Fall 2011, EE123 Digital Signal Processing

Lecture 6

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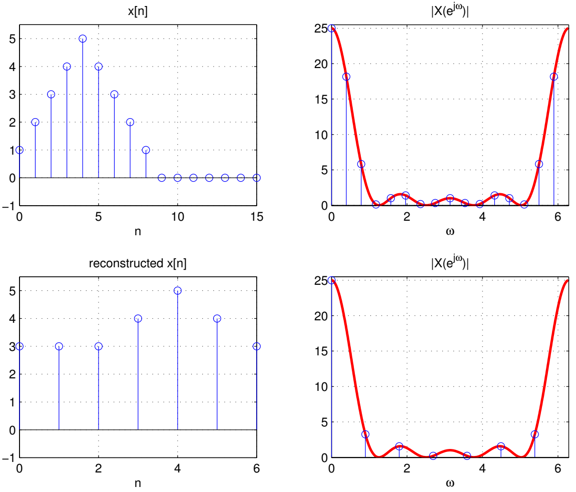
DFT and Sampling the DTFT

$$X(e^{j\omega}) = e^{-j4\omega}\frac{\sin^2(5\omega/2)}{\sin^2(\omega/2)}$$



Circular Convolution as Matrix Operation

Circular convolution:

$$
\begin{aligned}
h[n] \circledN x[n] &= \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & & h[2] \\ & & \vdots & \\ h[N-1] & h[N-2] & & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N] \end{bmatrix} \\
&= H_c x
\end{aligned}
$$

- H_c is a circulant matrix
- The columns of the DFT matrix are Eigen vectors of circulant matrices.
- Eigen vectors are DFT coefficients. How can you show?

Circular Convolution as Matrix Operation

- Diagonalize:

$$W_N H_c W_n^{-1} = \begin{bmatrix} H[0] & 0 \cdots & 0 \\ 0 & H[1] \cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix}$$

- Right-multiply by W_N

$$W_N H_c = \begin{bmatrix} H[0] & 0 \cdots & 0 \\ 0 & H[1] \cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix} W_N$$

- Multiply both sides by x

$$W_N H_c x = \begin{bmatrix} H[0] & 0 \cdots & 0 \\ 0 & H[1] \cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix} W_N x$$

- We are interested in efficient computing methods for the DFT and inverse DFT:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, \ldots, N-1 \\
x[n] &= \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad n = 0, \ldots, N-1
\end{aligned}
$$

where

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)}.$$

- Recall that we can use the DFT to compute the inverse DFT:

$$\mathcal{DFT}^{-1}\{X[k]\} = \frac{1}{N}\left(\mathcal{DFT}\{X^*[k]\}\right)^*$$

  Hence, we can just focus on efficient computation of the DFT.

- Straightforward computation of an N-point DFT (or inverse DFT) requires N^2 complex multiplications.

- *Fast Fourier transform algorithms* enable computation of an N-point DFT (or inverse DFT) with the order of just $N \cdot \log_2 N$ complex multiplications.
  This can represent a huge reduction in computational load, especially for large N.

N	N^2	$N \cdot \log_2 N$	$\frac{N^2}{N \cdot \log_2 N}$
16	256	64	4.0
128	16,384	896	18.3
1,024	1,048,576	10,240	102.4
8,192	67,108,864	106,496	630.2
6×10^6	36×10^{12}	135×10^6	2.67×10^5

* 6Mp image size

- Most FFT algorithms exploit the following properties of W_N^{kn}:
    - Conjugate Symmetry

$$W_N^{k(N-n)} = W_N^{-kn} = (W_N^{kn})^*$$

    - Periodicity in n and k:

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{(k+N)n}$$

    - Power:

$$W_N^2 = W_{N/2}$$

- Most FFT algorithms decompose the computation of a DFT into successively smaller DFT computations.
  - *Decimation-in-time* algorithms decompose $x[n]$ into successively smaller subsequences.
  - *Decimation-in-frequency* algorithms decompose $X[k]$ into successively smaller subsequences.
- We mostly discuss decimation-in-time algorithms here.

Assume length of $x[n]$ is power of 2 ($N = 2^\nu$). If smaller zero-pad to closest power.

Decimation-in-Time Fast Fourier Transform

- We start with the DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, \ldots, N-1$$

- Separate the sum into even and odd terms:

$$X[k] = \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n \text{ odd}} x[n] W_N^{kn}$$

These are two DFT's, each with half of the samples.

Decimation-in-Time Fast Fourier Transform

Let $n = 2r$ (n even) and $n = 2r + 1$ (n odd):

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{(2r+1)k} \\ &= \sum_{r=0}^{(N/2)-1} x[2r] W_N^{2rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_N^{2rk} \end{aligned}$$

- Note that:

$$W_N^{2rk} = e^{-j\left(\frac{2\pi}{N}\right)(2rk)} = e^{-j\left(\frac{2\pi}{N/2}\right)rk} = W_{N/2}^{rk}$$

Remember this trick, it will turn up often.

- Hence:

$$\begin{aligned} X[k] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk} \\ &\triangleq G[k] + W_N^k H[k], \qquad k = 0, \ldots, N-1 \end{aligned}$$

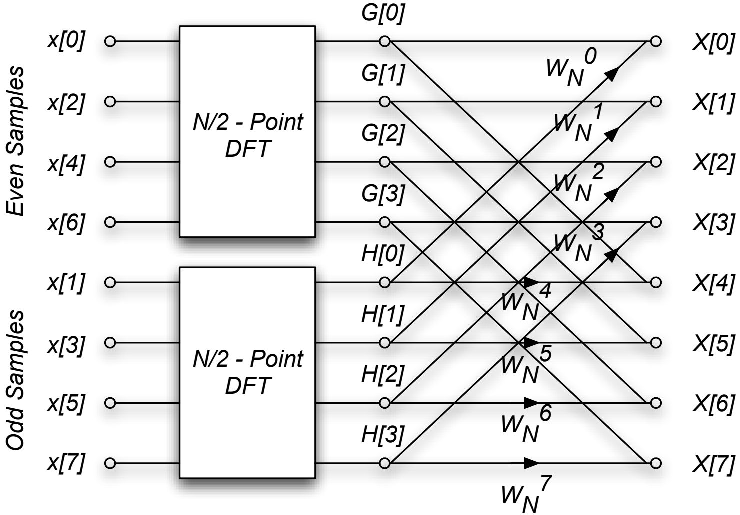
where we have defined:

$$G[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} \qquad \Rightarrow \text{DFT of even idx}$$

$$H[k] \triangleq \sum_{r=0}^{(N/2)-1} x[2r+1] W_{N/2}^{rk} \qquad \Rightarrow \text{DFT of odd idx}$$

Decimation-in-Time Fast Fourier Transform

An 8 sample DFT can then be diagrammed as



- Both $G[k]$ and $H[k]$ are periodic, with period $N/2$. For example

$$
\begin{aligned}
G[k+N/2] &= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{r(k+N/2)} \\
&= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} W_{N/2}^{r(N/2)} \\
&= \sum_{r=0}^{(N/2)-1} x[2r] W_{N/2}^{rk} \\
&= G[k]
\end{aligned}
$$

so

$$
\begin{aligned}
G[k+(N/2)] &= G[k] \\
H[k+(N/2)] &= H[k]
\end{aligned}
$$

Decimation-in-Time Fast Fourier Transform

- The periodicity of $G[k]$ and $H[k]$ allows us to further simplify.
- For the first $N/2$ points we calculate $G[k]$ and $W_N^k H[k]$, and then compute the sum

$$X[k] = G[k] + W_N^k H[k] \qquad\qquad \forall\{k : 0 \le k < \frac{N}{2}\}.$$

How does periodicity help for $\frac{N}{2} \le k < N$?

Decimation-in-Time Fast Fourier Transform

$$X[k] = G[k] + W_N^k H[k] \qquad \forall \{k : 0 \leq k < \frac{N}{2}\}.$$

- for $\frac{N}{2} \leq k < N$:

$$W_N^{k+(N/2)} = ?$$

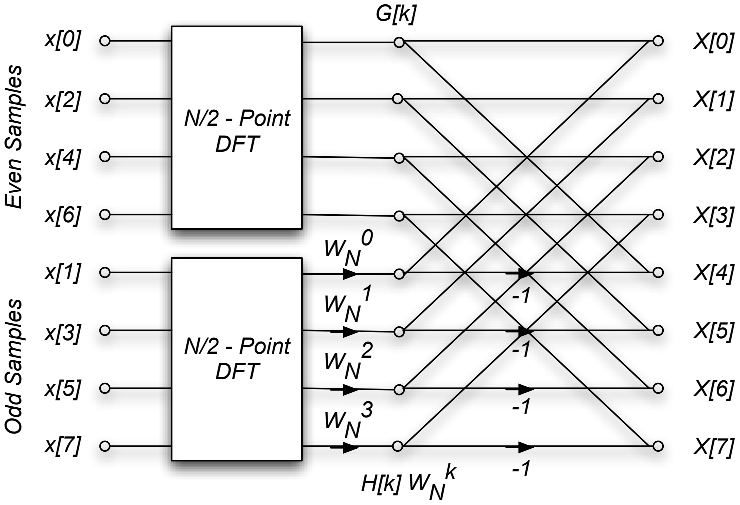
$$X[k + (N/2)] = ?$$

$$X[k + (N/2)] = G[k] - W_N^k H[k]$$

We previously calculated $G[k]$ and $W_N^k H[k]$.

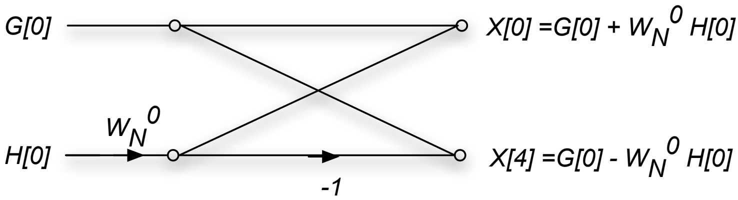
Now we only have to compute their difference to obtain the second half of the spectrum. No additional multiplies are required.

- The N-point DFT has been reduced two $N/2$-point DFTs, plus $N/2$ complex multiplications. The 8 sample DFT is then:



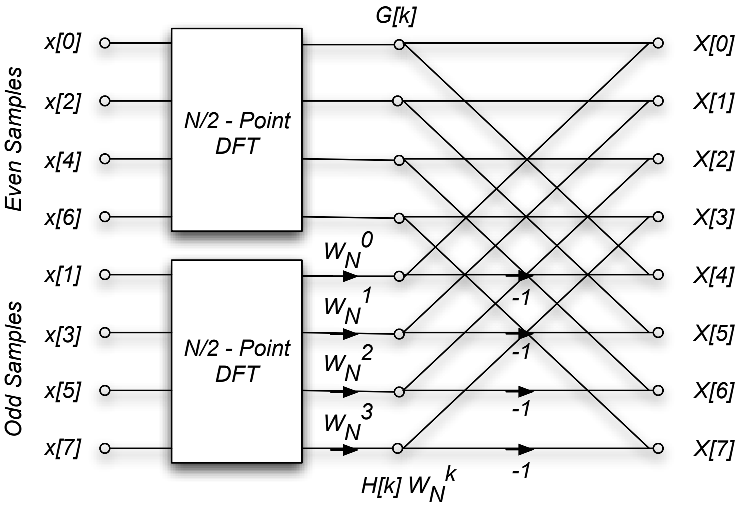
Decimation-in-Time Fast Fourier Transform

- Note that the inputs have been reordered so that the outputs come out in their proper sequence.
- We can define a *butterfly operation*, e.g., the computation of $X[0]$ and $X[4]$ from $G[0]$ and $H[0]$:



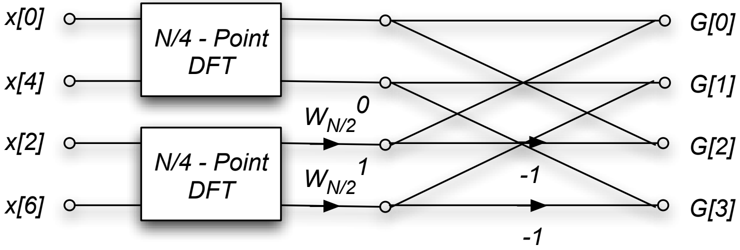
This is an important operation in DSP.

- Still $O(N^2)$ operations..... What shall we do?



Decimation-in-Time Fast Fourier Transform

- We can use the same approach for each of the $N/2$ point DFT's. For the $N = 8$ case, the $N/2$ DFTs look like



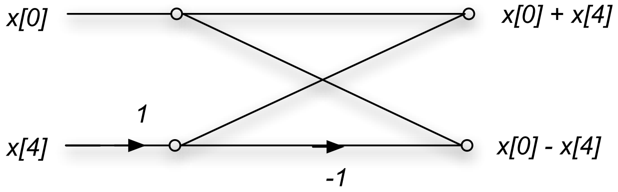
*Note that the inputs have been reordered again.

Decimation-in-Time Fast Fourier Transform

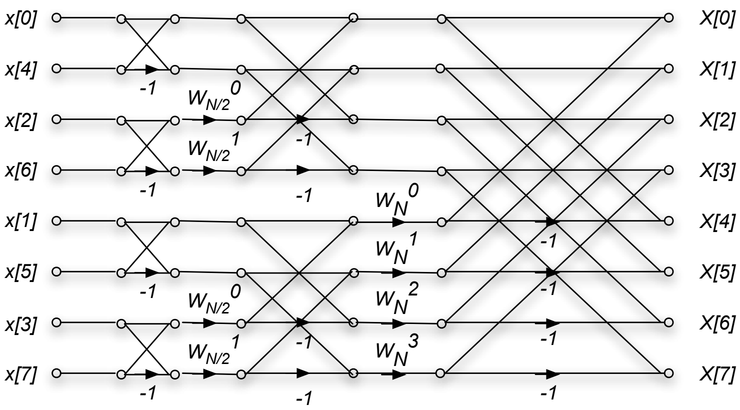
- At this point for the 8 sample DFT, we can replace the $N/4 = 2$ sample DFT's with a single butterfly.
  The coefficient is

$$W_{N/4} = W_{8/4} = W_2 = e^{-j\pi} = -1$$

The diagram of this stage is then



Combining all these stages, the diagram for the 8 sample DFT is:



This the decimation-in-time FFT algorithm.

Decimation-in-Time Fast Fourier Transform

- In general, there are $\log_2 N$ stages of decimation-in-time.
- Each stage requires $N/2$ complex multiplications, some of which are trivial.
- The total number of complex multiplications is $(N/2)\log_2 N$.
- The order of the input to the decimation-in-time FFT algorithm must be permuted.
  - First stage: split into odd and even. Zero low-order bit first
  - Next stage repeats with next zero-lower bit first.
  - Net effect is reversing the bit order of indexes

This is illustrated in the following table for $N = 8$.

Decimal	Binary	Bit-Reversed Binary	Bit-Reversed Decimal
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Decimation-in-Frequency Fast Fourier Transform

The DFT is

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

If we only look at the even samples of $X[k]$, we can write $k = 2r$,

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)}$$

We split this into two sums, one over the first $N/2$ samples, and the second of the last $N/2$ samples.

$$X[2r] = \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2r(n+N/2)}$$

Decimation-in-Frequency Fast Fourier Transform

But $W_N^{2r(n+N/2)} = W_N^{2rn} W_N^N = W_N^{2rn} = W_{N/2}^{rn}$.
We can then write

$$\begin{aligned} X[2r] &= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2r(n+N/2)} \\ &= \sum_{n=0}^{(N/2)-1} x[n] W_N^{2rn} + \sum_{n=0}^{(N/2)-1} x[n+N/2] W_N^{2rn} \\ &= \sum_{n=0}^{(N/2)-1} \left(x[n] + x[n+N/2]\right) W_{N/2}^{rn} \end{aligned}$$

This is the $N/2$-length DFT of first and second half of $x[n]$ summed.

Decimation-in-Frequency Fast Fourier Transform

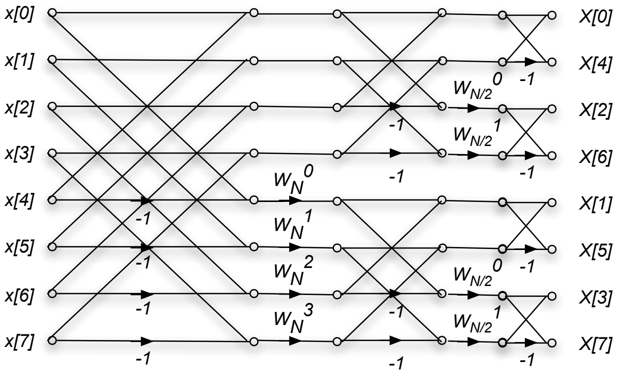
$$\begin{aligned} X[2r] &= \mathrm{DFT}_{\frac{N}{2}}\left\{(x[n] + x[n+N/2])\right\} \\ X[2r+1] &= \mathrm{DFT}_{\frac{N}{2}}\left\{(x[n] - x[n+N/2])\, W_N^n\right\} \end{aligned}$$

(By a similar argument that gives the odd samples)

Continue the same approach is applied for the $N/2$ DFTs, and the $N/4$ DFT's until we reach simple butterflies.

Decimation-in-Frequency Fast Fourier Transform

The diagram for and 8-point decimation-in-frequency DFT is as follows

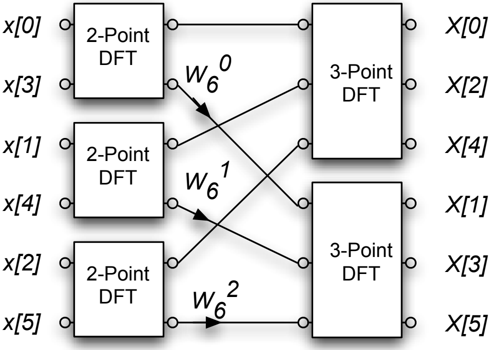


This is just the decimation-in-time algorithm reversed!
The inputs are in normal order, and the outputs are bit reversed.

Non-Power-of-2 FFT's

A similar argument applies for any length DFT, where the length N is a composite number.

For example, if $N = 6$, a decimation-in-time FFT could compute three 2-point DFT's followed by two 3-point DFT's



Non-Power-of-2 FFT's

Good component DFT's are available for lengths up to 20 or so. Many of these exploit the structure for that specific length. For example, a factor of

$$W_N^{N/4} = e^{-j\frac{2\pi}{N}(N/4)} = e^{-j\frac{\pi}{2}} = -j \qquad \textit{Why?}$$

just swaps the real and imaginary components of a complex number, and doesn't actually require any multiplies.
Hence a DFT of length 4 doesn't require any complex multiplies.
Half of the multiplies of an 8-point DFT also don't require multiplication.
Composite length FFT's can be very efficient for any length that factors into terms of this order.

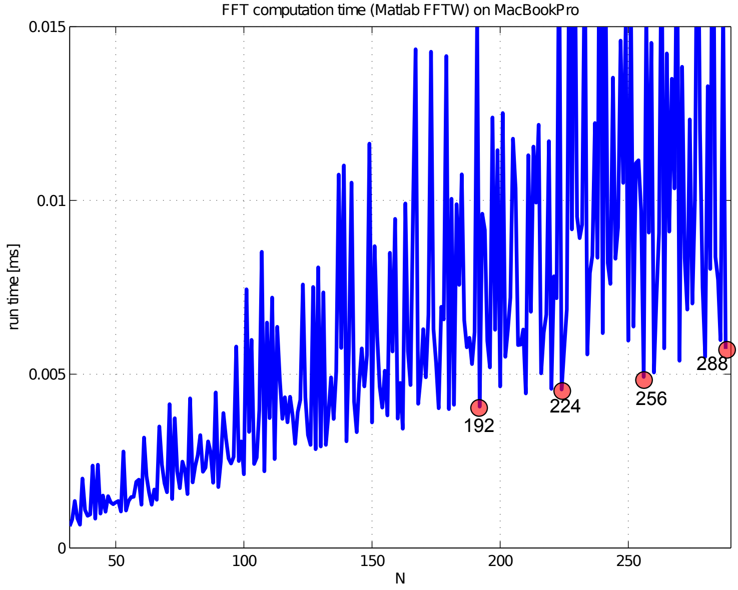
For example $N = 693$ factors into

$$N = (7)(9)(11)$$

each of which can be implemented efficiently. We would perform

- 9×11 DFT's of length 7
- 7×11 DFT's of length 9, and
- 7×9 DFT's of length 11

- Historically, the power-of-two FFTs were much faster (better written and implemented).
- For non-power-of-two length, it was faster to zero pad to power of two.
- Recently this has changed. The free FFTW package implements very efficient algorithms for almost any filter length. Matlab has used FFTW since version 6


FFT computation time (Matlab FFTW) on MacBookPro
run time [ms]
N
0.015
0.01
0.005
0
50
100
150
200
250
192
224
256
288

FFT as Matrix Operation

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

- W_N is fully populated $\Rightarrow$ N^2 entries.

FFT as Matrix Operation

$$\begin{pmatrix} X[0] \\ \vdots \\ X[k] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^{00} & \cdots & W_N^{0n} & \cdots & W_N^{0(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{k0} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_N^{(N-1)0} & \cdots & W_N^{(N-1)n} & \cdots & W_N^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[n] \\ \vdots \\ x[N-1] \end{pmatrix}$$

- W_N is fully populated $\Rightarrow N^2$ entries.
- FFT is a decomposition of W_N into a more sparse form:

$$F_N = \begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} W_{N/2} & 0 \\ 0 & W_{N/2} \end{bmatrix} \begin{bmatrix} \text{Even-Odd Perm.} \\ \text{Matrix} \end{bmatrix}$$

- $I_{N/2}$ is an identity matrix. $D_{N/2}$ is a diagonal with entries $1,\ W_N,\ \cdots, W_N^{N/2-1}$

FFT as Matrix Operation

Example: $N = 4$

$$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & W_4 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -W_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$