oct28,05

FIR Filter Design using Windows

$N = M + 1$
\# of Taps.
$2\alpha + 1 = N$
$\alpha = \frac{N-1}{2}$

Kaiser Window:

$$W(n) = \begin{cases} I_0\left[\underbrace{\beta'\left(1 - \left(\frac{n-\alpha}{\alpha}\right)^2\right)^{\frac{1}{2}}}_{x} \right] & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

I_0 = zeroth order modified Bessel function.

$$I_0(x) = 1 + \frac{x^2}{2^2 (1!)^2} + \frac{x^4}{2^4 (2!)^2} + \frac{x^6}{2^6 (3!)^2} + \dots$$

Soln to the following diff eqn:

$$\frac{x^2\, dy^2}{dx^2} + x \frac{dy}{dx} - (x^2 + n^2)\, y = 0$$

nth order modified Bessel fn.

β' controls the shape of Kaiser window allowing trade off between sidelobe and mainlobe.

Design Using Kaiser Window

(1) $\Delta\omega = \omega_s - \omega_p$: Transition width.

(2) ripple $= \delta \longrightarrow A = -20 \log_{10} \delta$

choose $\alpha = \frac{M}{2} = \frac{N-1}{2}$ and β' as follows.

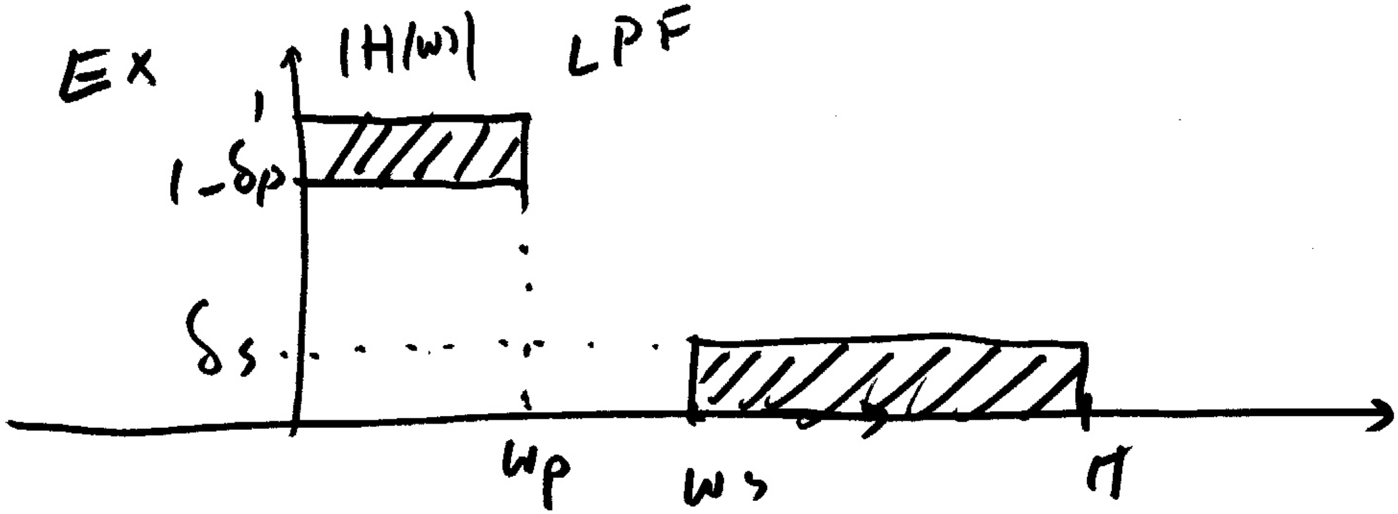
(a) $M = 2\alpha = \dfrac{A - 8}{2.285\,\Delta\omega}$

(b)

$$\beta' = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5842(A - 21)^{0.4} & 21 \leq A < 50 \\ 0 & A < 21 \end{cases}$$

2.1

Ex



linear phase filter.

$H_d(\omega) \longrightarrow h_d(n)$

$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} \, e^{j(\alpha\omega - \beta)} \leftarrow$ generalized LP filter.

$\beta = 0 \longrightarrow$ Type I or Type II $\longrightarrow$ Both capable of LPF.

Specification:

$\omega_p = 0.4\pi$

$\omega_s = 0.6\pi$

$$\longrightarrow \left| \begin{array}{l} \delta_p = 0.01 \\ \delta_s = 0.001 \end{array} \right\} \rightarrow \delta = 0.001$$

$\Delta\omega = \omega_s - \omega_p = 0.2\pi$

$A = -20 \log_{10} \delta = -60$

plug in (a), (b): To compute M, β'

$M = 37 \Rightarrow$ # of Taps $= M + 1 = 38 \Rightarrow$ Type II.

Fig 7.25

Windows : No fine grain control over δ_s, δ_p, $\Delta\omega$.

OPTimal FIR filter Design

Type I Generalized Linear Phase Filter.

LPF

$$H(\omega) = H_m(\omega)\, e^{j\beta}\, e^{-j\alpha\omega}$$

Assume Type I $\Longrightarrow \beta = 0$,

$N = \#$ of Taps is odd.
$= 2M + 1$

$$h(n) = h(N - n - 1)$$

$$H_m(\omega) = \boxed{\sum_{n=0}^{M} \underline{a(n)} \cos(\omega n) \triangleq G(\omega)}$$

$a(0) = h(0)$

$a(n) = 2h(M-n)$ all other n.

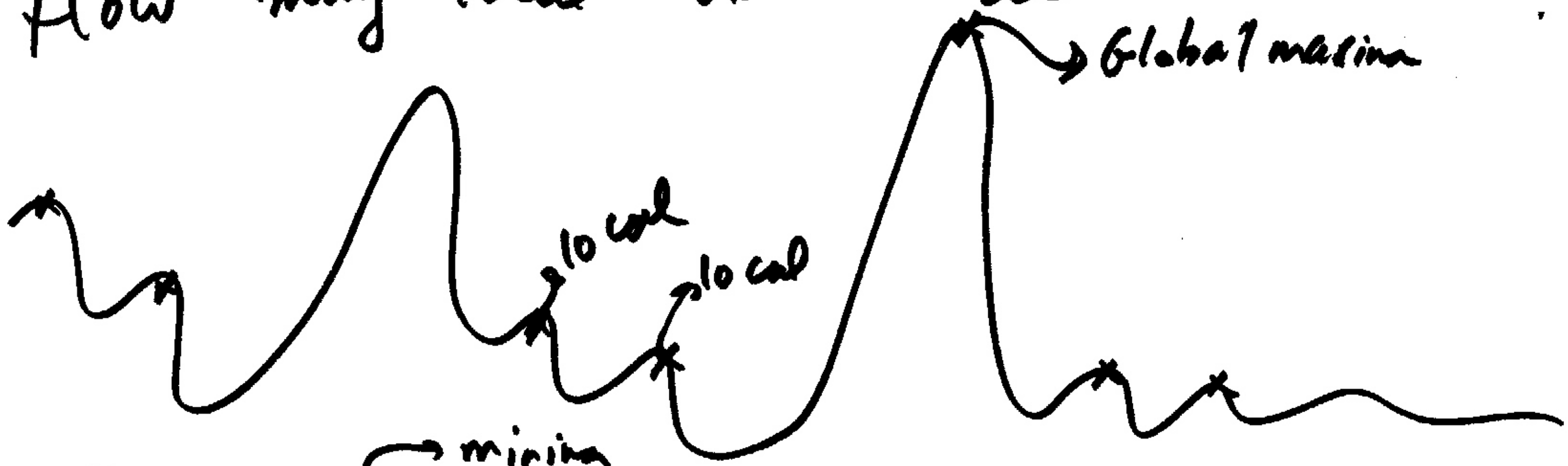
$$H(\omega) = \sum_{n=0}^{N-1} \underline{h(n)}\, e^{-j\omega n}$$

$$H(\omega) = G(\omega)\, e^{-j\alpha\omega}$$

Observations on $G(\omega)$

① $G(\omega)$ is a continuous fn of ω and is as many times differentiable as you want.

② How many local extrema does $G(\omega)$ have?



Extrema → minima, maxima.

Express $\cos(\omega n)$ as sum of powers of $\cos\omega$.

$$\cos(2\omega) = 2\cos^2\omega - 1$$

$$\cos(3\omega) = \cos(2\omega + \omega) = \cos 2\omega \cos\omega - \sin 2\omega \sin\omega$$

$$= \cos\omega\left[2\cos^2\omega - 1\right] - 2\sin^2\omega\cos\omega$$

$$= 2\cos^3\omega - \cos\omega - 2\cos\omega\left[1 - \cos^2\omega\right]$$

$$= 4\cos^3\omega - 3\cos\omega$$

Generally $\cos(\omega n)$ as sum of powers of $\cos\omega$

$$\cos(\omega n) = \sum_{i=0}^{n} \eta_i (\cos\omega)^i$$

Tchebychev

"Chebyshev"

$$G(\omega) = \sum_{n=0}^{M} a(n) \left[\sum_{i=0}^{n} \eta_i \; (\cos\omega)^i \right]$$

$$G(\omega) = \sum_{n=0}^{M} \gamma(n) \; (\cos\omega)^n$$

γ depends on η and a

To compute local extrema of $G(\omega)$
take the derivative and set to zero.

$$\frac{dG(\omega)}{d\omega} = 0 \Rightarrow \sum_{n=0}^{M} \gamma(n) \; n \left(\cos\omega\right)^{n-1} \left(-\sin\omega\right)$$

$$= 0 \implies$$

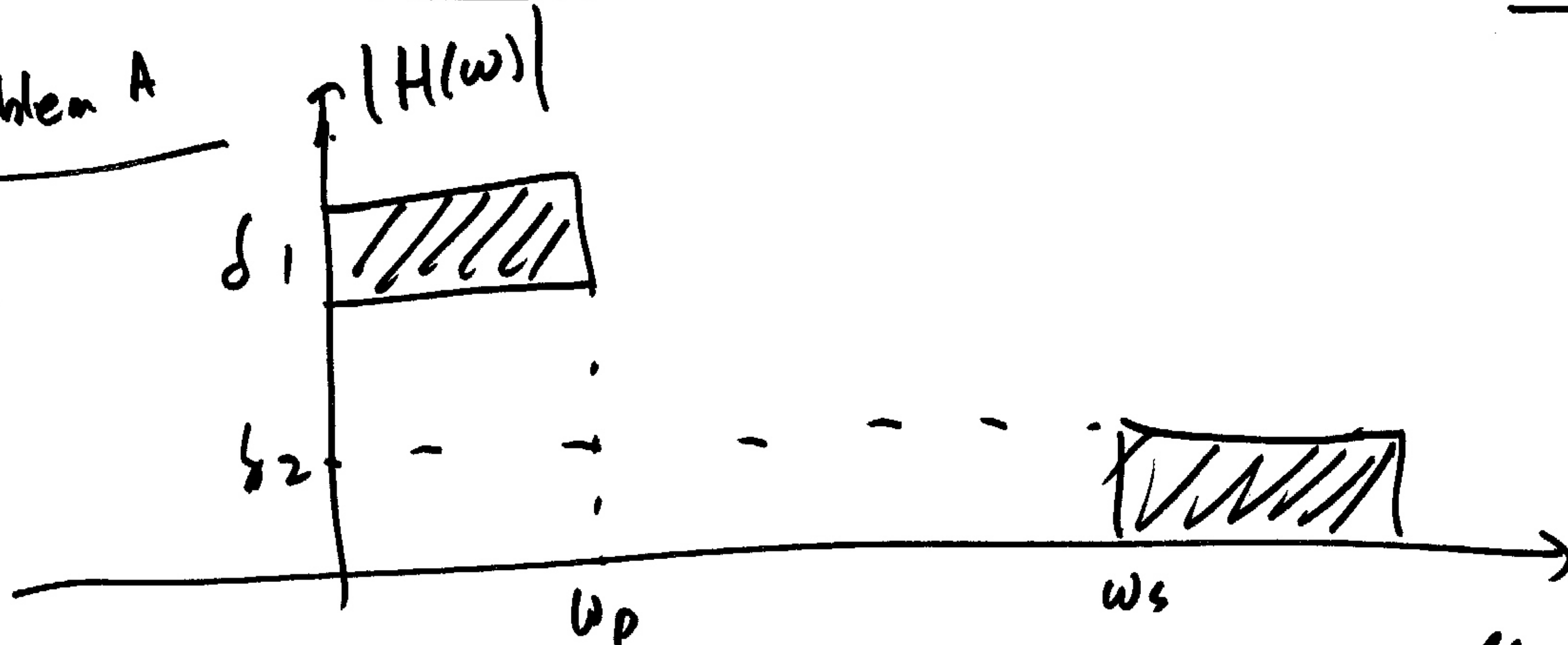
$\sin\omega = 0 \Longrightarrow \omega = 0, \pi$

$$\underbrace{\sum_{n=0}^{M} \gamma(n)\, n\, (\cos\omega)^{n-1}}_{\text{Polynomial in } \cos\omega = x} = 0 \longrightarrow \text{Max of } M-1 \text{ real zeros.}$$

$\Rightarrow$ $\boxed{\text{Total \# of local extrema for } G(\omega) \text{ is } (M-1) + 2 = M+1}$

Problem Statement Optimal FIR Filter Design

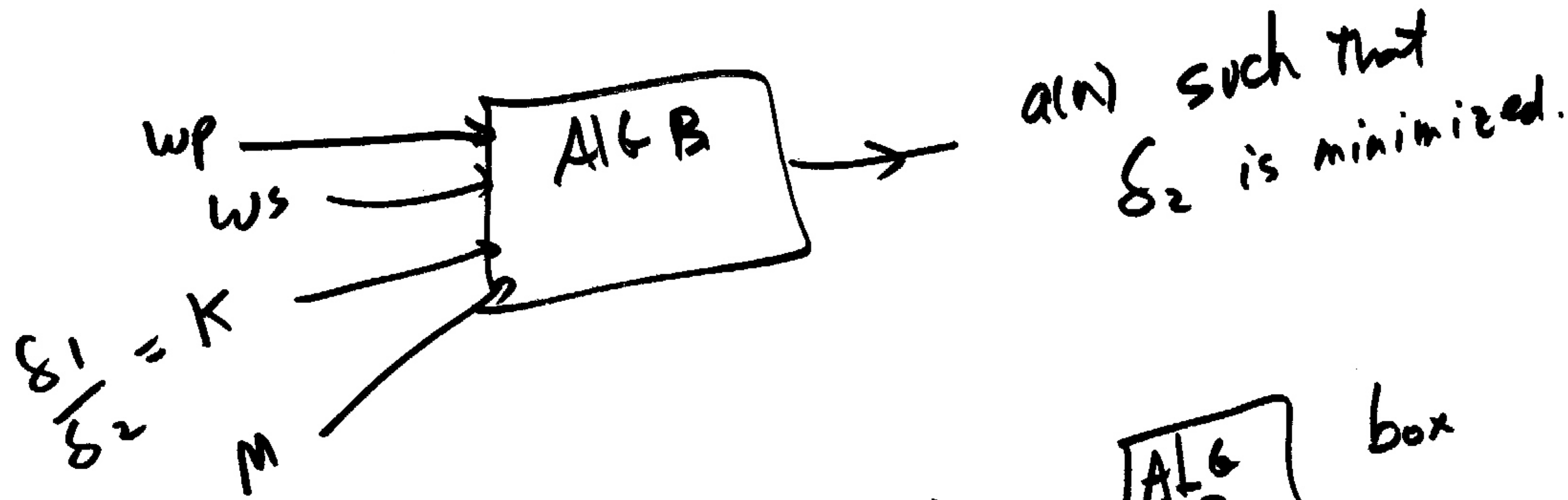
Problem A



Given ω_s, ω_p, δ_1, δ_2, Determine coeff of $G(\omega)$ i.e. $a(n)$ such that M is minimized

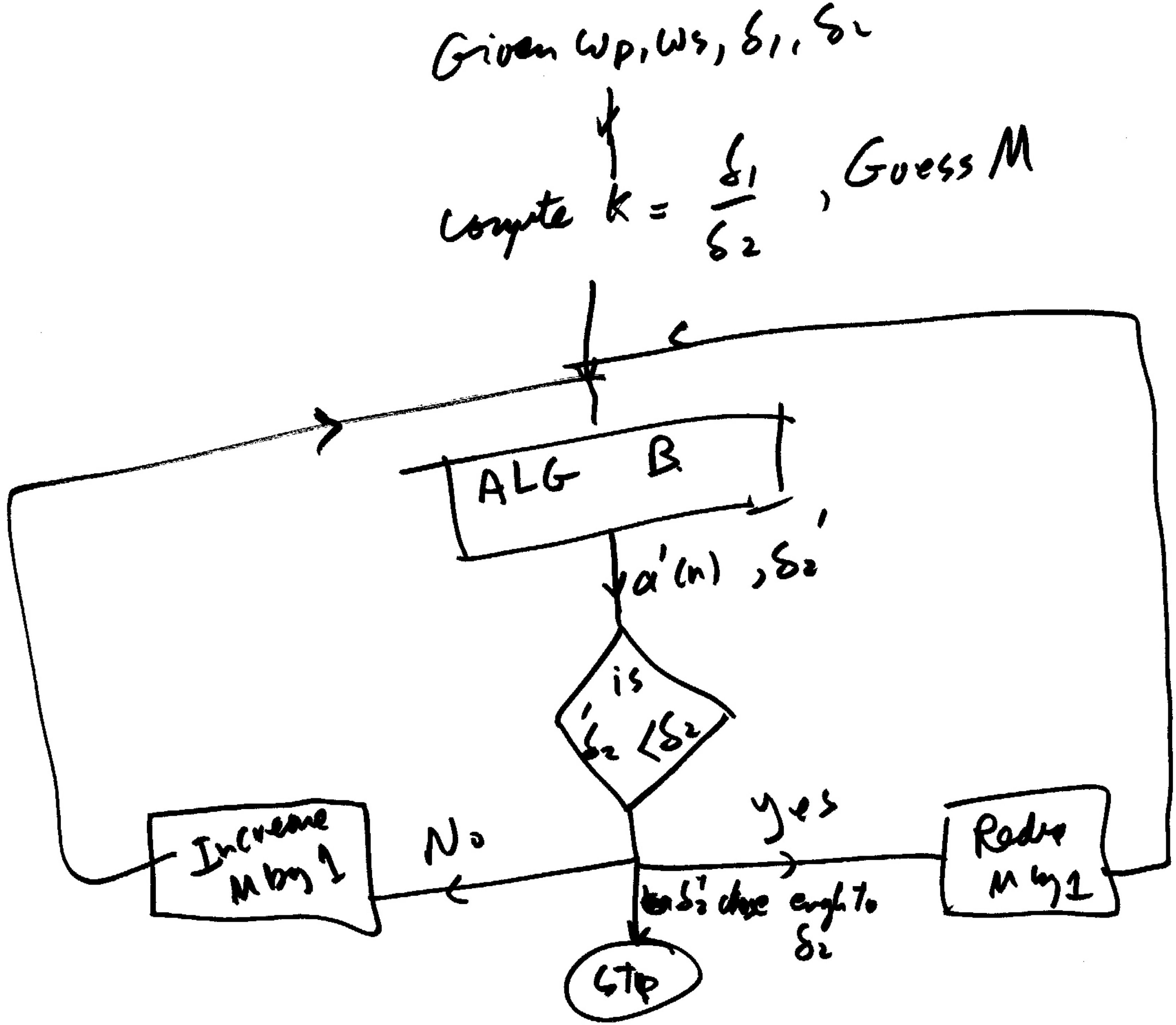
Problem B Given w_p, w_s, M, $K = \frac{\delta_1}{\delta_2}$

Determin $a(n)$ such that δ_2 is minimized.



box $\boxed{\text{ALG B}}$ box

Show that if I had a box that solves Problem B, Then I can use it To solve Problem A.



Given $\omega_p, \omega_s, \delta_1, \delta_2$
Compute $k = \frac{\delta_1}{\delta_2}$, Guess M
ALG B
$\alpha'(n)$, δ_2'
is $\delta_2' < \delta_2$
Yes
No
Increase M by 1
Reduce M by 1
δ_2' close enough to δ_2
Stop

Problem C: $E(\omega) = W(\omega)\left[G(\omega) - D(\omega)\right]$

where $W(\omega)$ = positive weighting function $\begin{cases} \frac{1}{K} & I_1 \\ 1 & I_2 \end{cases}$

$$G(\omega) = \sum_{n=0}^{M} a(n) \cos(\omega n)$$

$D(\omega)$ = Desired Freq. Response $= \begin{cases} 1 & I_1 \\ 0 & I_2 \end{cases}$

$$\boxed{\textbf{Problem C}: \text{ Find } a(n) \text{ To minimize } \max_{\omega \in F} |E(\omega)|}$$

where $F = I_1 \cup I_2$

is a subset of a closed Interval $0 \le \omega \le \pi$

I_1 = passband, I_2 = stopband.