilde{w}(t)]^2 dt \\
&\quad = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{m=-(1+\epsilon)\tau/2T}^{+(1+\epsilon)\tau/2T} \sum_{n=-(1+\epsilon)\tau/2T}^{+(1+\epsilon)\tau/2T} \mathbb{E}\left[w[m]w[n]\right] \int_{-\tau/2}^{\tau/2} \mathrm{sinc}(t/T-m)\mathrm{sinc}(t/T-n)dt \\
&\quad = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{m=-(1+\epsilon)\tau/2T}^{+(1+\epsilon)\tau/2T} \sum_{n=-(1+\epsilon)\tau/2T}^{+(1+\epsilon)\tau/2T} \mathbb{E}\left[w[m]w[n]\right] \delta_{nm} \\
&\quad = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{m=-(1+\epsilon)\tau/2T}^{+(1+\epsilon)\tau/2T} \mathbb{E}\left[w[m]^2\right] \\
&\quad = \lim_{\tau\to\infty} \frac{1}{\tau T} \sum_{m=-(1+\epsilon)\tau/2}^{+(1+\epsilon)\tau/2} \mathbb{E}\left[w[m]^2\right] \\
&\quad = \lim_{n\to\infty} \frac{1}{nT} \sum_{m=-(1+\epsilon)n/2}^{+(1+\epsilon)n/2} \mathbb{E}\left[w[m]^2\right]
\end{aligned}$$

As this result for holds for any $\epsilon > 0$ we let $\epsilon \to 0$ to get

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \mathbb{E}[\tilde{w}(t)]^2 dt \quad = \lim_{n\to\infty} \frac{1}{nT} \sum_{m=-n/2}^{+n/2} \mathbb{E}\left[w[m]^2\right].$$

(c) We can essentially use the same proof, we just need to show that if for any $\epsilon > 0$, $|m| > (1+\epsilon)\tau/2T$ or $|n| > (1+\epsilon)\tau/2T$, then

$$\left|\int_{-\tau/2}^{\tau/2} g(t/T-m)g(t/T-n)dt\right| \to 0 \tag{1}$$

as $\tau \to \infty$. As the functions $g(t/T - m)$ are orthonormal, they have finite energy, i.e.

$$\int_{-\infty}^{\infty} g^2(t)dt < \infty.$$

This guarantees the property (1) holds.

4. (a) D is also measured in seconds. The variable t represents the number of seconds after the signal enters the channel. The parameter D dictates the amount of memory in the channel.

(b) The impulse response is non-zero for all $t > 0$, an infinite duration. However, we can define the ϵ-delay spread as the smallest value of t such that a fraction $1 - \epsilon$ of the total energy is contained in the impulse response up to time t, i.e.

$$\epsilon\text{-delay spread} = \min_{\text{s.t. } \int_0^t |h(\tau)|^2 d\tau > (1-\epsilon) \int_0^\infty |h(\tau)|^2 d\tau} t$$

For the channel in question, the ϵ-delay spread is

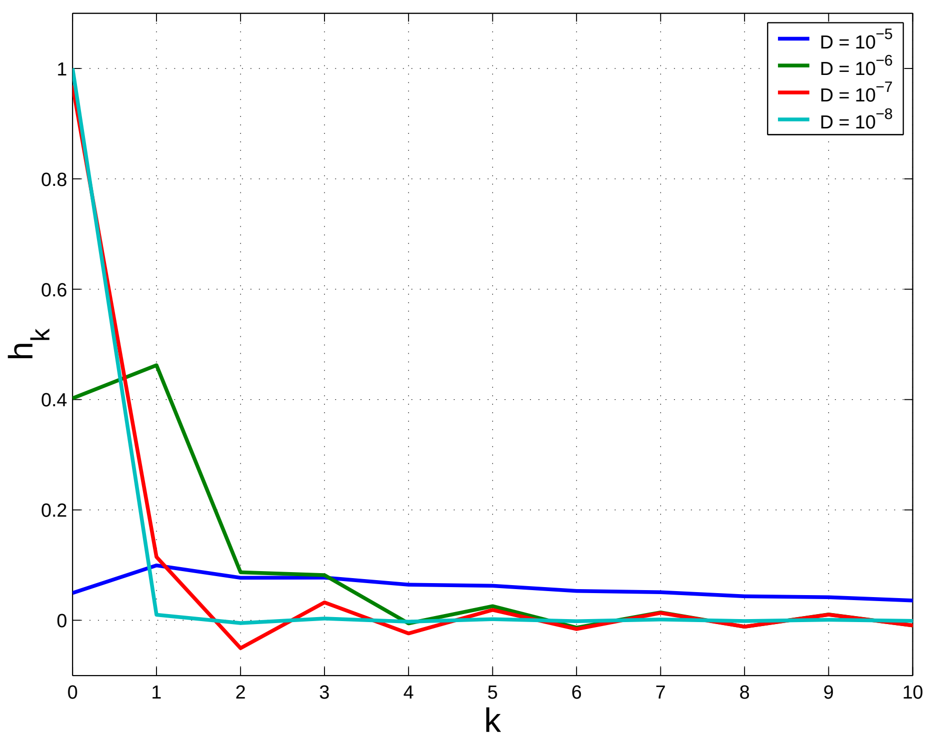
$$\frac{D}{2} \log_2 \frac{1}{\epsilon}$$

which is proportional to D, as expected.

c) The discrete time channel response is

$$h_k = \int_0^\infty \frac{1}{D} e^{-t/D} \text{sinc}\left(k - \frac{t}{T}\right) dt$$

with $T = 10^{-6}$ seconds, we have the following plot



The discrete time channel is effectively memoryless for $D = 10^{-7}$ and $D = 10^{-8}$. This is consistent with what was taught in class, as for these values of D, most of the symbol energy arrives in the first tap.