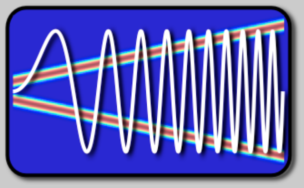
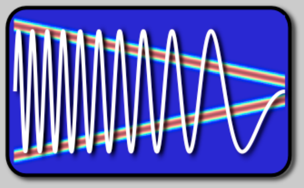


EE123



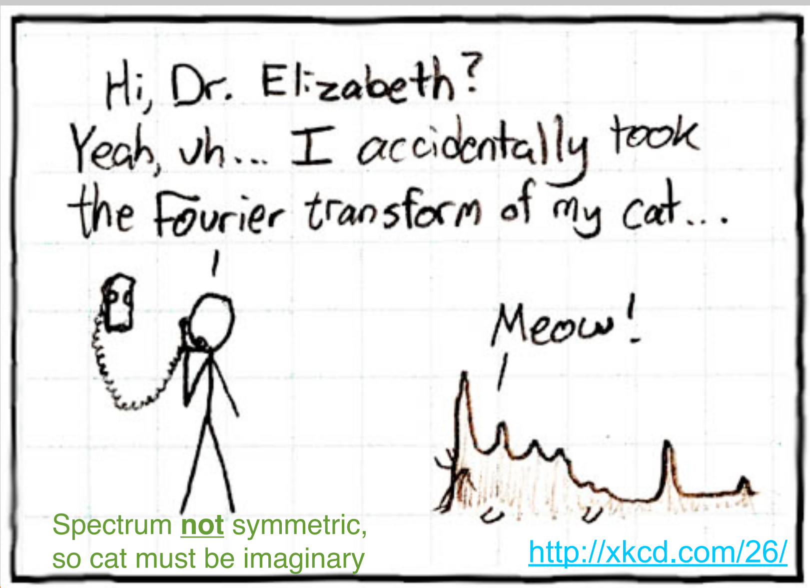
Digital Signal Processing

Lecture 10

Announcements

- Office hours TA 1-2pm
- Midterm 10/12
  - –All Material so far including wavelets
- Office hours attendance weak

How do you know this guy is insane?



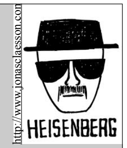
Spectrum **not** symmetric,
so cat must be imaginary

http://xkcd.com/26/

Last Time

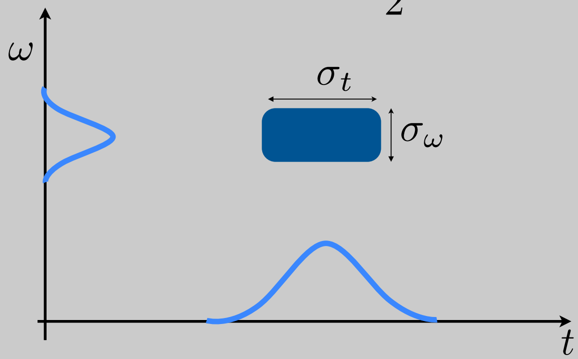
- Frequency Tiling (Heisenberg boxes)
- Short-Time Fourier Transform
  - – Equal area tiling
- Wavelets
  - – Adaptive tiling

Heisenberg Boxes



- Time-Frequency uncertainty principle

$$\sigma_t \cdot \sigma_\omega \geq \frac{1}{2}$$



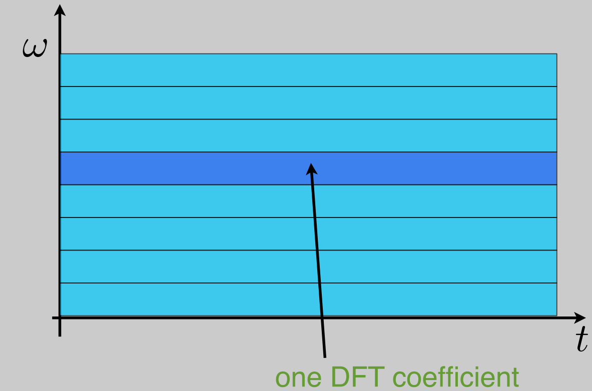
DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$\Delta\omega = \frac{2\pi}{N}$$

$$\Delta t = N$$

$$\Delta\omega \cdot \Delta t = 2\pi$$



DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

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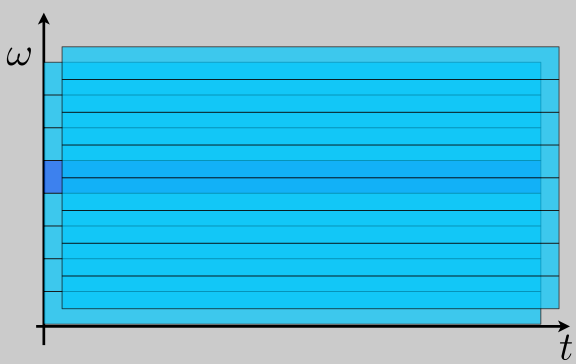


Question: What is the effect of zero-padding?
Answer: Overlapped Tiling!

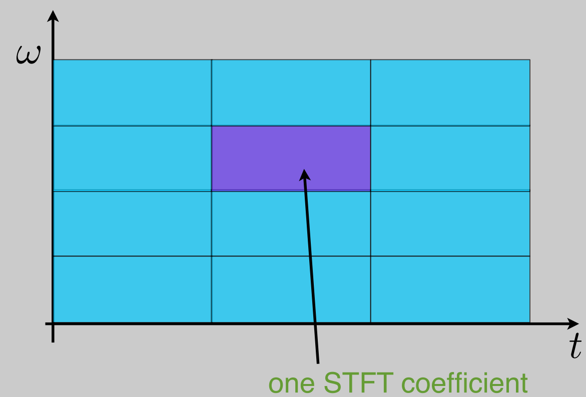
Discrete STFT

optional (R)

$$X[r,k] = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

$$\Delta t = L$$

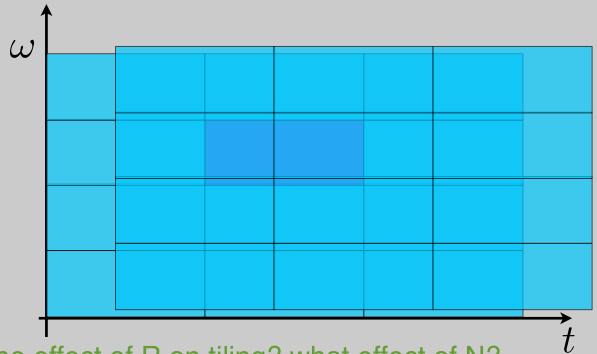


Discrete STFT

$$X[r,k] = \sum_{m=0}^{L-1} x[r\overset{\text{optional}}{R} + m]w[m]e^{-j2\pi km/N}$$

$$\Delta\omega = \frac{2\pi}{L}$$

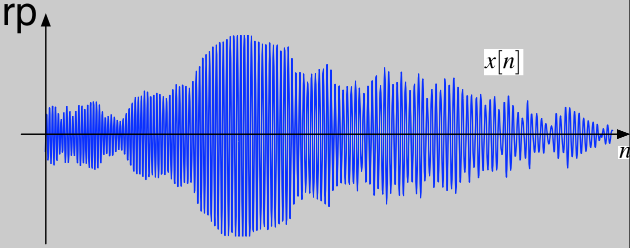
$$\Delta t = L$$

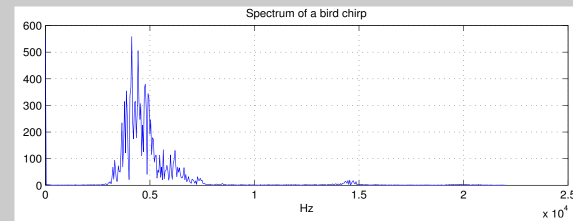


Question: What is the effect of R on tiling? what effect of N?
Answer: Overlapping in time of frequency or both!

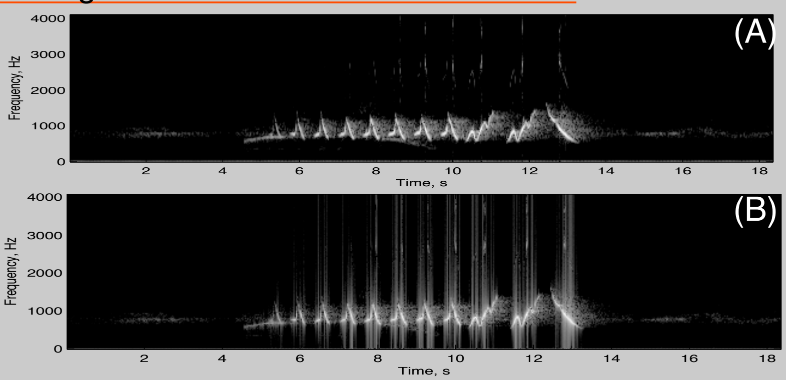
Spectrogram

- Recall bird chirp





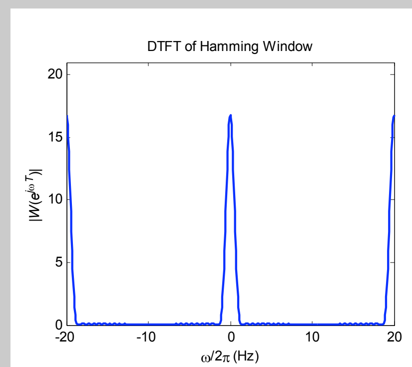
Spectrogram

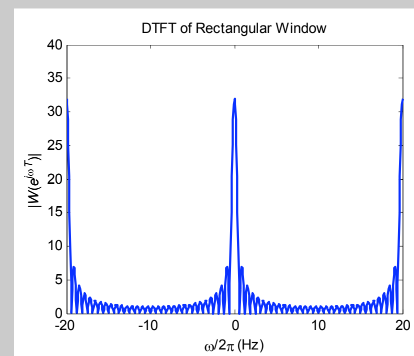


- What is the difference between the spectrograms?

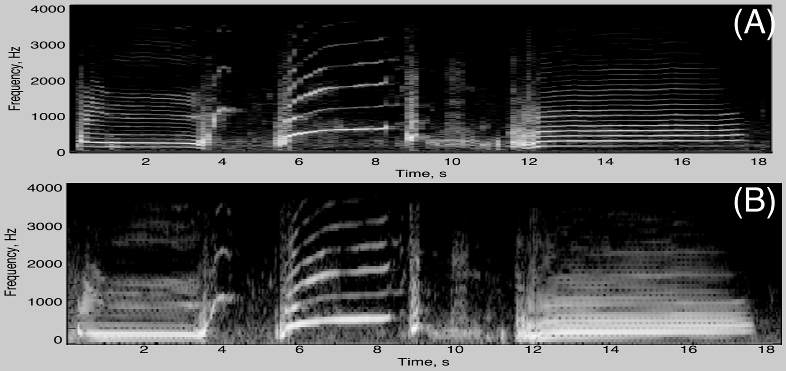
a) Window size B<A
b) Window size B>A
c) Window type is different
d) (A) uses overlapping window

Sidelobes of Hann vs rectangular window





Spectrogram



- What is the difference between the spectrograms?

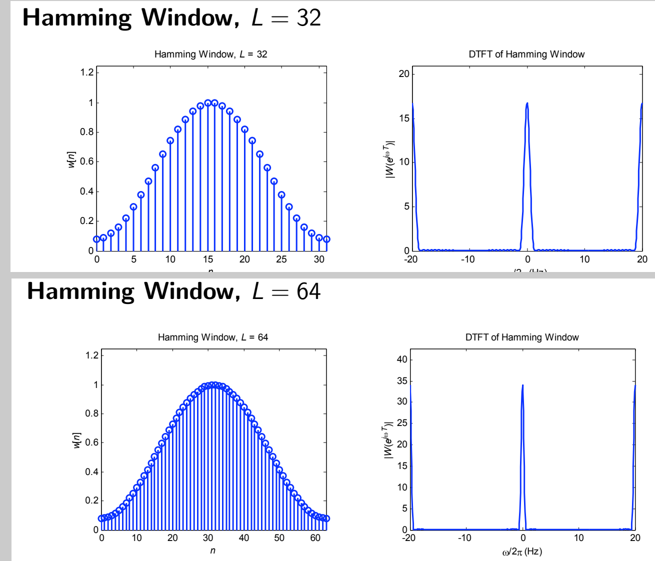
a) Window size B<A

b) Window size B>A

c) Window type is different

d) (A) uses overlapping window

Spectrogram

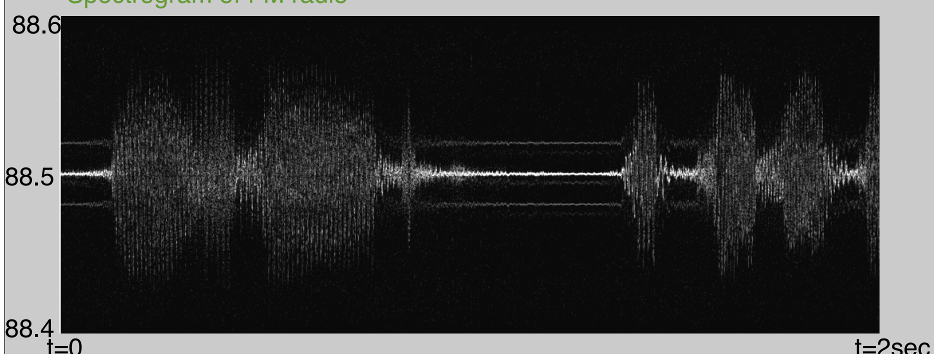


Lab 2

$$y_c(t) = A\cos\left(2\pi f_c t + 2\pi\Delta f \int_0^t x(\tau)d\tau\right)$$

$$y[n] = y(nT) = A\exp\left(j2\pi\Delta f \int_0^{nT} x(\tau)d\tau\right)$$

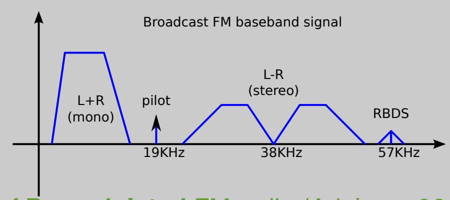
Spectrogram of FM radio



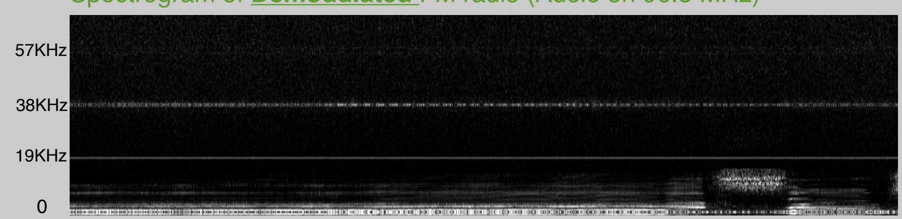
Lab 2

$$y[n] = y(nT) = A\exp\left(j2\pi\Delta f \int_0^{nT} x(\tau)d\tau\right)$$

$$x(t) = \underbrace{(L+R)}_{\text{mono}} + \underbrace{0.1\cdot\cos(2\pi f_p t)}_{\text{pilot}} + \underbrace{(L-R)\cos(2\pi(2f_p)t)}_{\text{stereo}} + \underbrace{0.05\cdot \text{RBDS}(t)\cos(2\pi(3f_p)t)}_{\text{digital RBDS}}.$$



Spectrogram of **Demodulated** FM radio (Adele on 96.5 MHz)



Limitations of Discrete STFT

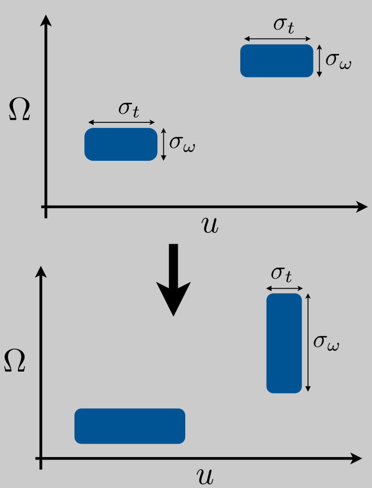
- Need overlapping ⇒ Not orthogonal
- Computationally intensive O(MN log N)
- Same size Heisenberg boxes

From STFT to Wavelets

- Basic Idea:
  - low-freq changes slowly - fast tracking unimportant
  - Fast tracking of high-freq is important in many apps.
  - Must adapt Heisenberg box to frequency
- Back to continuous time for a bit.....

From STFT to Wavelets

- Continuous time



$$Sf(u,\Omega) = \int_{-\infty}^{\infty} f(t) w(t-u) e^{-j\Omega t} dt$$

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*\left(\frac{t-u}{s}\right) dt$$

*Morlet - Grossmann

From STFT to Wavelets

$$Wf(u,s) = \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{s}} \Psi^*\left(\frac{t-u}{s}\right) dt$$

- The function Ψ is called a mother wavelet
  - Must satisfy:

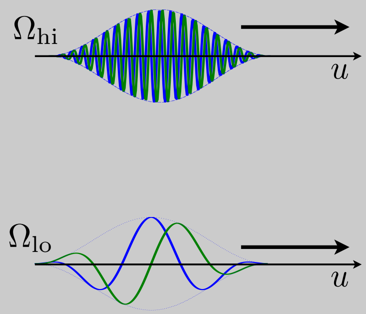
$$\int_{-\infty}^{\infty} |\Psi(t)|^2 dt = 1$$ ⇒ unit norm

$$\int_{-\infty}^{\infty} \Psi(t) dt = 0$$ ⇒ Band-Pass

STFT and Wavelets “Atoms”

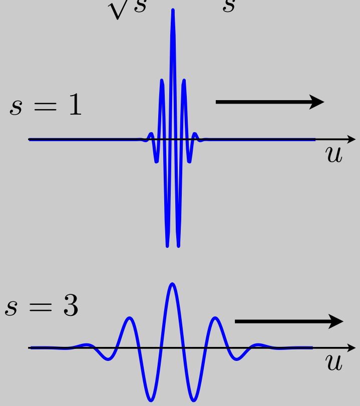
STFT Atoms

$w(t-u)e^{j\Omega t}$



Wavelet Atoms

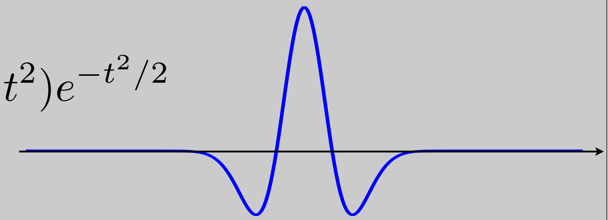
$\frac{1}{\sqrt{s}}\Psi(\frac{t-u}{s})$



Examples of Wavelets

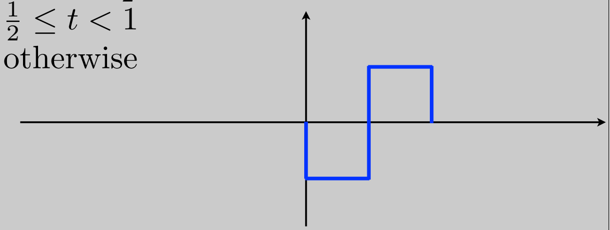
- Mexican Hat

$$\Psi(t) = (1-t^2)e^{-t^2/2}$$

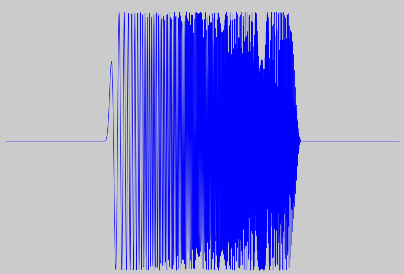


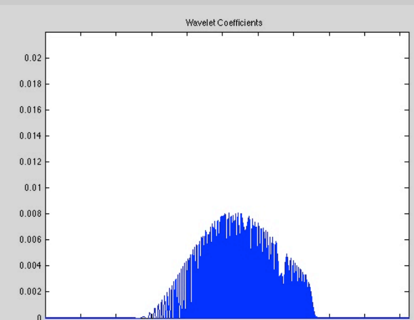
- Haar

$$\Psi(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Example: Wavelet of Chirp





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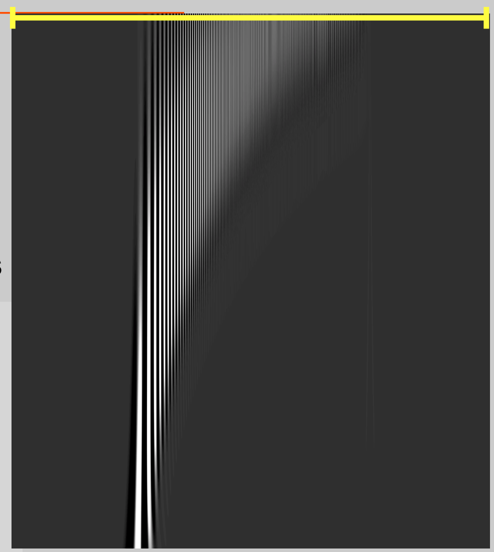

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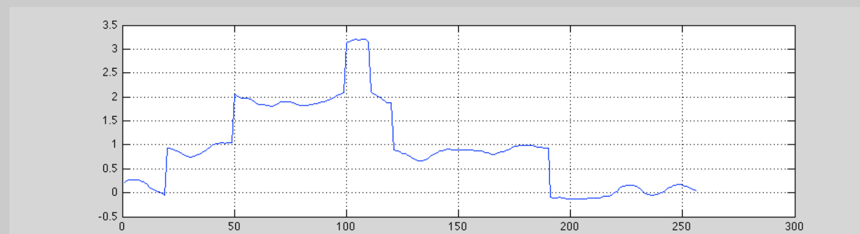
Example 2: “Bumpy” Signal



SombreroWavelet

log(s)

u

Wavelets Transform

- Can be written as linear filtering

$$
\begin{aligned}
Wf(u,s) &= \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} f(t)\Psi^*(\frac{t-u}{s})dt \\
&= \left\{ f(t) * \overline{\Psi}_s(t) \right\}(u)
\end{aligned}
$$

$$\overline{\Psi}_s = \frac{1}{\sqrt{s}}\Psi(\frac{t}{s})$$

- Wavelet coefficients are a result of bandpass filtering

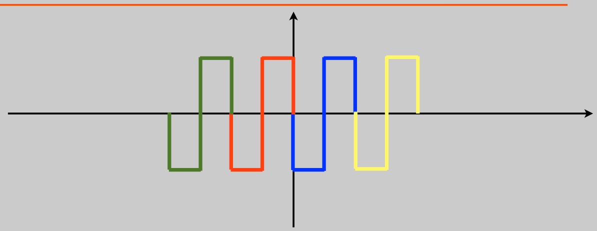
Wavelet Transform

- Many different constructions for different signals
  - –Haar good for piece-wise constant signals
  - –Battle-Lemarie' : Spline polynomials
- Can construct Orthogonal wavelets
  - – For example: dyadic Haar is orthonormal

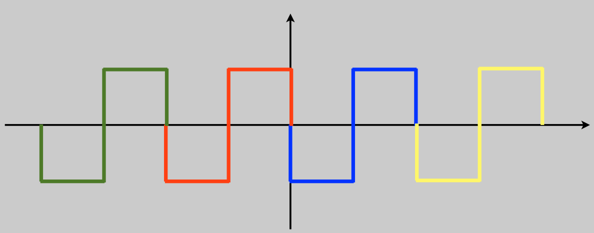
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$

$$i = [1,\ 2,\ 3, \cdots]$$

Orthonormal Haar



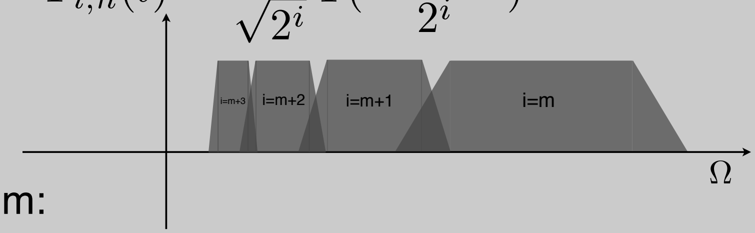
Same scale non-overlapping



Orthogonal between scales

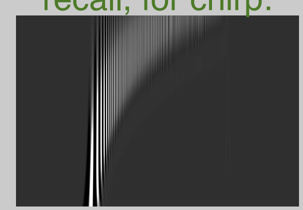
Scaling function

$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi(\frac{t-2^i n}{2^i})$$



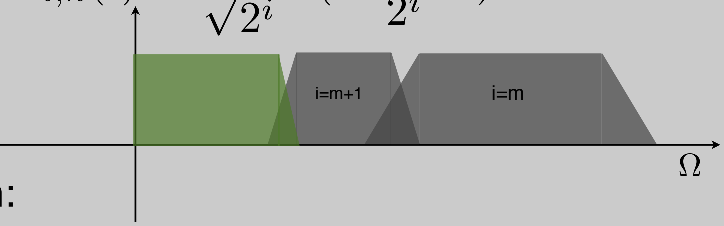
- Problem:
  - –Every stretch only covers half remaining bandwidth
  - –Need Infinite functions

recall, for chirp:



Scaling function

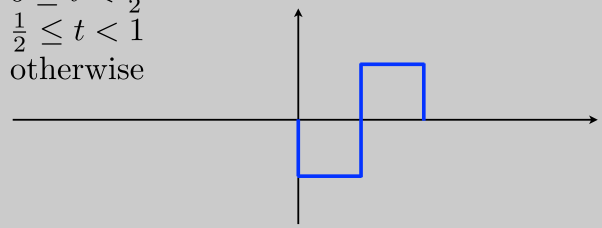
$$\overline{\Psi}_{i,n}(t) = \frac{1}{\sqrt{2^i}}\Psi\left(\frac{t-2^i n}{2^i}\right)$$



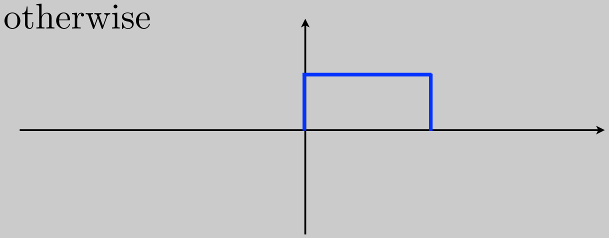
- Problem:
  - Every stretch only covers half remaining bandwidth
  - Need Infinite functions
- Solution:
  - Plug low-pass spectrum with a scaling function $\overline{\Phi}$

Haar Scaling function

$$\Psi(t) = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\Phi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$



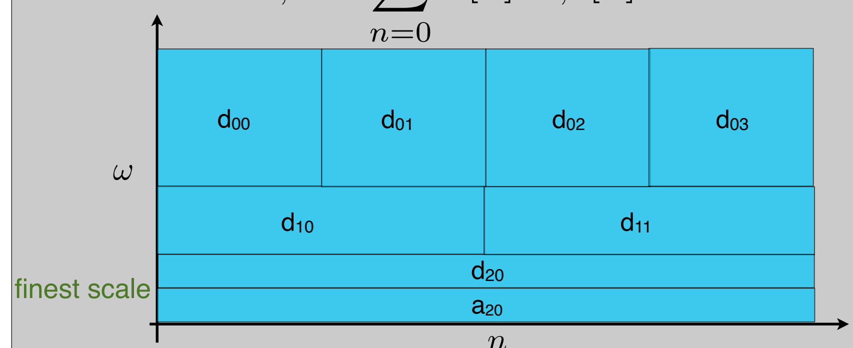
Back to Discrete

- Early 80's, theoretical work by Morlett, Grossman and Meyer (math, geophysics)
- Late 80's link to DSP by Daubechies and Mallat.
- From CWT to DWT not so trivial!
- Must take care to maintain properties

Discrete Wavelet Transform

$$d_{s,u} = \sum_{n=0}^{N-1} x[n]\Psi_{s,u}[n]$$

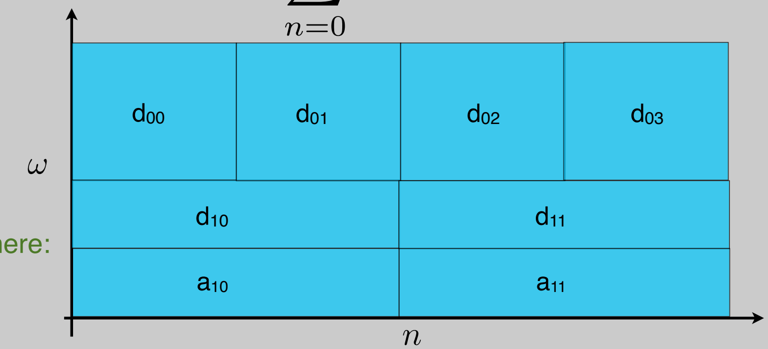
$$a_{s,u} = \sum_{n=0}^{N-1} x[n]\Phi_{s,u}[n]$$



Discrete Wavelet Transform

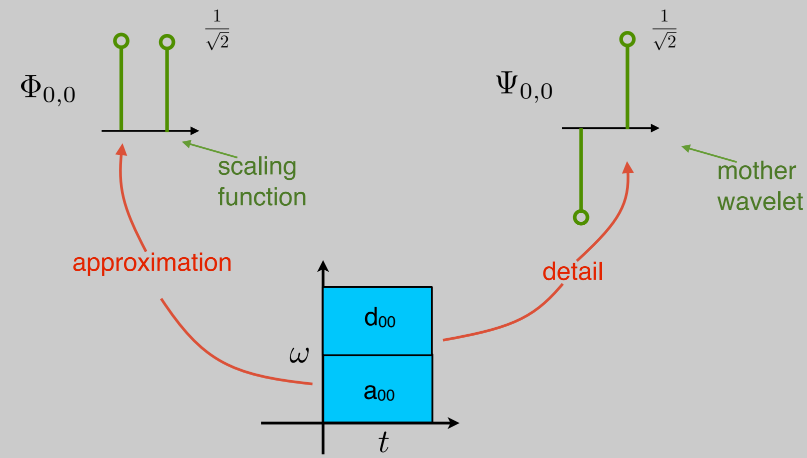
$$d_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Psi_{\mathrm{s,u}}[n]$$

$$a_{\mathrm{s,u}} = \sum_{n=0}^{N-1} x[n]\Phi_{\mathrm{s,u}}[n]$$



Example: Discrete Haar Wavelet

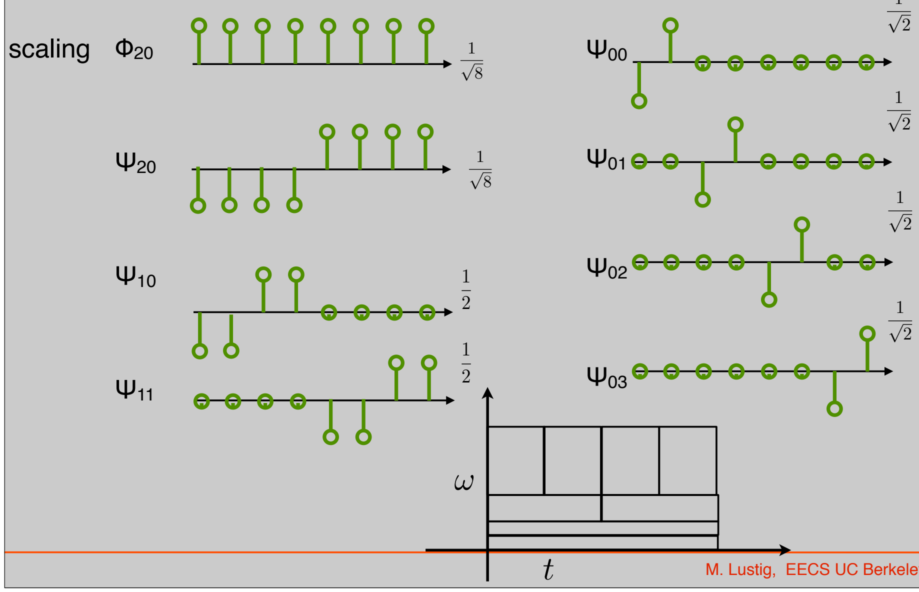
Haar for n=2



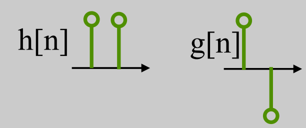
Equivalent to DFT_2!

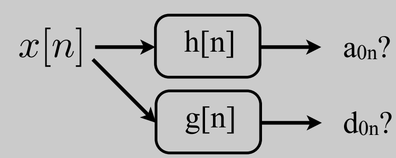
Discrete Orthogonal Haar Wavelet

Haar for n=8

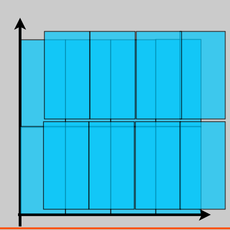


Fast DWT with Filter Banks

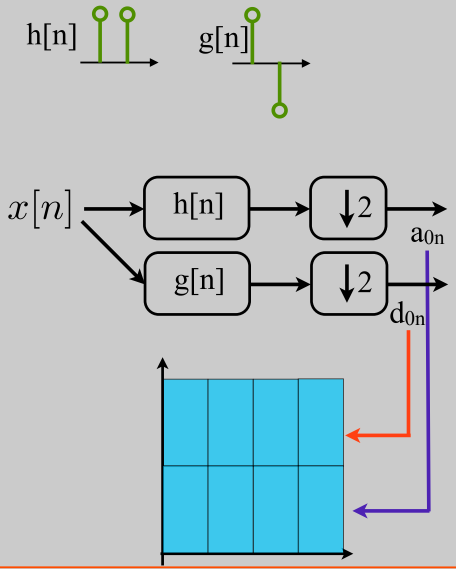




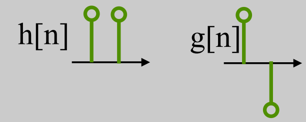
not quite...
too many
coefficients



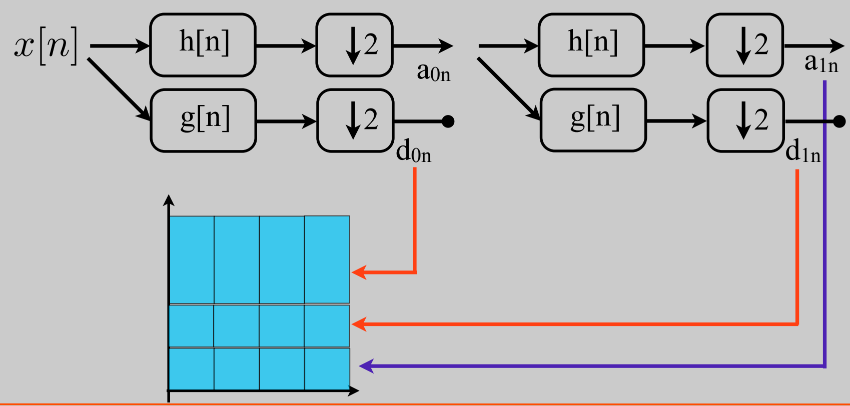
Fast DWT with Filter Banks



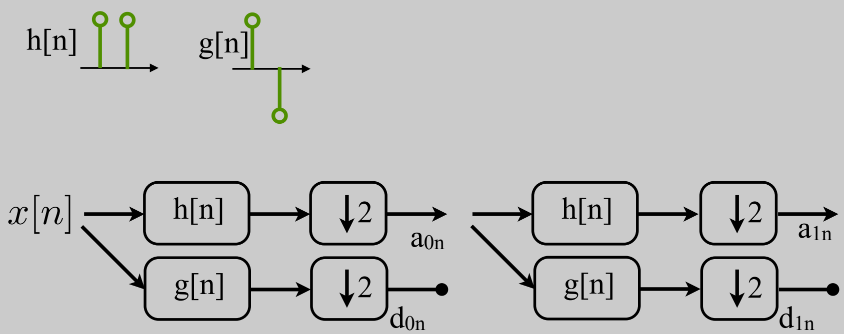
Fast DWT with Filter Banks



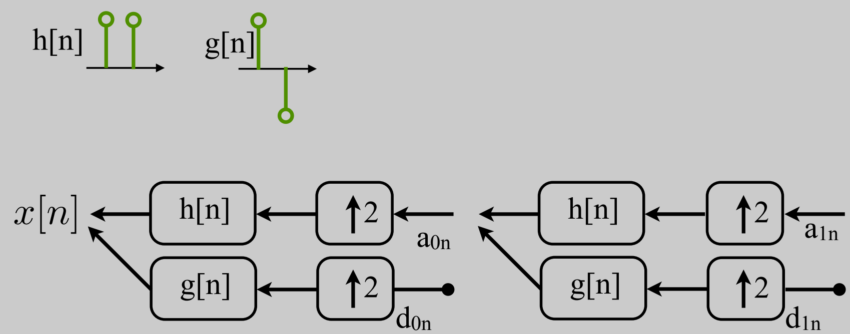
complexity:
N + N/2 + N/4 + N/8 +...+ = 2N
=O(N)



Decomposition

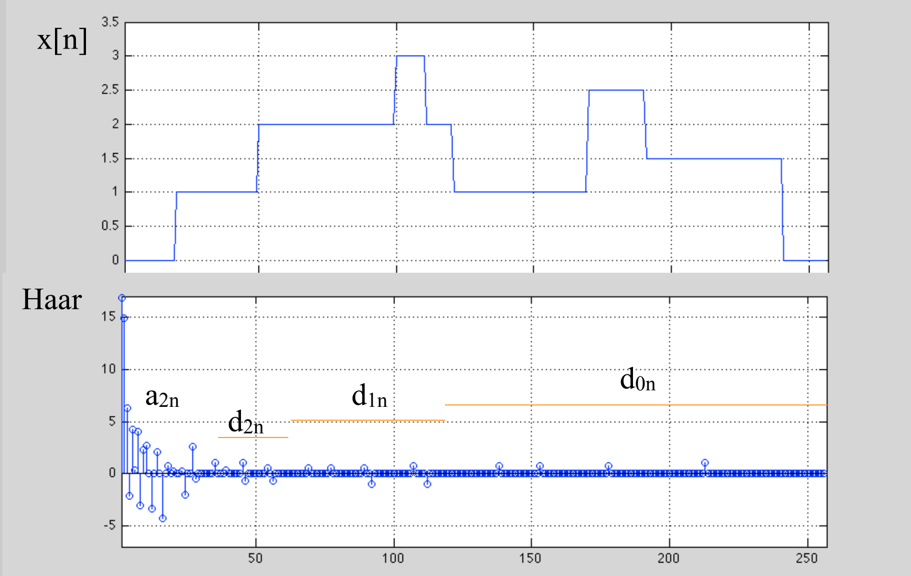


Reconstruction

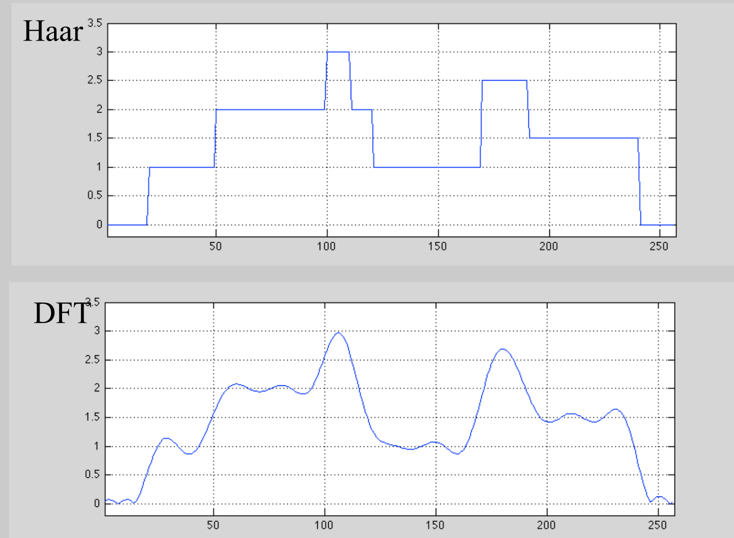


Just flip arrows!

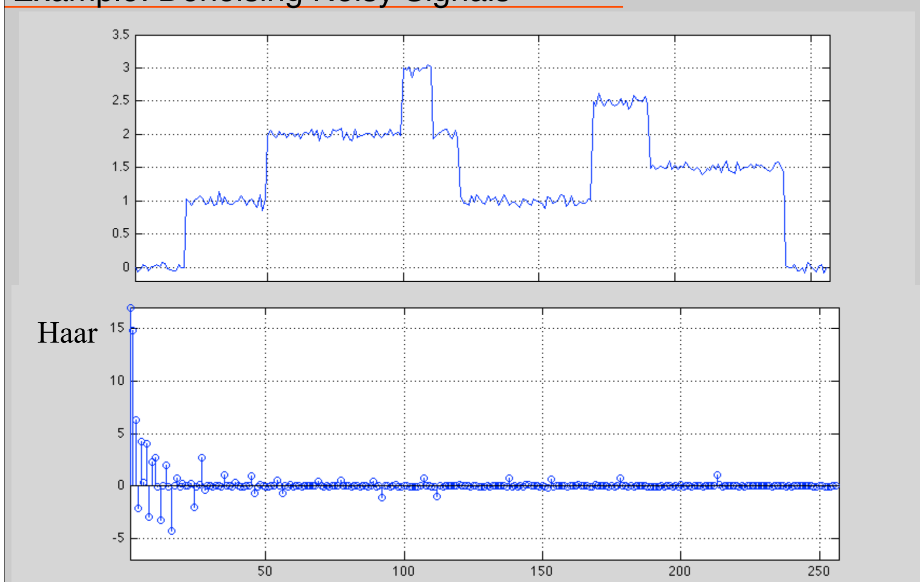
Haar DWT Example



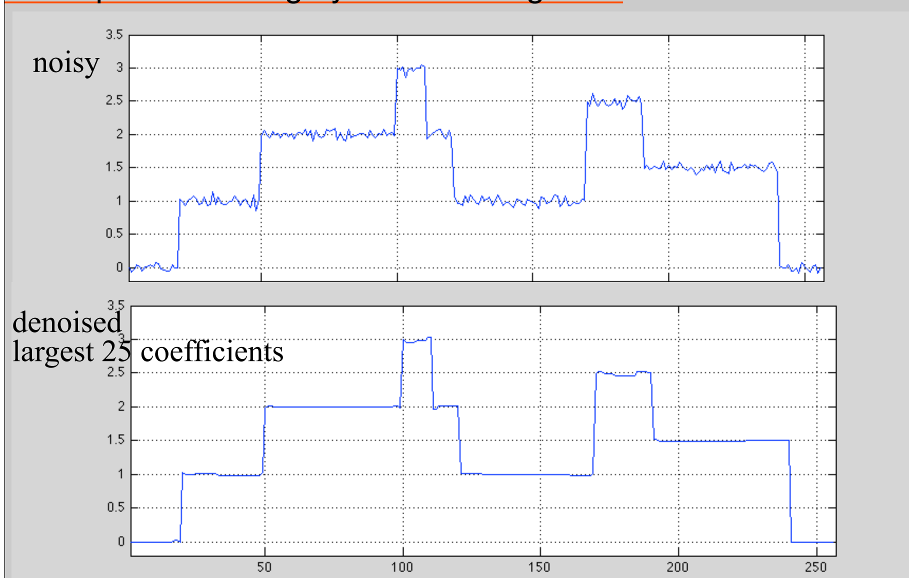
Approximation from 25/256 coefficients

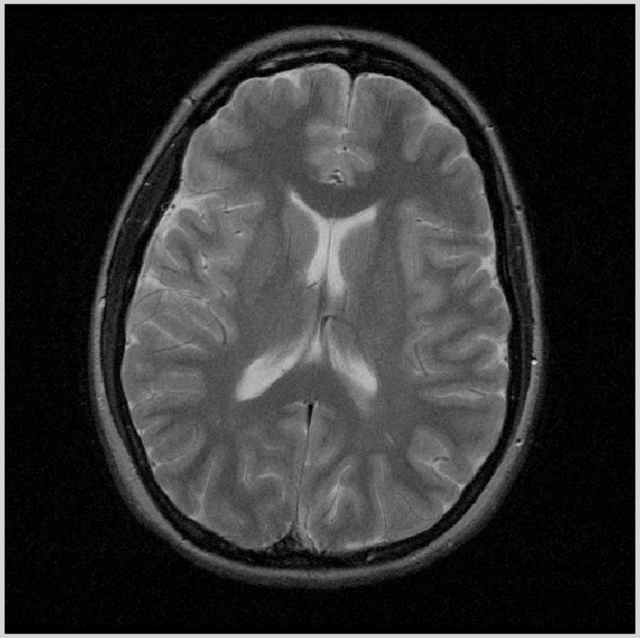


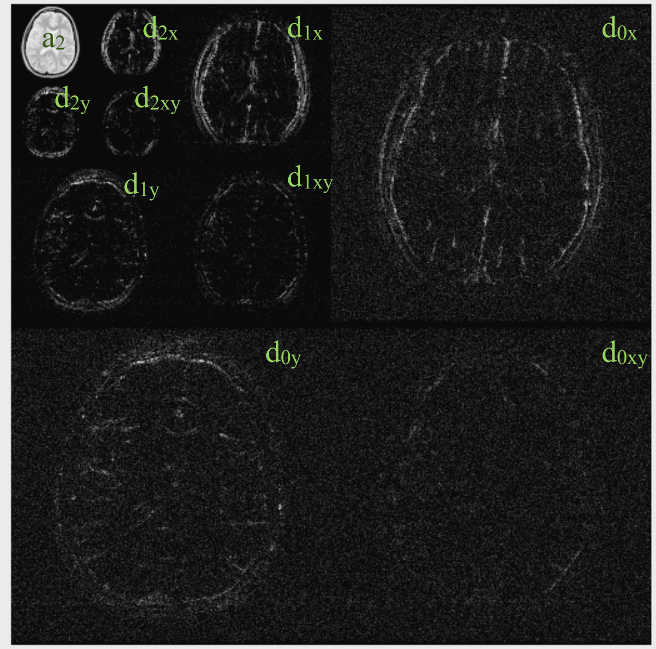
Example: Denoising Noisy Signals

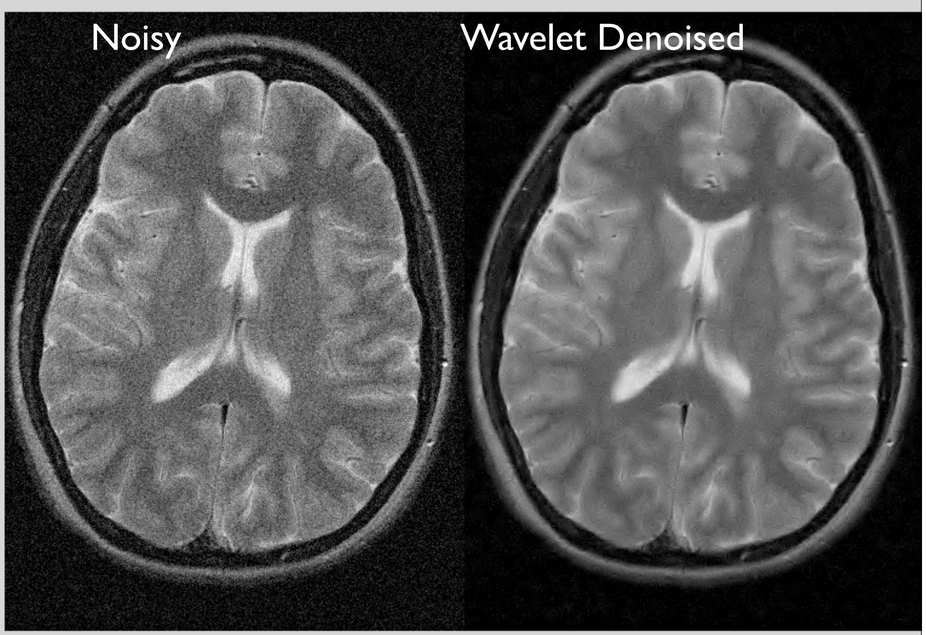


Example: Denoising by Thresholding









Approximation/Compression

0.000 % coefficients