EE123
Digital Signal Processing

Lecture 9

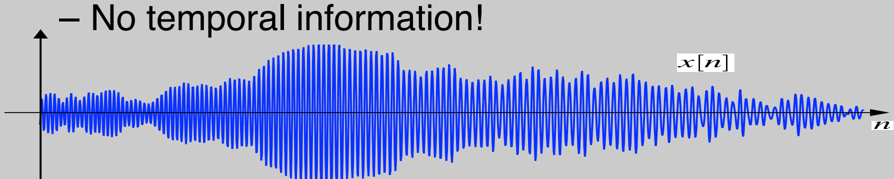
Discrete Transforms (Finite)

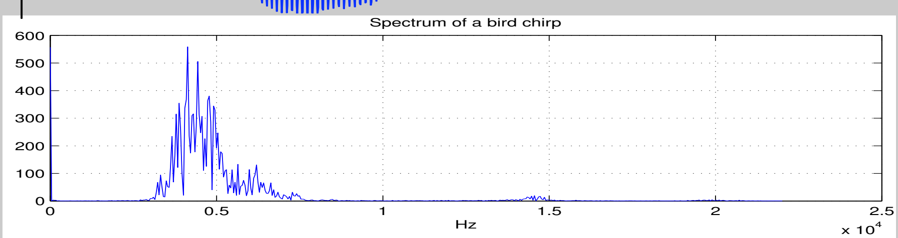
- DFT is only one out of a LARGE class of transforms
- Used for:
  - –Analysis
  - –Compression
  - –Denoising
  - –Detection
  - –Recognition
  - –Approximation (Sparse)

Sparse representation has been one of the hottest research topics in the last 15 years in sp

Example of spectral analysis

- Spectrum of a bird chirping
  - Interesting,.... but...
  - Does not tell the whole story
  - No temporal information!





Time Dependent Fourier Transform

- To get temporal information, use part of the signal around every time point

$$X[n,\omega) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\omega m}$$

*Also called Short-time Fourier Transform (STFT)

- Mapping from 1D ⇒ 2D, n discrete, *w* cont.
- Simply slide a window and compute DTFT

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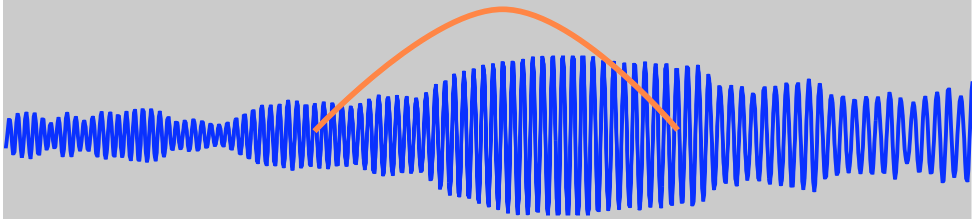
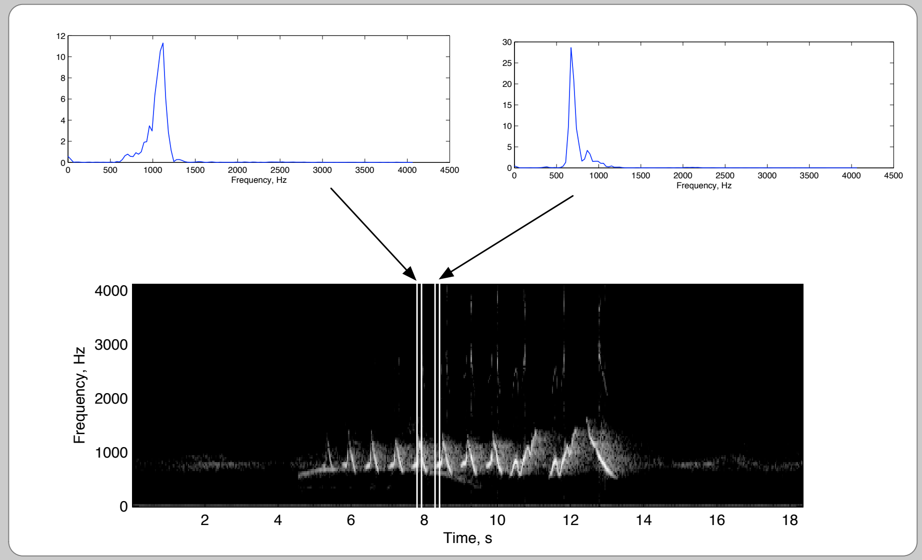


Spectrogram



Discrete Time Dependent FT

$$X_r[k] = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j2\pi km/N}$$

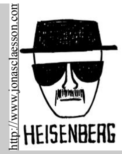
- L - Window length
- R - Jump of samples
- N - DFT length

- Tradeoff between time and frequency resolution

Discrete Transforms (Finite)

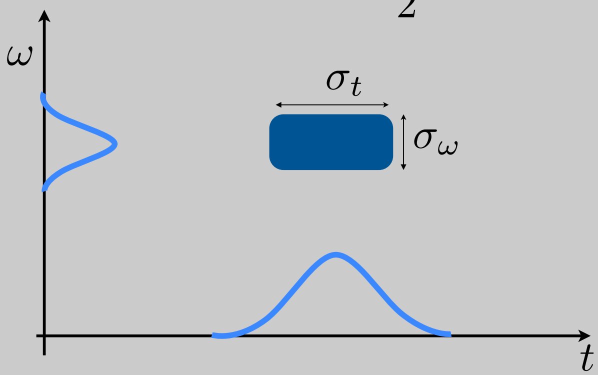
- Today:
  - Start with DFT ⇒ Frequency only
  - Short-time DFT ⇒ Time-Frequency
  - Wavelets ⇒ More flexible/better Time-frequency
  - Wavelets ⇒ Sparsity ⇒
    - ⇒ Compression
    - ⇒ denoising
    - ⇒ approximation

Heisenberg Boxes



- Time-Frequency uncertainty principle

$$\sigma_t \cdot \sigma_\omega \geq \frac{1}{2}$$



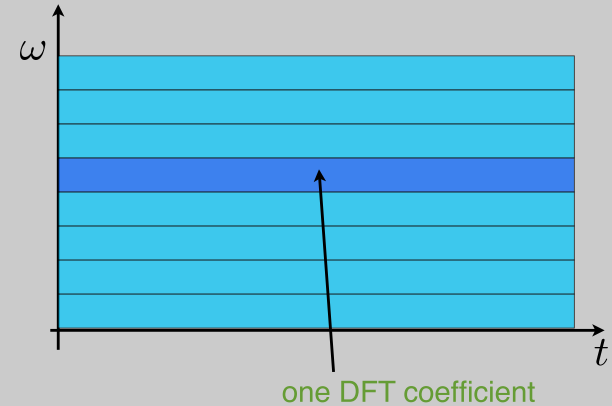
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$\Delta\omega = \frac{2\pi}{N}$$

$$\Delta t = N$$

$$\Delta\omega \cdot \Delta t = 2\pi$$



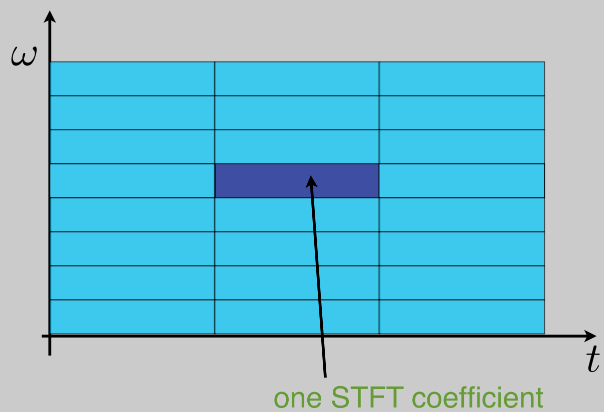
Discrete STFT

optional

$$X[r,k] = \sum_{m=0}^{L-1} x[rR + m] w[m] e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$



STFT Reconstruction

$$x[rR + m] w_L[m] = \frac{1}{N} \sum_{k=0}^{N-1} X[n,k] e^{j2\pi km/N}$$

- For non-overlapping windows, R=L :

$$x[n] = \frac{x[n - rL]}{w_L[n - rL]}$$

$$rL \leq n \leq (r+1)R - 1$$

- What is the problem?

STFT Reconstruction

$$x[rR+m]w_L[m] = \frac{1}{N}\sum_{k=0}^{N-1} X[n,k]e^{j2\pi km/N}$$

- For non-overlapping windows, R=L :

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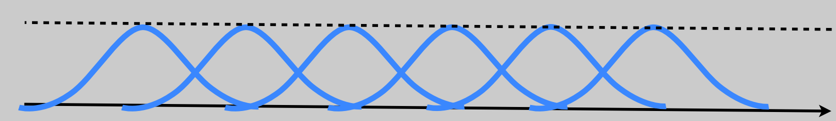
$$rL \le n \le (r+1)R-1$$

- For stable reconstruction must overlap window 50% (at least)

STFT Reconstruction

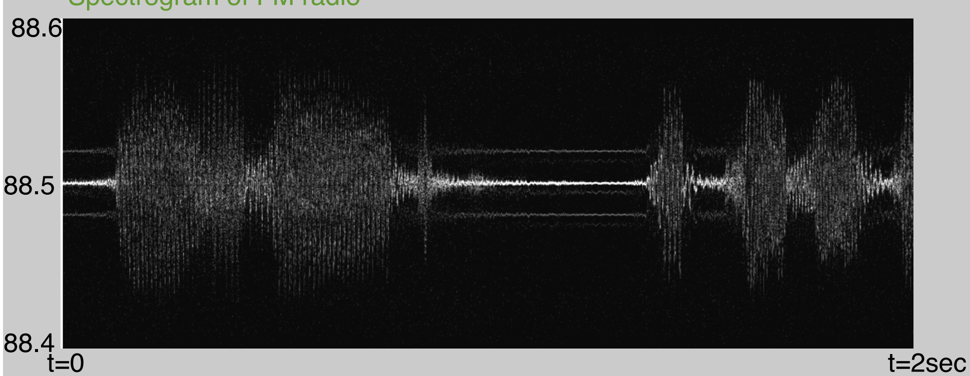
- For stable reconstruction must overlap window 50% (at least)
- For Hann, Bartlett reconstruct with overlap and add. No division!



Applications

- Time Frequency Analysis

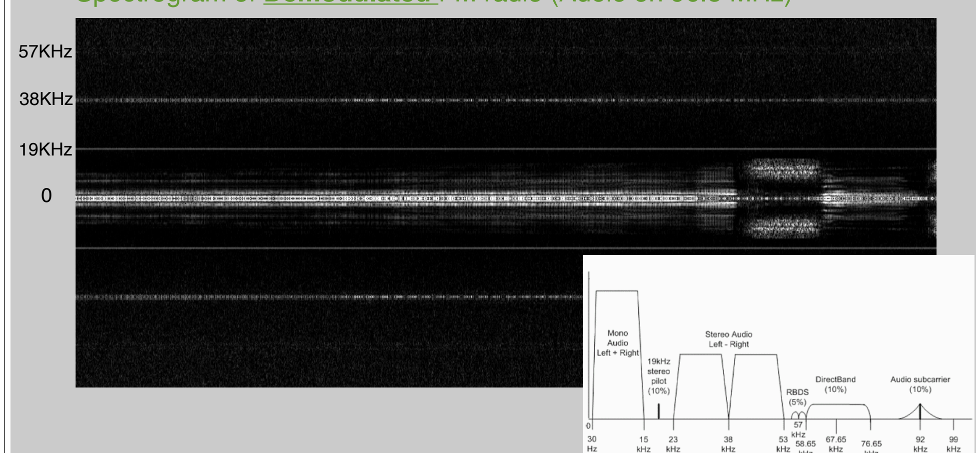
Spectrogram of FM radio



Applications

- Time Frequency Analysis

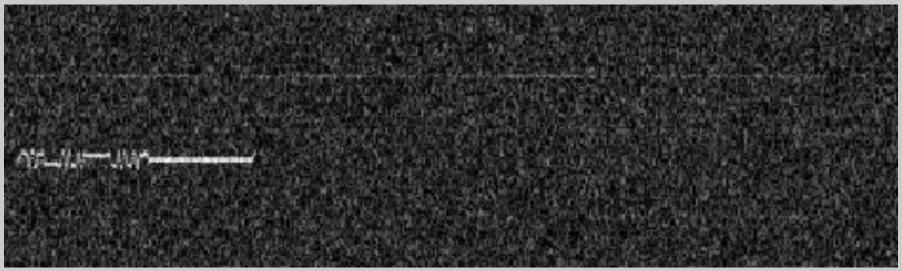
Spectrogram of **Demodulated** FM radio (Adele on 96.5 MHz)



Applications

- Time Frequency Analysis

Spectrogram of digital communications - Frequency Shift Keying

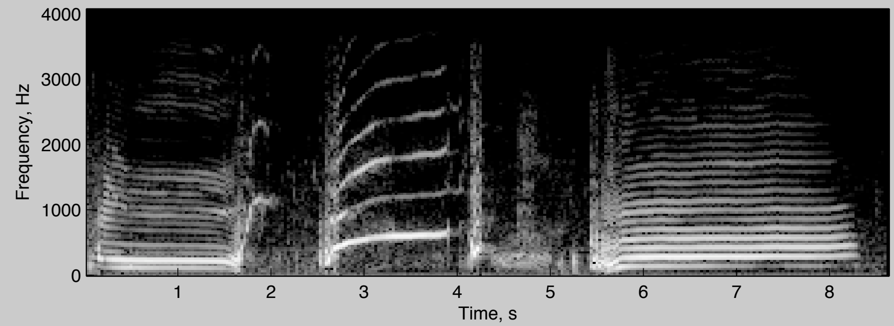


t=0 t=1sec

Applications

- Time Frequency Analysis

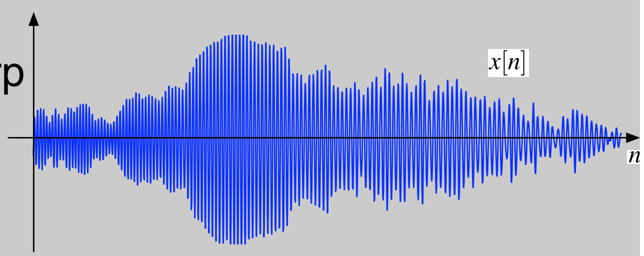
Spectrogram of Orca whale

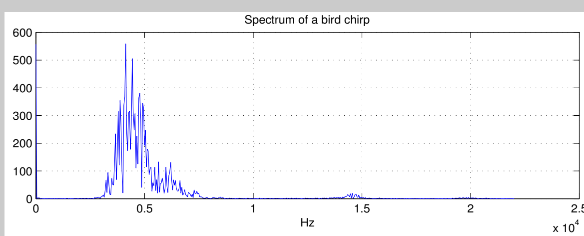


Applications

- Noise removal

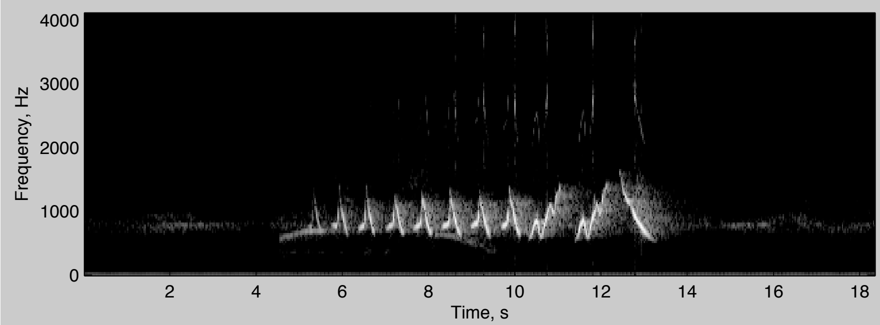
- Recall bird chirp





Application

- Denoising of Sparse spectrograms



- Spectrum is sparse! can implement adaptive filter, or just threshold!

Limitations of Discrete STFT

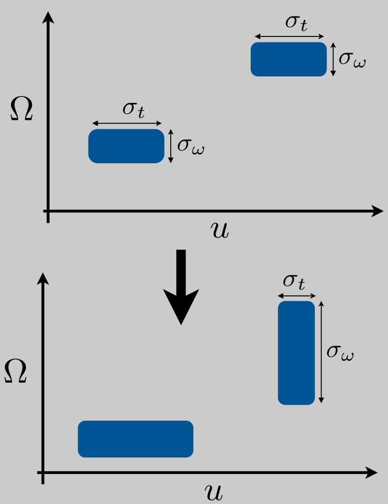
- Need overlapping ⇒ Not orthogonal
- Computationally intensive O(MN log N)
- Same size Heisenberg boxes

From STFT to Wavelets

- Basic Idea:
  - low-freq changes slowly - fast tracking unimportant
  - Fast tracking of high-freq is important in many apps.
  - Must adapt Heisenberg box to frequency
- Back to continuous time for a bit.....

From STFT to Wavelets

- Continuous time



$$Sf(u,\Omega) = \int_{-\infty}^{\infty} f(t)w(t-u)e^{-j\Omega t}dt$$

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

*Morlet - Grossmann

From STFT to Wavelets

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\Psi^*(\frac{t-u}{s})dt$$

- The function Ψ is called a mother wavelet
  - Must satisfy:

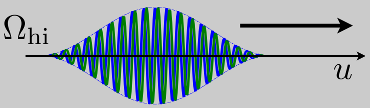
$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1$$ ⇒ unit norm

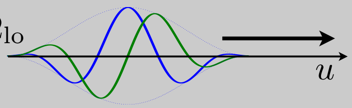
$$\int_{-\infty}^{\infty} \Psi(t)dt = 0$$ ⇒ Band-Pass

STFT and Wavelets “Atoms”

STFT Atoms

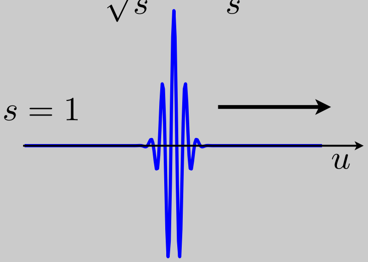
$$w(t-u)e^{j\Omega t}$$

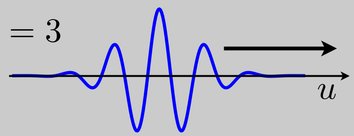




Wavelet Atoms

$$\frac{1}{\sqrt{s}}\Psi\left(\frac{t-u}{s}\right)$$

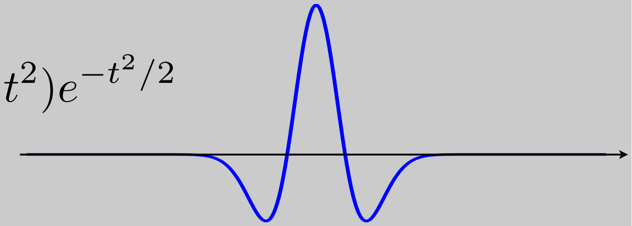




Examples of Wavelets

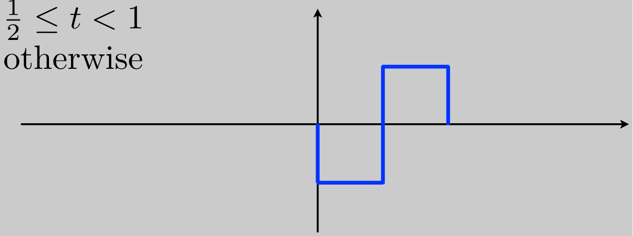
- Mexican Hat

$$\Psi(t) = (1-t^2)e^{-t^2/2}$$

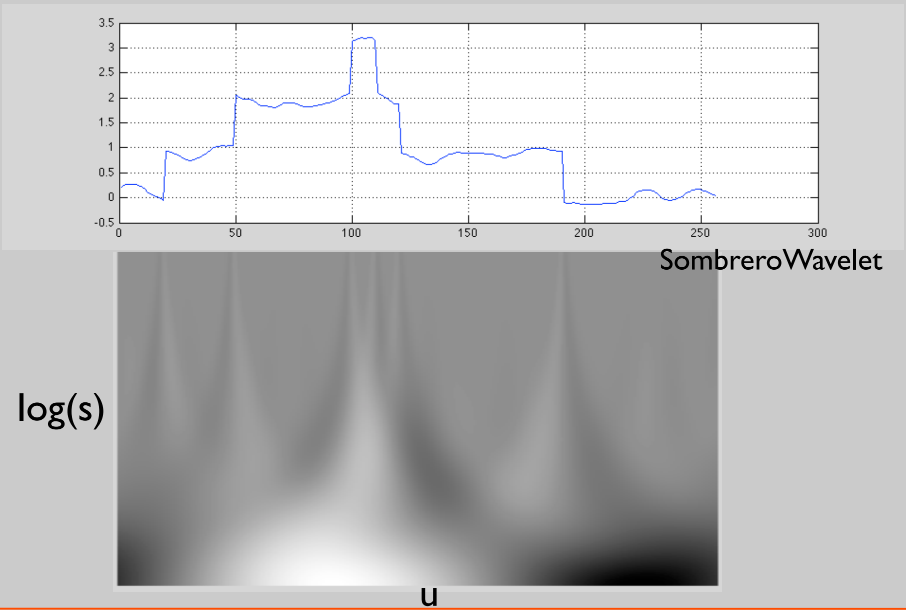


- Haar

$$\Psi(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Example: Mexican Hat



Wavelets Transform

- Can be written as linear filtering

$$\begin{aligned} Wf(u,s) &= \frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} f(t)\Psi^*\left(\frac{t-u}{s}\right)dt \\ &= \left\{f(t) * \overline{\Psi}_s(t)\right\}(u) \end{aligned}$$

$$\overline{\Psi}_s = \frac{1}{\sqrt{s}}\Psi\left(\frac{t}{s}\right)$$

- Wavelet coefficients are a result of bandpass filtering

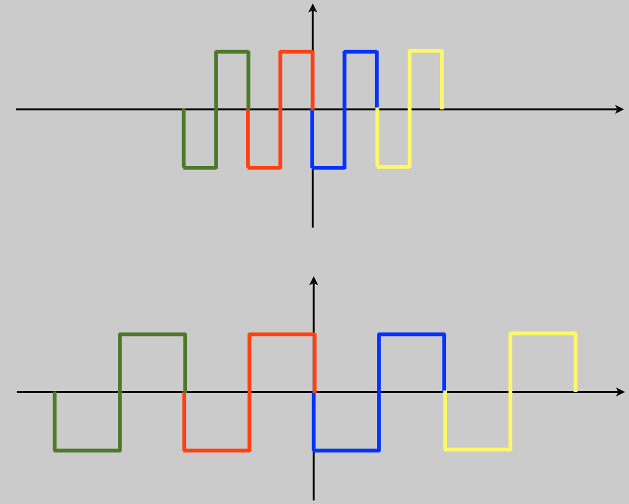
Wavelet Transform

- Many different constructions for different signals
  - Haar good for piece-wise constant signals
  - Battle-Lemarie' : Spline polynomials
- Can construct Orthogonal wavelets
  - For example: dyadic Haar is orthonormal

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$

$i = [1,\ 2,\ 3, \cdots]$

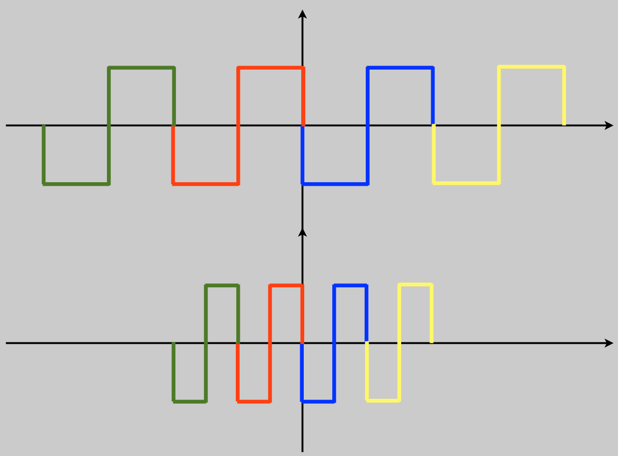
Orthonormal Haar



Same scale non-overlapping

Orthogonal between scales

Orthonormal Haar

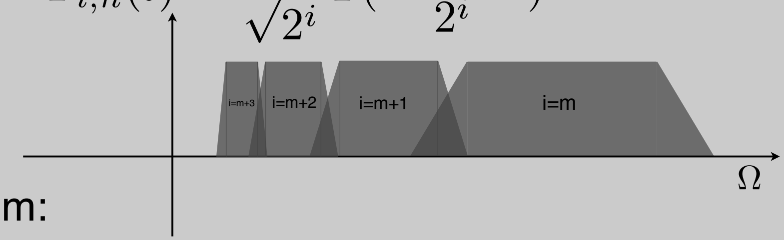


Same scale non-overlapping

Orthogonal between scales

Scaling function

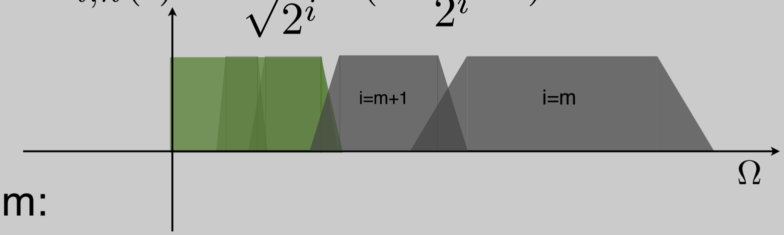
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$



- Problem:
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions
- Solution:
  - Plug low-pass spectrum with a scaling function

Scaling function

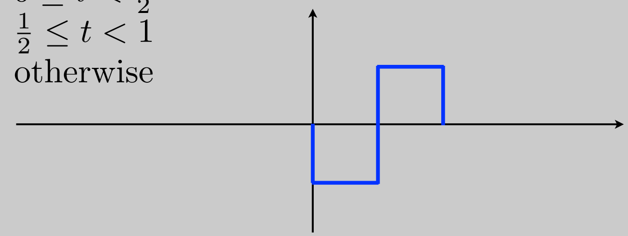
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi\left(\frac{t-2^i n}{2^i}\right)$$



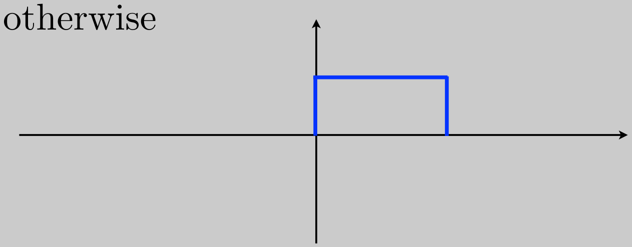
- Problem:
  - –Every stretch only covers half remaining bandwidth
  - –Need Infinite functions
- Solution:
  - –Plug low-pass spectrum with a scaling function $\overline{\Phi}$

Haar Scaling function

$$\Psi(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\Phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

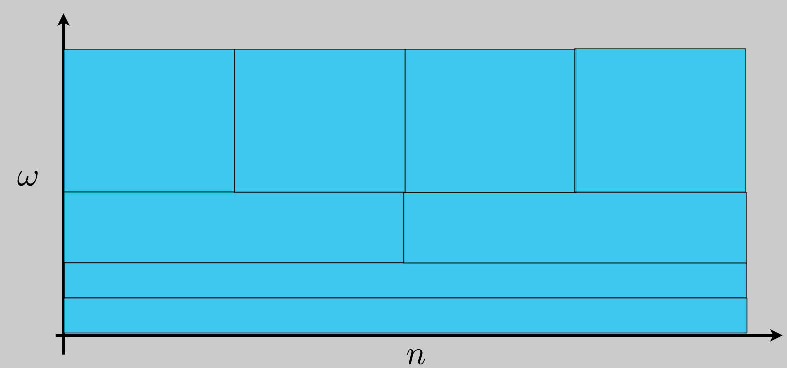


Back to Discrete

- Early 80's, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80's link to DSP by Daubechies and Mallat.
- From CWT to DWT not so trivial!
- Must take care to maintain properties

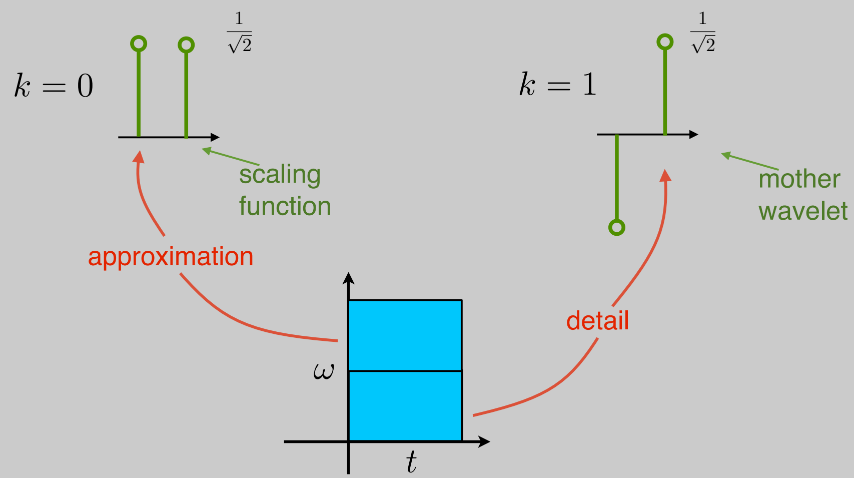
Discrete Wavelet Transform

$$W[k] = \sum_{n=0}^{N-1} x[n]\Psi_k[n]$$



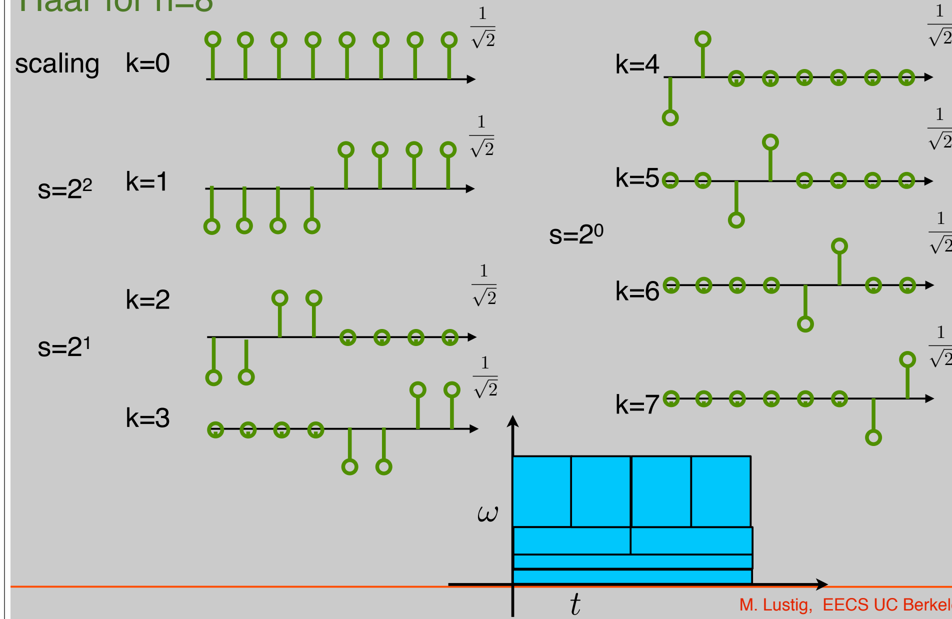
Example: Discrete Haar Wavelet

Haar for n=2



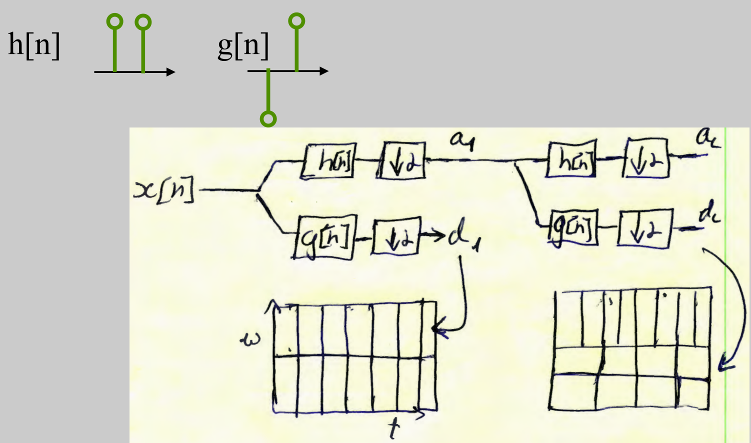
Example: Discrete Haar Wavelet

Haar for n=8



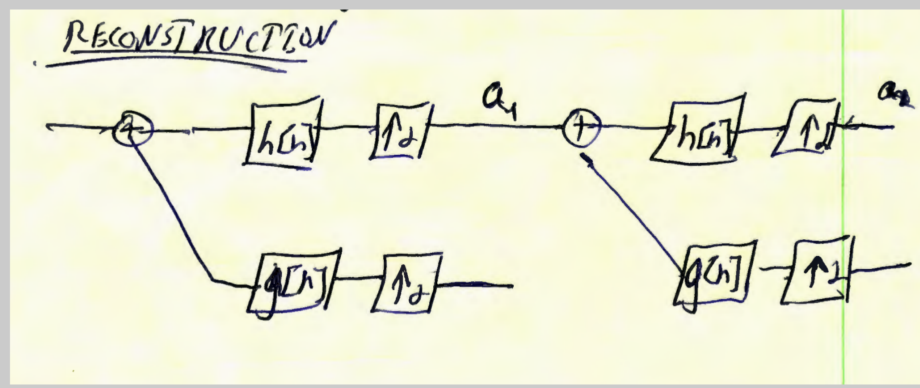
Fast DWT

- Fast Pyramidal Decomposition

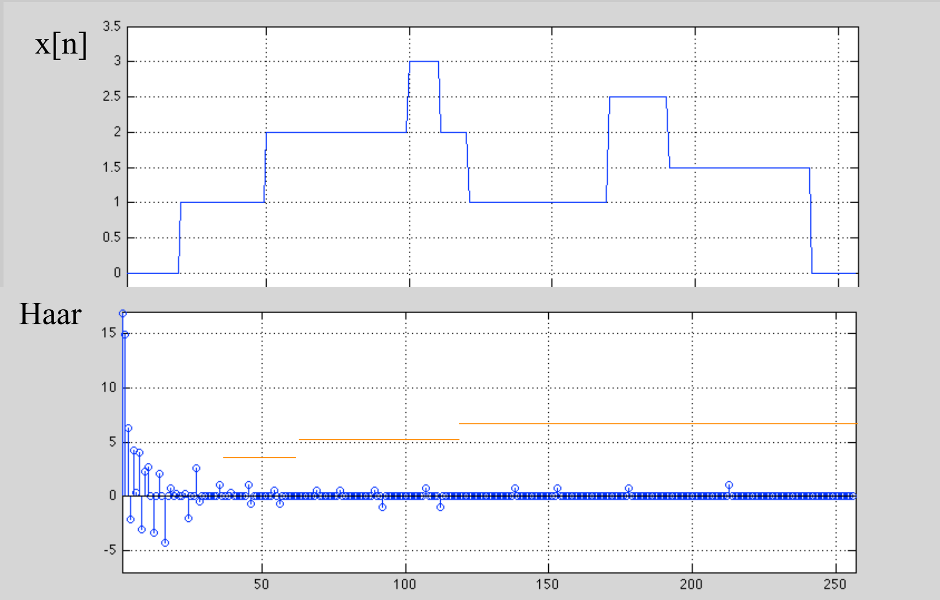


- Complexity is O(N), less than FFT

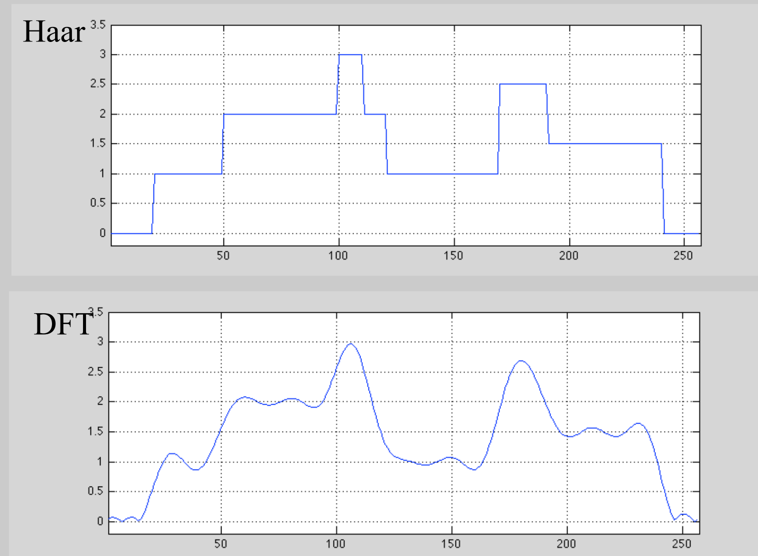
Fast DWT



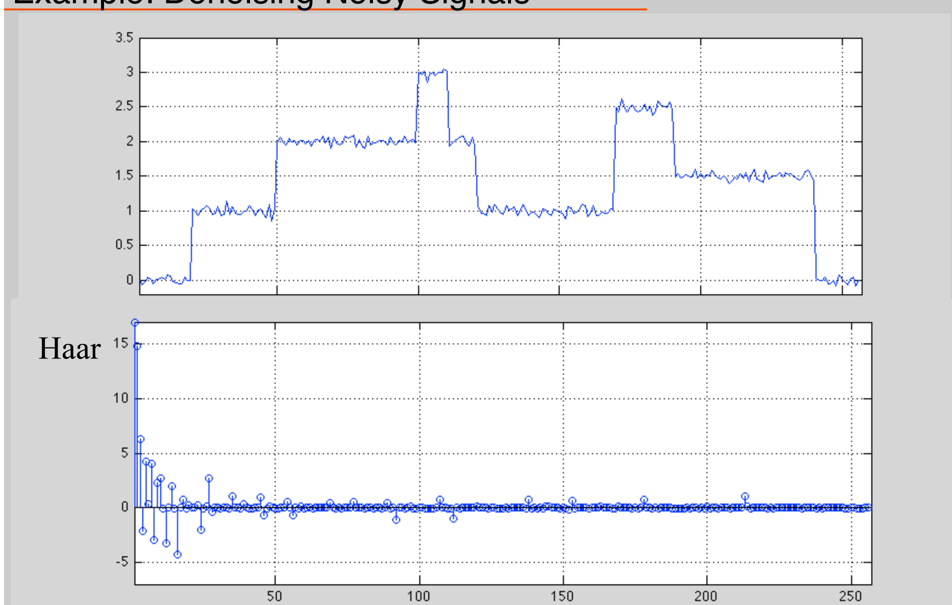
Haar DWT Example



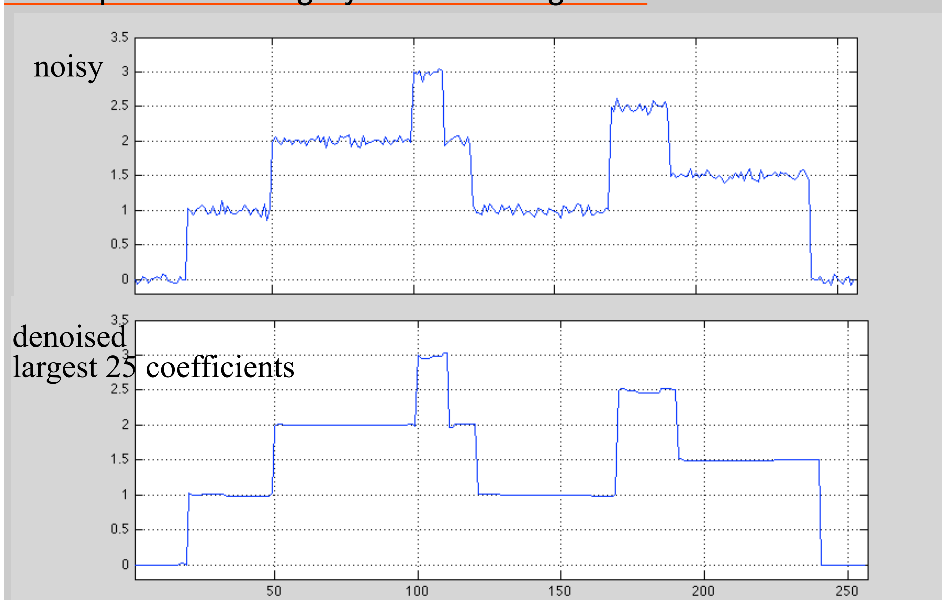
Approximation from 25/256 coefficients

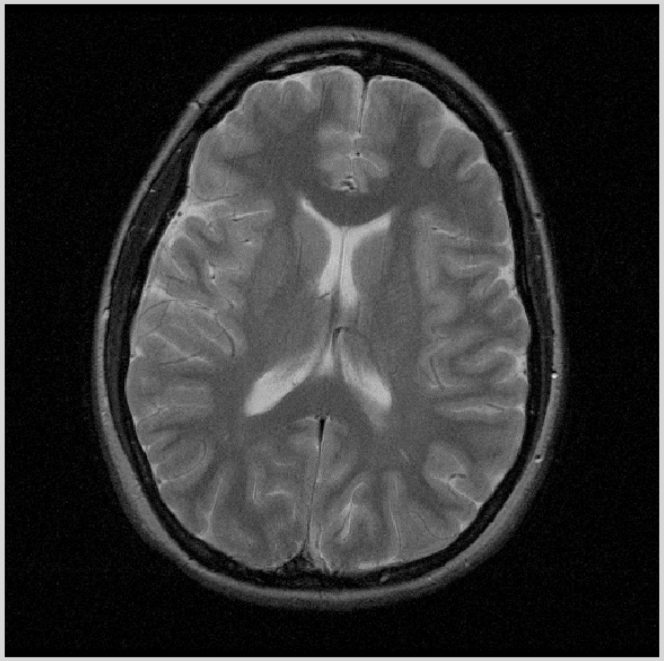


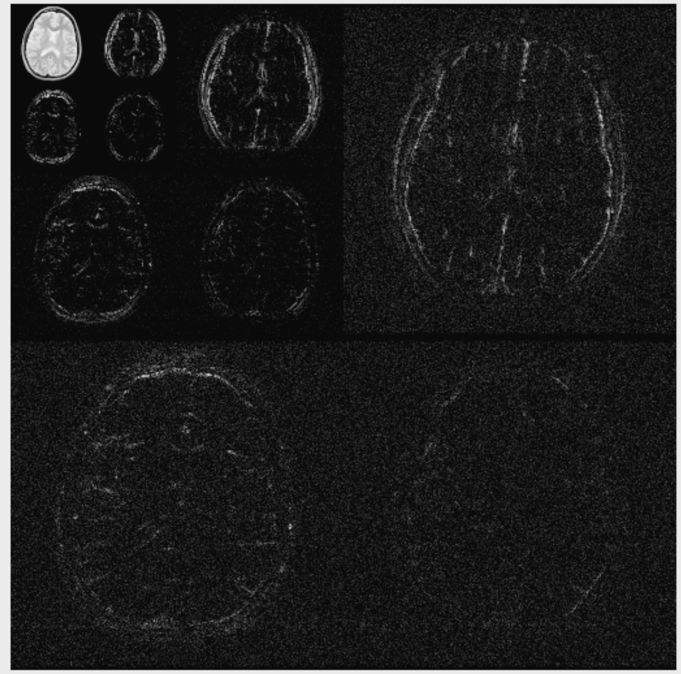
Example: Denoising Noisy Signals

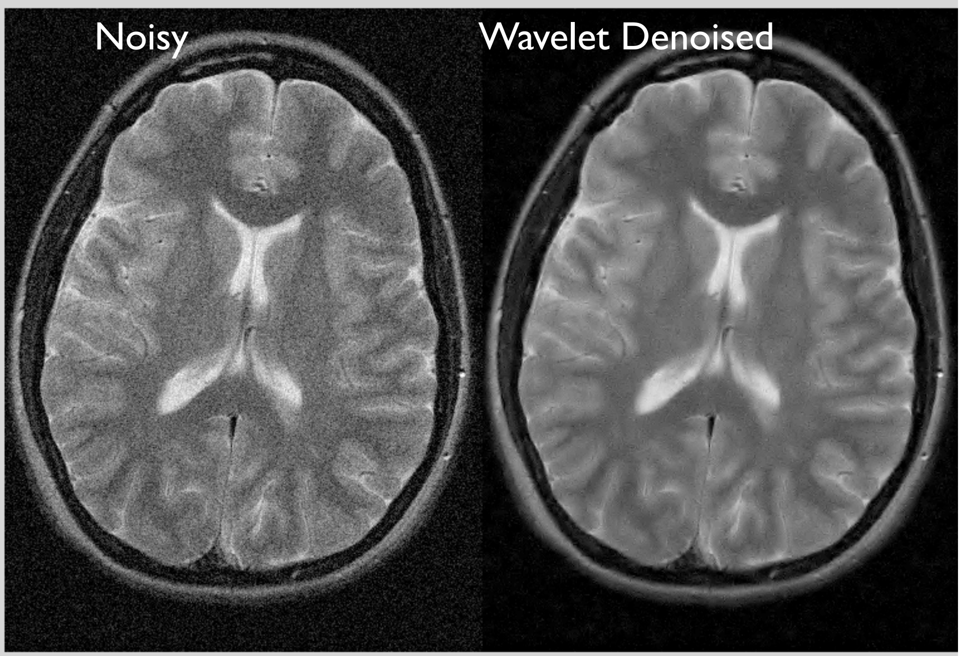


Example: Denoising by Thresholding








Noisy
Wavelet Denoised