EE 121 - Introduction to Digital Communications
Homework 9 Solutions

May 16, 2008

1. (a) If we do no transmit anything on symbol time 1 then

$$\begin{pmatrix} y[0] \\ y[1] \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} x[0] + \begin{pmatrix} w[0] \\ w[1] \end{pmatrix}$$

in which case the matched filter receiver projects $x[0]$ onto the vector $[h_0, h_1]^T$, i.e.

$$\hat{x}[0] = h_0 y[0] + h_1 y[1]$$

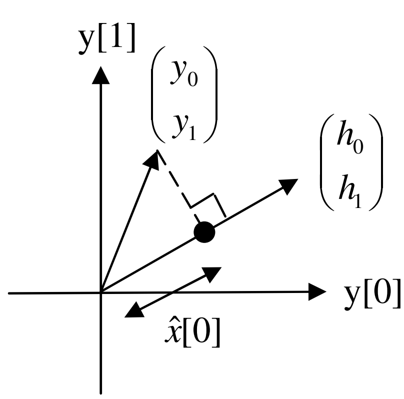
A picture illustrating this operation is given below. We can write the estimate of $x[0]$ in terms of $x[0]$ and the noise like so

$$\hat{x}[0] = (h_0^2 + h_1^2)x[0] + h_0 w[0] + h_1 w[1].$$

As $w[0]$ and $w[1]$ are i.i.d $\mathcal{N}(0, \sigma^2)$ r.v.'s, $h_0 w[0] + h_1 w[1] \sim \mathcal{N}(0, (h_0^2 + h_1^2)\sigma^2)$ and thus the error probability is precisely

$$\begin{aligned} \Pr(\mathcal{E}) &= Q\left(\sqrt{\frac{(h_0^2 + h_1^2)^2}{h_0^2 + h_1^2}\mathsf{SNR}}\right) \\ &= Q\left(\sqrt{(h_0^2 + h_1^2)\mathsf{SNR}}\right) \end{aligned}$$

where $\mathsf{SNR} \equiv E/\sigma^2$.



(b) The matched filter estimate of $x[0]$ can now be written as

$$\hat{x}[0] = (h_0^2 + h_1^2)x[0] + h_0 h_1 x[1] + h_0 w[0] + h_1 w[1].$$

where the $h_1 h_0 x[1]$ term is the interference. We compute the average error probability of detecting $x[0]$ by conditioning on the interference. Due to the symmetry we can assume without loss of generality, that $x[1] = +\sqrt{E}$ was transmitted. We have

$$\Pr(\mathcal{E}) = \frac{1}{2}\Pr\left(\mathcal{E}\,\middle|\,x_1 = +\sqrt{E}, x_2 = +\sqrt{E}\right) + \frac{1}{2}\Pr\left(\mathcal{E}\,\middle|\,x_1 = +\sqrt{E}, x_2 = -\sqrt{E}\right).$$

As the matched filter detector for $x[0]$ treats the interference as noise, the error probability is

$$\Pr(\mathcal{E}) = \frac{1}{2}Q\left(\sqrt{\frac{(h_0^2 + h_1^2 + h_0 h_1)^2}{h_0^2 + h_1^2}\mathsf{SNR}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{(h_0^2 + h_1^2 - h_0 h_1)^2}{h_0^2 + h_1^2}\mathsf{SNR}}\right).$$

(c) At high-SNR

$$\Pr(\mathcal{E}) \approx \frac{1}{2}Q\left(\sqrt{\frac{(h_0^2 + h_1^2 - h_0 h_1)^2}{h_0^2 + h_1^2}\mathsf{SNR}}\right).$$

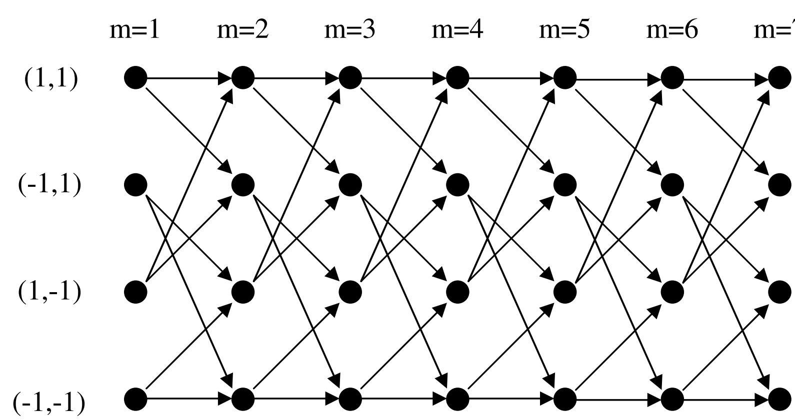
More generally $\Pr(\mathcal{E}) \to 0$ as $\mathsf{SNR} \to \infty$ for this example, which is desirable. This is because we are using BPSK on a two tap ISI channel. With a sufficient number of taps or a sufficient number of constellation points, this conclusion will not hold and we will instead find that $\Pr(\mathcal{E}) \to$ constant > 0 as $\mathsf{SNR} \to \infty$.

(d) The MLSD has an error performance which is at least as good as the zero-forcing equalizer where $\hat{x}[0] = y[0]$. This means

$$\Pr(\mathcal{E}) < Q\left(\sqrt{h_0^2\mathsf{SNR}}\right)$$

which for any fixed $h_0^2 > 0$, goes to zero as SNR goes to infinity.

2. The received sequence is $y[0] = 1.1$, $y[1] = 2.9$, $y[2] = 0.5$, $y[3] = -3.6$, $y[4] = 2.6$, $y[5] = -1.2$, $y[6] = -1.9$ and $y[7] = 2.3$. The trellis diagram is shown below.



At stage 1 there are four possible paths, $(1,1),(1,-1),(-1,1),(-1,-1)$, with costs

$$\begin{aligned}
C(1,1) &= (1.5\times 1-1.1)^2+(1.5\times 1+0.6\times 1-2.9)^2 = 0.80\\
C(1,-1) &= (1.5\times 1-1.1)^2+(-1.5\times 1+0.6\times 1-2.9)^2 = 14.60\\
C(-1,1) &= (-1.5\times 1-1.1)^2+(1.5\times 1-0.6\times 1-2.9)^2 = 10.76\\
C(-1,-1) &= (1.5\times 1-1.1)^2+(1.5\times 1+0.6\times 1-2.9)^2 = 31.76
\end{aligned}$$

At stage 2 we pick the least costly two paths entering each vertex in the trellis at $m=2$.

$$\begin{aligned}
C(1,1,1) &= C(1,1)+(1.5\times 1+0.6\times 1-0.5)^2 = 0.80+2.56 = 3.36\\
C(1,1,-1) &= C(1,1)+(-1.5\times 1+0.6\times 1-0.5)^2 = 0.80+1.96 = 2.76\\
C(1,-1,1) &= C(1,-1)+(1.5\times 1-0.6\times 1-0.5)^2 = 14.60+0.16 = 14.76\\
C(1,-1,-1) &= C(1,-1)+(-1.5\times 1-0.6\times 1-0.5)^2 = 14.60+6.76 = 21.36
\end{aligned}$$

As all four paths have $\hat{x}[0]=1$ in common, we can decode this symbol immediately. At stage three we have

$$\begin{aligned}
C(1,1,1,1) &= 3.36+(1.5+0.6+3.6)^2 = 35.85\\
C(1,1,1,-1) &= 3.36+(-1.5+0.6+3.6)^2 = 10.65\\
C(1,1,-1,1) &= 2.76+(1.5-0.6+3.6)^2 = 23.01\\
C(1,1,-1,-1) &= 2.76+(-1.5-0.6+3.6)^2 = 5.01
\end{aligned}$$

As all four paths have the same second symbol, we can immediately decode $\hat{x}[1]=1$. At stage four we have

$$\begin{aligned}
C(1,1,-1,1,1) &= 23.01+(1.5+0.6-2.6)^2 = 23.26\\
C(1,1,-1,1,-1) &= 23.01+(-1.5+0.6-2.6)^2 = 35.26\\
C(1,1,-1,-1,1) &= 5.01+(1.5-0.6-2.6)^2 = 7.90\\
c(1,1,-1,-1,-1) &= 5.01+(-1.5-0.6-2.6)^2 = 27.10
\end{aligned}$$

Thus we can immediately decode $\hat{x}[2]=-1$. At stage five we have

$$\begin{aligned}
C(1,1,-1,-1,1,1) &= 7.90+(1.5+0.6+1.2)^2 = 18.79\\
C(1,1,-1,-1,1,-1) &= 7.90+(-1.5+0.6+1.2)^2 = 7.99\\
C(1,1,-1,-1,-1,1) &= 27.10+(1.5-0.6+1.2)^2 = 31.51\\
C(1,1,-1,-1,-1,-1) &= 27.10+(-1.5-0.6+1.2)^2 = 27.91
\end{aligned}$$

Thus $\hat{x}[3]=-1$. At stage six we compute

$$\begin{aligned}
C(1,1,-1,-1,1,1,1) &= 18.79+(1.5+0.6+1.9)^2 = 34.79\\
C(1,1,-1,-1,1,1,-1) &= 18.79+(-1.5+0.6+1.9)^2 = 19.79\\
C(1,1,-1,-1,1,-1,1) &= 7.99+(1.5-0.6+1.9)^2 = 15.83\\
C(1,1,-1,-1,1,-1,-1) &= 7.99+(-1.5-0.6+1.9)^2 = 8.03
\end{aligned}$$

Thus $\hat{x}[4] = 1$. At stage seven we compute

$$\begin{aligned}
C(1,1,-1,-1,1,-1,1,1) &= 15.83 + (1.5 + 0.6 - 2.3)^2 = 15.87 \\
C(1,1,-1,-1,1,-1,1,-1) &= 15.83 + (-1.5 + 0.6 - 2.3)^2 = 26.07 \\
C(1,1,-1,-1,1,-1,-1,1) &= 8.03 + (1.5 - 0.6 - 2.3)^2 = 9.99 \\
C(1,1,-1,-1,1,-1,-1,-1) &= 8.03 + (-1.5 - 0.6 - 2.3)^2 = 27.39
\end{aligned}$$

The most probable path, and hence the path the Viterbi algorithm outputs, is then $\hat{x}[0] = 1$, $\hat{x}[1] = 1$, $\hat{x}[2] = -1$, $\hat{x}[3] = -1$, $\hat{x}[4] = 1$, $\hat{x}[5] = -1$, $\hat{x}[6] = -1$, $\hat{x}[7] = 1$. Comparing this to the transmitted sequence we have an error for $x[2]$, but all other symbols are decoded correctly.

3. (a) $0, +\sqrt{E}, -\sqrt{E}, 0, 0, +\sqrt{E}, 0, -\sqrt{E}, 0.$

(b) R = 1 bits/symbol. The average energy is $E/2$.

(c) We effectively transmit $+\sqrt{E}$ and $-\sqrt{E}$ each with probability 1/4, and 0 with probability 1/2. The symbol by symbol detection rule can therefore be found just by considering $|y[m]|$, in which case we have two symbols $\sqrt{E}$ and 0, each occurring with probability 1/2. The ML detection rule is then

$$\hat{b}[m] = \begin{cases} 0, & \text{if } |y[m]| < a \\ 1, & \text{if } |y[m]| \geq a. \end{cases}$$

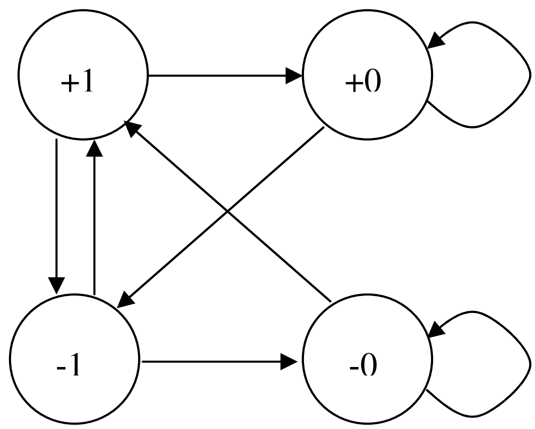
To find a we solve

$$\begin{aligned}
p_{|y[m]|\,\big|\,b[m]}(a|0) &= p_{|y[m]|\,\big|\,b[m]}(a|1) \\
\Rightarrow p_{w[m]}(a) + \Rightarrow p_{w[m]}(-a) &= p_{w[m]}(a - \sqrt{E}) + p_{w[m]}(a + \sqrt{E}) \\
\Rightarrow 2p_{w[m]}(a) &= p_{w[m]}(a - \sqrt{E}) + p_{w[m]}(a + \sqrt{E}) \\
\Rightarrow 2e^{-a^2/2\sigma^2} &= e^{-(\sqrt{E}-a)^2/2\sigma^2} + e^{-(\sqrt{E}+a)^2/2\sigma^2}.
\end{aligned}$$

a is then given by the solution to the last equation (it cannot be expressed explicitly). The error probability is

$$\begin{aligned}
\Pr(\mathcal{E}) &= \frac{1}{2}\Pr(\mathcal{E}|b[m] = 0) + \frac{1}{2}\Pr(\mathcal{E}||b[m]| = \sqrt{E}) \\
&= Q\left(\frac{a}{\sigma}\right) + \frac{1}{2}Q\left(\frac{\sqrt{E} - a}{\sigma^2}\right) + \frac{1}{2}Q\left(\frac{\sqrt{E} + a}{\sigma^2}\right)
\end{aligned}$$

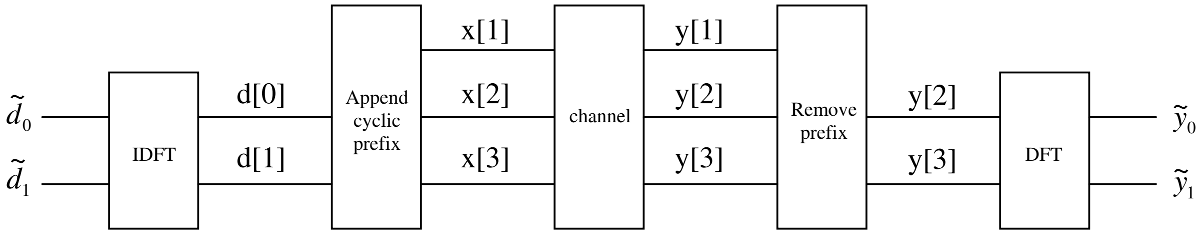
(d) Symbol by symbol detection is not optimal because we are ignoring knowledge of whether a $b[m] = 1$ will more likely correspond to a $+\sqrt{E}$ than a $-\sqrt{E}$. To do MLSD we use the state space $\mathcal{S} = \{+1, +0, -0, -1\}$ so that the state $s[m] \in \mathcal{S}$ for each time $m = 0, 1, \ldots$. State '+1' corresponds to transmitting a $+\sqrt{E}$, state '-1' corresponds to transmitting a $-\sqrt{E}$, state '+0' corresponds to transmitting a 0 when the previous non-zero transmission was a $+\sqrt{E}$, and lastly, state '-0' corresponds to transmitting a 0 when the previous non-zero transmission was a $-\sqrt{E}$. The state transition diagram is shown below. All transitions occur with probability 1/2. We can draw a corresponding trellis diagram and compute costs in the obvious way.



4. Let $y[r] = (x * h)[r] = \sum_{m=0}^{N-1} h[m]x[r - m \mod N]$. Then

$$
\begin{aligned}
Y[k] &= \sqrt{N}\sum_{r=0}^{N-1} y[r]e^{-j2\pi rk/N} \\
&= \sqrt{N}\sum_{r=0}^{N-1}\sum_{m=0}^{N-1} h[m]x[r-m \mod N]e^{-j2\pi rk/N} \\
&= \sqrt{N}\sum_{m=0}^{N-1} h[m]\sum_{r=0}^{N-1} x[r-m \mod N]e^{-j2\pi rk/N} \\
&= \sqrt{N}\sum_{m=0}^{N-1} h[m]\sum_{n=-m}^{N-1-m} x[n \mod N]e^{-j2\pi(n+m)k/N} \\
&= \sqrt{N}\sum_{m=0}^{N-1} h[m]e^{-j2\pi mk/N}\sum_{n=-m}^{N-1-m} x[n \mod N]e^{-j2\pi nk/N} \\
&= \sqrt{N}\sum_{m=0}^{N-1} h[m]e^{-j2\pi mk/N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} \\
&= \sqrt{N}\sum_{m=0}^{N-1} h[m]e^{-j2\pi mk/N}X[k] \\
&= \sqrt{N}H[k]X[k]
\end{aligned}
$$

5. (a)



where

$$d[0] = \frac{\tilde{d}[0]}{\sqrt{2}} + \frac{\tilde{d}[1]}{\sqrt{2}}$$
$$d[1] = \frac{\tilde{d}[0]}{\sqrt{2}} - \frac{\tilde{d}[1]}{\sqrt{2}}$$

and

$$\tilde{y}[0] = \sqrt{2}y[2] + \sqrt{2}y[3]$$
$$\tilde{y}[1] = \sqrt{2}y[2] - \sqrt{2}y[3]$$

corresponding to IDFT and DFT operations, respectively.

(b) $\tilde{h}_0 = h_0/\sqrt{N} + h_1/\sqrt{N} = \frac{3}{2\sqrt{2}}$ and $\tilde{h}_0 = h_0/\sqrt{N} - h_1/\sqrt{N} = 3/2\sqrt{2} = \frac{1}{2\sqrt{2}}$.

(c) Not very efficient as the prefix is half the length of the data block, i.e. 66 percent efficiency. We could improve efficiency by increasing the number of tones.

6. (a) The Doppler spread at $v = 120$ km/h at $f_c = 5$ Ghz is $2f_c v/c = 2 \times 5 \times 10^9 \times 33.33/3 \times 10^8 = 1.111$KHz. With $W = 5$Mhz of bandwidth and N tones, the inter-carrier spacing is $\approx W/N$. Thus we must have $1111 \leq 0.01 \times W/N \Rightarrow N \leq 45$ tones. So choose $N = 45$ tones. The number of channels taps is then $\approx 10^{-5} \times 2W = 50$. So we use a cyclic prefix of length 49.

(b) The overhead is $49/(49 + 45) = 52.6$ percent. The data rate is then $2 \times 0.474 \approx 0.95$ bits/symbol or about 4.74 Mbits/sec.

(c) In an indoor wireless system the maximum velocity would likely decrease, leading to a smaller doppler spread and the ability to pack more tones into the frequency band. The delay spread would also decrease as a consequence of the shorter path lengths involved. This leads to fewer taps. Both effects would lead to an improvement in efficiency, i.e. a reduction of overhead.