EE123

Digital Signal Processing

Lecture 03

Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Why one is sum and the other integral?

Why use one over the other?

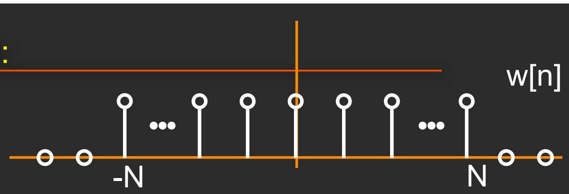
Alternative

$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi f k}$$

$$x[n] = \int_{-\pi}^{\pi} X(f)e^{j2\pi f n}df$$

Example 1:

w[n] "window"



DTFT:

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=-N}^{N} e^{-j\omega k} \\ &= e^{-j\omega N}\left(1 + e^{j\omega} + \cdots + e^{j\omega 2N}\right) \end{aligned}$$

Recall: $1 + p + p^2 + \cdots + p^M = \dfrac{1-p^{M+1}}{1-p}$ $\quad p = e^{j\omega}$, $\quad M = 2N$

Example 1 cont.

DTFT:

Example 1 cont.

DTFT:

$$W(e^{j\omega}) = e^{-j\omega N}\left(1 + e^{j\omega} + \cdots + e^{j\omega 2N}\right)$$

$$= e^{-j\omega N}\frac{1 - e^{j\omega(2N+1)}}{1 + e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega N}e^{j\omega}}{1 - e^{j\omega}} \qquad \frac{\times\ e^{-j\frac{\omega}{2}}}{\times\ e^{-j\frac{\omega}{2}}}$$

$$= \frac{e^{-j\omega(N+\frac{1}{2})} - e^{j\omega(N+\frac{1}{2})}}{e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}}$$

$$= \frac{\sin[(N+\frac{1}{2})\omega]}{\sin(\frac{\omega}{2})}$$

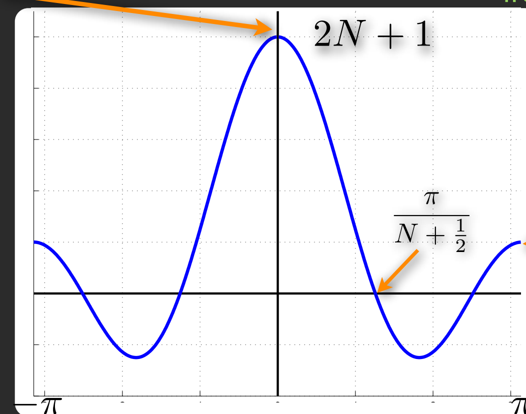
periodic sinc

Example 1 cont.

$$W(e^{j\omega}) = \frac{\sin[(N+\frac{1}{2})\omega]}{\sin(\frac{\omega}{2})} \rightarrow (2N+1) \text{ as } \omega \rightarrow 0$$

from l'Hôpital

also, Σx[n]



=1, why?

Properties of the DTFT

Periodicity: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Conjugate Symmetry:

$$X^*(e^{j\omega}) = X(e^{-j\omega})$$ if x[n] is real

$$\mathcal{R}e\left\{X(e^{-j\omega})\right\} = \mathcal{R}e\left\{X(e^{j\omega})\right\}$$

$$\mathcal{I}m\left\{X(e^{-j\omega})\right\} = -\mathcal{I}m\left\{X(e^{j\omega})\right\}$$

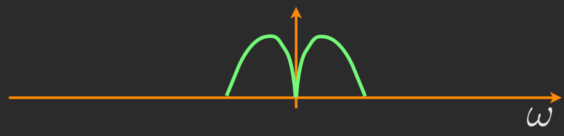
Big deal for: MRI, Communications, more....

Half Fourier Imaging in MR

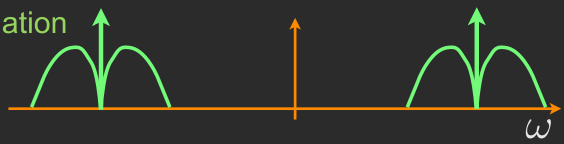
k-space (Raw Data) Image

SSB Modulation

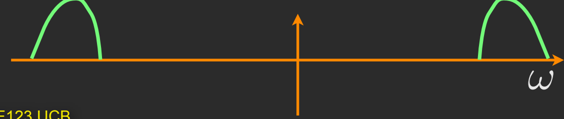
Real Baseband signal has conjugate symmetric spectrum



AM modulation



SSB-SC reduced power, half bandwidth



Properties of the DTFT cont.

Time-Reversal

$$x[n] \leftrightarrow X(e^{i\omega})$$
$$x[-n] \leftrightarrow X(e^{-i\omega})$$
$$= X^*(e^{j\omega}) \text{ if } x[n] \in \mathcal{R}eal$$

If x[n] = x[-n] and x[n] is real, then:

$$X(e^{j\omega}) = X^*(e^{j\omega})$$
$$\rightarrow X(e^{j\omega}) \in \mathcal{R}eal$$

Q: Suppose:

$$x[n] \leftrightarrow X(e^{j\omega})$$
$$? \leftrightarrow \mathcal{R}e\left\{X(e^{j\omega})\right\}$$

A: Decompose x[n] to even and odd functions

$$x[n] = x_e[n] + x_o[n]$$

$$x_e[n] := \frac{1}{2}(x[n] + x[-n])$$

$$x_o[n] := \frac{1}{2}(x[n] - x[-n])$$

$$x_e[n] + x_o[n] \rightarrow \mathcal{R}e\left\{X(e^{j\omega})\right\} + j\mathcal{I}m\left\{X(e^{j\omega})\right\}$$

Properties of the DTFT cont.

Time-Freq Shifting/modulation:

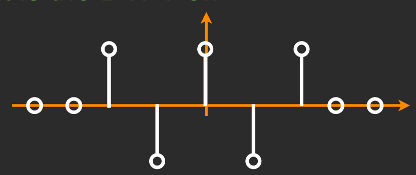
$$x[n] \leftrightarrow X(e^{j\omega})$$

Good for MRI! Why

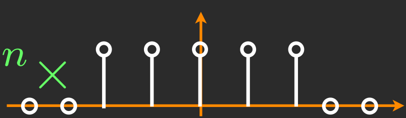
$$x[n-n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$$
$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_o)})$$

Example 2

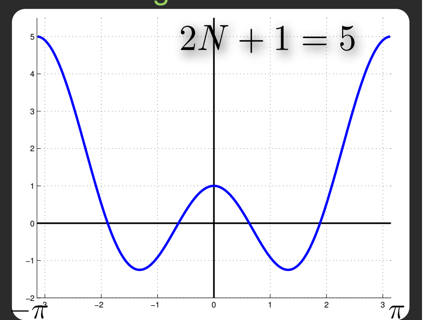
What is the DTFT of:



$e^{j\pi n} \times$



High Pass Filter



See 2.9 for more properties

Frequency Response of LTI Systems

Check response to a pure frequency:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0(n-k)}$$

$$= \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k} \right)}_{H\left(e^{j\omega}\right)\big|_{\omega=\omega_0}} e^{j\omega_0 n}$$

Frequency Response of LTI Systems

Check response to a pure frequency:

$$H(e^{j\omega}) = \text{DTFT}\{h[n]\}$$

$$y[n] = H\left(e^{j\omega}\right)\big|_{\omega=\omega_0} e^{j\omega_0 n}$$

Output is the same pure frequency, scaled and phase-shifted!

$e^{j\omega_0 n}$ is an eigen function of LTI systems

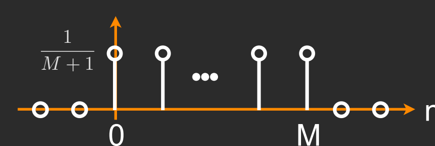
Recall eigen vectors satisfy: $A\nu = \lambda\nu$

Example 3

Frequency response of a causal moving average filter

$$y[n] = \frac{x[n-M] + \cdots + x[n]}{M+1}$$

Q: What type of filter is it? A: Low-Pass



$$h[n] = \frac{1}{M+1} w[n - \frac{M}{2}]$$

Example 3 Cont.

Frequency response of a causal moving average filter

$$h[n] = \frac{1}{M+1} w[n - \frac{M}{2}]$$

Same as example 1, only: Shifted by N, divided by M+1, M=2N

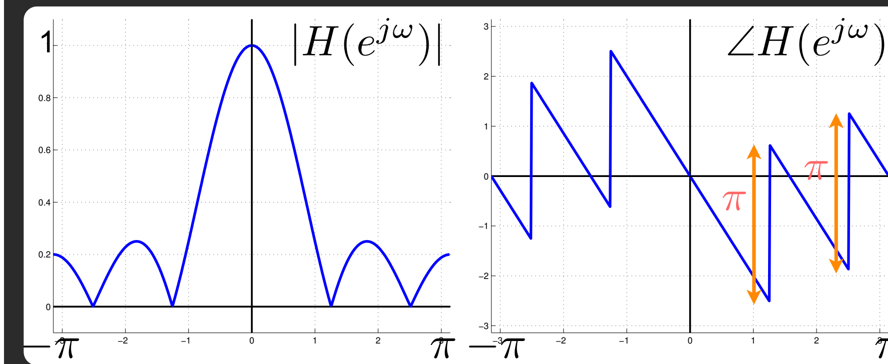
$$H(e^{j\omega}) = \frac{e^{-j\omega\frac{M}{2}}}{M+1} \cdot \frac{\sin\left(\left(\frac{M}{2}+\frac{1}{2}\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Example 3 Cont.

Frequency response of a causal moving average filter

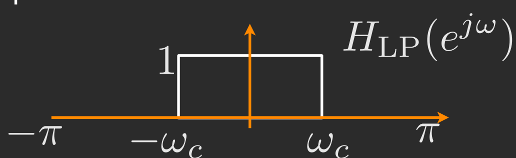
$$H(e^{j\omega}) = \frac{e^{-j\omega\frac{M}{2}}}{M+1} \cdot \frac{\sin\left(\left(\frac{M}{2}+1\right)\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Not a sinc!



Example 4:

Impulse Response of an Ideal Low-Pass Filter



$$\begin{aligned} h_{\mathrm{LP}}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{jw})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \end{aligned}$$

Example 4

Impulse Response of an Ideal Low-Pass Filter

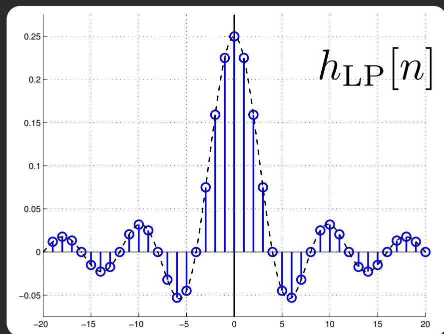
$$\begin{aligned} h_{\mathrm{LP}}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{jw})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \frac{1}{2\pi jn} e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = 2j\sin(w_c n) \\ &= \frac{\sin(w_c n)}{\pi n} \end{aligned}$$

Example 4

Impulse Response of an Ideal Low-Pass Filter

$$h_{\mathrm{LP}}[n] = \frac{\sin(w_c n)}{\pi n}$$

sampled "sinc"



Non causal! Truncate and shift right to make causal

Example 4

Impulse Response of an Ideal Low-Pass Filter

Non causal! Truncate and shift right to make causal

How does it changes the frequency response?

Truncation:

$$\tilde{h}_{\mathrm{LP}}[n] = w_N[n] \cdot h_{\mathrm{LP}}[n]$$

property 2.9.7:

$$\tilde{H}_{\mathrm{LP}}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\mathrm{LP}}(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

Periodic convolution

Example 4

We get "smearing" of the frequency response