Fall 2012, EE123 Digital Signal Processing

Lecture 4

Miki Lustig, UCB

September 4, 2012

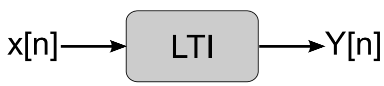
The Z-Transform

Used for:

- Analysis of LTI systems
- Solving difference equations
- Determining system stability
- Finding frequency response of stable systems

Eigen Functions of LTI System

Consider an LTI system with an impulse response $h[n]$:



- We already showed that $x[n] = e^{j\omega n}$ are eigen-functions.
- What if $x[n] = z^n = re^{j\omega n}$

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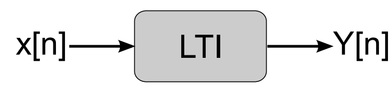

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- What if $x[n] = z^n = re^{j\omega n}$

  Calculate using Convolution:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \\
&= \left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n = H(z) z^n
\end{aligned}
$$

Eigen Functions of LTI System

$x[n] = z^n$ are also eigen-functions of LTI systems.

Transfer Function

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

- $H(z)$ exists for larger classs of $h[n]$ than $H(e^{j\omega})$

The Z Transform

Bilateral Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $z = re^{j\omega}$

$$X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \mathcal{DTFT}\{x[n]\}$$

Region of Convergence (ROC)

- The ROC is the set of values of z for which the sum

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

converges.

Region of Convergence (ROC)

- Example 1: Right-sided sequence $x[n] \;=\; a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

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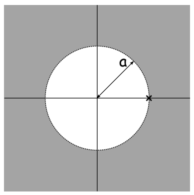
Recall:

$$1 + x + x^2 + \cdots = \frac{1}{1-x}, \text{if} |x| < 1$$

So,

$$X(z) = \frac{1}{1-az^{-1}}, \qquad \text{ROC} = \{z: \; |z| > |a|\}$$



Region of Convergence (ROC)

- Example 2: $x[n] \;=\; \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

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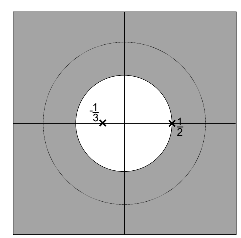
$$\begin{aligned} \text{ROC} &= \{z: \; |z| > \frac{1}{2}\} \;\cap\; \{z: \; |z| > \frac{1}{3}\} \\ &= \{z: \; |z| > \frac{1}{2}\} \end{aligned}$$



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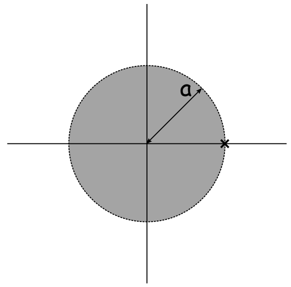
If $|a^{-1}z| < 1$, *i.e.*, $|z| < |a|$ then,

$$\begin{aligned} X(z) &= 1 - \frac{1}{1 - a^{-1}z} \\ &= \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} \end{aligned}$$

Region of Convergence (ROC)

- Expression $X[z] = \frac{1}{1-az^{-1}}$ the same as Example 1
- ROC $= \{z : \; |z| < |a|\}$ is different



- The Z-transform without ROC does not uniquely define a sequence!

Region of Convergence (ROC)

- Example 4: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$

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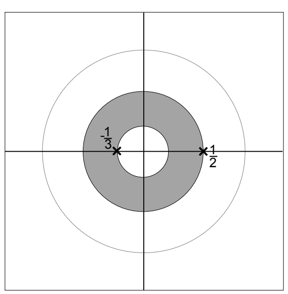
- Example 4: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$

$$\begin{aligned} X(z) &= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \quad \text{same as example 2} \\ \text{ROC} &= \{z: \ |z| < \frac{1}{2}\} \ \cap \ \{z: \ |z| > \frac{1}{3}\} \\ &= \{z: \ \frac{1}{3} < |z| < \frac{1}{2}\} \end{aligned}$$



Region of Convergence (ROC)

- Example 5: $x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$

$$\begin{aligned} \text{ROC} &= \{z: \ |z| > \frac{1}{2}\} \ \cap \ \{z: \ |z| < \frac{1}{3}\} \\ &= 0 \end{aligned}$$

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- Example 6: $x[n] = a^n$, two sided $a \neq 0$

$$\begin{aligned} \text{ROC} &= \{z: \ |z| > a\} \ \cap \ \{z: \ |z| < a\} \\ &= 0 \end{aligned}$$

Region of Convergence (ROC)

- Example 7: Finite sequence $x[n] = a^n u[n] u[-n+M-1]$

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$$\begin{aligned} X[z] &= \sum_{n=0}^{M-1} a^n z^{-n} \quad \text{Finite, always converges} \\ &= \frac{1-a^M z^{-M}}{1-az^{-1}} \quad \text{zero cancels the pole} \\ &= \prod_{k=1}^{M-1} (1 - a e^{j\frac{2\pi k}{M}} z^{-1}) \end{aligned}$$

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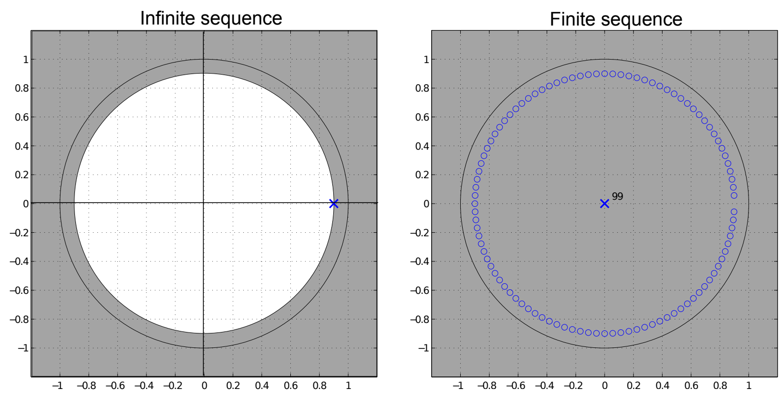


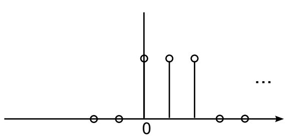
Properties of the ROC

Section 3.2

1. A ring or a disk in the Z-plane, centered at the origin
2. DTFT converges <u>iff</u> ROC includes the unit circle
3. ROC can't contain poles

Properties of the ROC

4. For finite duration sequences, ROC is the entire Z-plane, except possibly $z = 0$ or $z = \infty$



$X(z) = 1 + z^{-1} + z^{-2}$ ROC excludes $z = 0$

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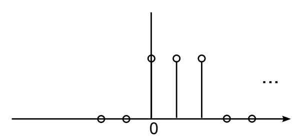

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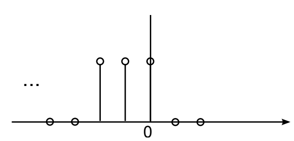

$X(z) = 1 + z^{1} + z^{2}$ ROC excludes $z = \infty$

Properties of the ROC

5. For right-sided sequences ($x[n] = 0 \quad \forall\ n < N_1$ for some N_1) ROC extends outward from the outer-most pole to infinity.
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Example 3

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Examples 1,2
6. For left-sided sequences ($x[n] = 0 \quad \forall\ n > N_2$ for some N_2) ROC extends inwards from the inner-most pole to zero.
Example 3

7. For two-sided sequences, ROC is a ring
Inner-bound: Pole with largest magnitude that contributes for $n > 0$
Outer-bound: Pole with smallest magnitude that contributes for $n < 0$ Examples 4,5,6

Several Properties of the Z Transform

- $x[n - n_d] \quad \leftrightarrow \quad z^{-n_d} X(z)$
- $z_0^n x[n] \quad \leftrightarrow \quad X(\frac{z}{z_0})$
- $nx[n] \quad \leftrightarrow \quad -z\frac{dX(z)}{dz}$
- $x[-n] \quad \leftrightarrow \quad X(z^{-1})$
- $x[n] * y[n] \quad \leftrightarrow \quad X(z)Y(z)$

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- $x[-n] \quad \leftrightarrow \quad X(z^{-1})$
- $x[n] * y[n] \quad \leftrightarrow \quad X(z)Y(z)$ ROC at least $\text{ROC}_x \cap \text{ROC}_y$

Inversion of the Z-Transform

- In general, we can invert by contour integration within the ROC:

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}$$

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the Z-transform
  - Power series expansion
  - Partial fraction expansion
  - Residue theorem

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- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the Z-transform
  - Power series expansion
  - Partial fraction expansion
  - Residue theorem
- Most useful is the inverse of rational polynomials

$$X(z) = \frac{B(z)}{A(z)}$$

Why?

Inversion of the Z-Transform

- Example: Long division

$$X(z) = \frac{2+z^{-1}}{1-\frac{1}{2}z^{-1}},$$

Inversion of the Z-Transform

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x[n] right/left sequencs?

- Arrange num/denum in ascending powers of z^{-1}

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What about ROC $= \{z: \ |z| < \frac{1}{2}\}$?

$$\begin{array}{r|l} & 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \cdots \\ \hline 1 - \frac{1}{2}z^{-1} \ ) & 2 + z^{-1} \\ & 2 - z^{-1} \\ \hline & \quad 2z^{-1} \\ & \quad 2z^{-1} - z^{-2} \\ \hline & \qquad\qquad z^{-2} \\ & \qquad\qquad z^{-2} - \frac{1}{2}z^{-3} \\ \hline & \qquad\qquad\qquad \frac{1}{2}z^{-3} \end{array}$$

Inversion of the Z-Transform

$$\begin{aligned} X(z) &= 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \ldots \\ &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ x[n] &= 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3] + \ldots \end{aligned}$$

Inversion of the Z-Transform

- Example: Partial Fraction Expansion

$$\begin{aligned} X(z) &= \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \\ &= \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} \\ &= \frac{A_1}{1 - \frac{1}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{2}z^{-1}} \end{aligned}$$

Inversion of the Z-Transform

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Find A_1 and A_2

$$\begin{aligned} A_1 &= (1-\frac{1}{4}z^{-1})X(z)\Big|_{z=\frac{1}{4}} = -1 \\ A_2 &= (1-\frac{1}{2}z^{-1})X(z)\Big|_{z=\frac{1}{2}} = 2 \end{aligned}$$

Inversion of the Z-Transform

Partial fraction expansion:

$$X(z) = \frac{-1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{2}z^{-1}}$$

From the tables:

Inversion of the Z-Transform

Partial fraction expansion:

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From the tables:

$$x[n] = \left[-(\frac{1}{4})^n + 2(\frac{1}{2})^n\right] u[n]$$ because right sided

Inversion of the Z-Transform

- Partial fraction expansion in general:

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Suppose real and unrepeated poles $d_1, \cdots, d_N$

- If $M < N$,

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$ Like example

Inversion of the Z-Transform

- If $M \geq N$,

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

$$\Rightarrow \quad x[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

Inversion of the Z-Transform

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$$\Rightarrow \quad x[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

- If d_k is a repeated pole of order S, replace $\frac{A_k}{1-d_k z^{-1}}$ with

$$\sum_{m=1}^{S} \frac{C_m}{(1 - d_k z^{-1})^m}$$

Inversion of the Z-Transform

- Example

$$\begin{aligned} X(z) &= \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \qquad M = N = 2 \\ &= B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}} \end{aligned}$$

Matching coefficients: $A_1 = -9,\ A_2 = 8,\ B_0 = 2$

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$$\begin{aligned} \text{ROC} &= \{z : \ |z| > 1\} \\ \Rightarrow \quad x[n] &= 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n] \end{aligned}$$