EE123
Digital Signal Processing

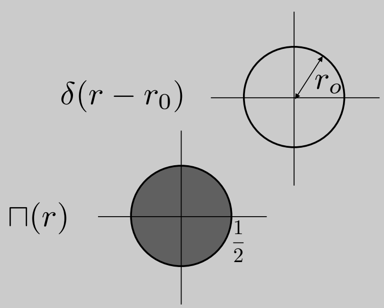
Lecture 11

Multi-Dimensional Signals

- Our world is more complex than 1D
- Images: f(x,y)
- Videos: f(x,y,t)
- Dynamic 3F scenes: f(x,y,z,t)
  - Medical Imaging
  - 3D Video
  - Computer Graphics
- We will focus on 2D

Continuous-Time 2D functions

- $\delta(x,y)$: Impulse at $x=0$, $y=0$
- $\delta(x)$: Impulse line (vertical or horizontal?)
- $\prod(x,y)$: 2D rect function
- $\cos(2\pi(f_x x + f_y y))$ - Spatial harmonic
- Circularly Symmetic:
  - $\delta(r)$: Impulse ring
  - $\prod(x,y)$: Pillbox



Spatial Frequency

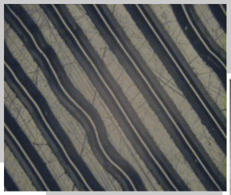
- What is a spatial frequency?
  - Complex Harmonic:

$$e^{j(\Omega_x x + \Omega_y y)} = e^{j2\pi(f_x x + f_y y)}$$

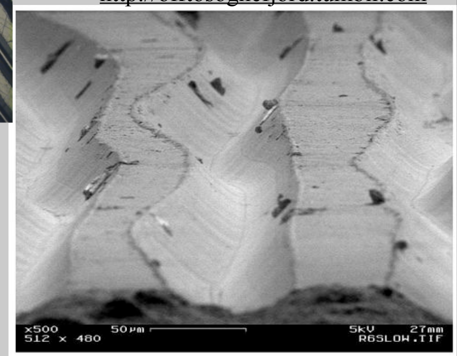
  - Units (for example):
    - x,y - cm
    - f_x, f_y - 1/cm
    - Ω_x, Ω_y - rad/cm

Spatial Frequency

- Vinyl Record
  - –Transforms a temporal signal to a spatial signal

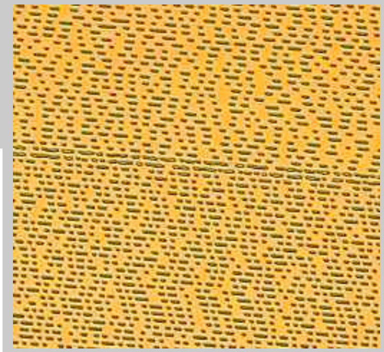


http://offtosognefjord.tumblr.com



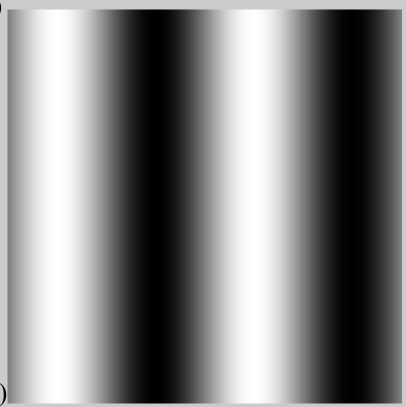
Spatial Frequency

- CD ROM
  - – encodes digital temporal signals to spatial signals



What is the frequency?

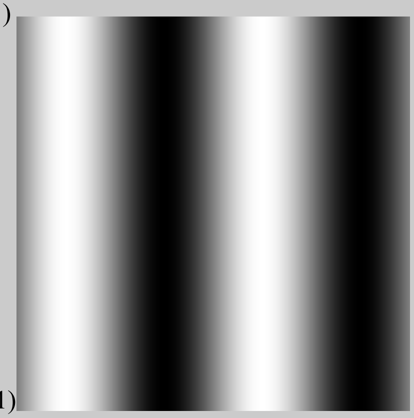
$$\sin(2\pi(f_x x + f_y y))$$



a) $f_x=2$, $f_y=2$
b) $f_x=1$, $fy=0$
c) $f_x = 4$, $f_y=0$
d) none of the above

What is the frequency?

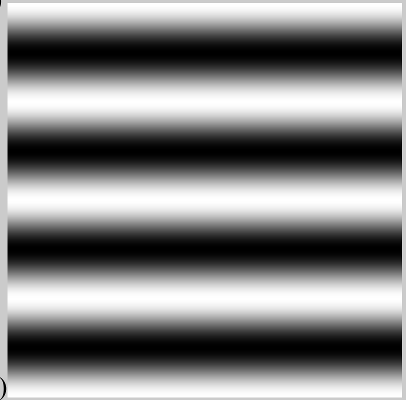
$$\sin(2\pi(f_x x + f_y y))$$



2 cycles for 2 cm $\Rightarrow f_x=1\ cm^{-1}$

What is the frequency?

$$\sin(2\pi(f_x x + f_y y)) \quad \text{or} \quad \cos(2\pi(f_x x + f_y y))$$



a) sin, $f_x=0$, $f_y=2$

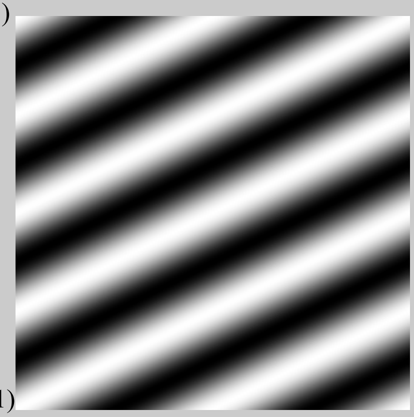
b) cos, $f_x=0$, $f_y=4$

c) cos, $f_x = 0$, $f_y=2$

d) none of the above

What is the frequency?

$$\cos(2\pi(f_x x + f_y y))$$



a) $f_x=1$, $f_y=2$

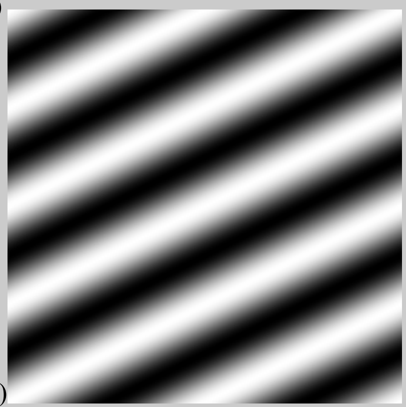
b) $f_x=4$, $f_y=2$

c) $f_x = 2$, $f_y=1$

d) $f_x=2$, $f_y=4$

What is the frequency?

$$\cos(2\pi(f_x x + f_y y))$$



a) $f_x=1$, $f_y=2$

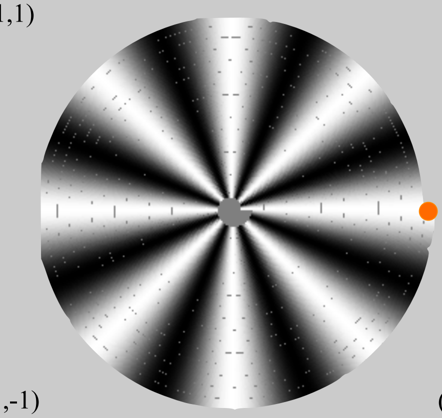
b) $f_x=4$, $f_y=2$

c) $f_x = 2$, $f_y=1$

d) $f_x=2$, $f_y=4$

What is the Temporal Frequency?

Vinyl rotates at 1 Hz



a) $\cos(2\pi 8t)$

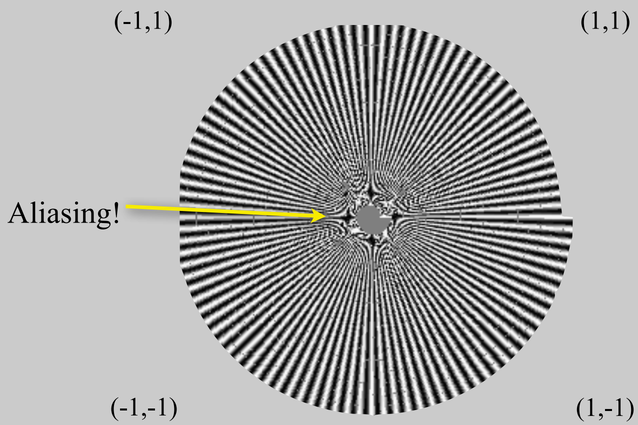
b) $\cos(2\pi 8t^2)$

c) $\cos(2\pi 4t)$

d) $\cos(2\pi 4t^2)$

What is the Temporal Frequency?

Vinyl rotates at 1 Hz



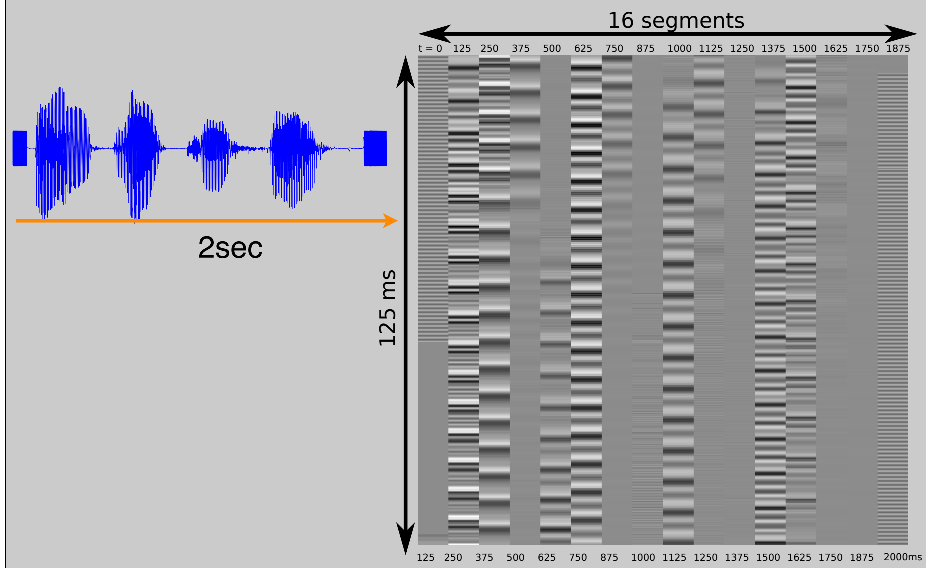
a) cos(2π100t)

b) cos(2π100t²)

c) cos(2π40t)

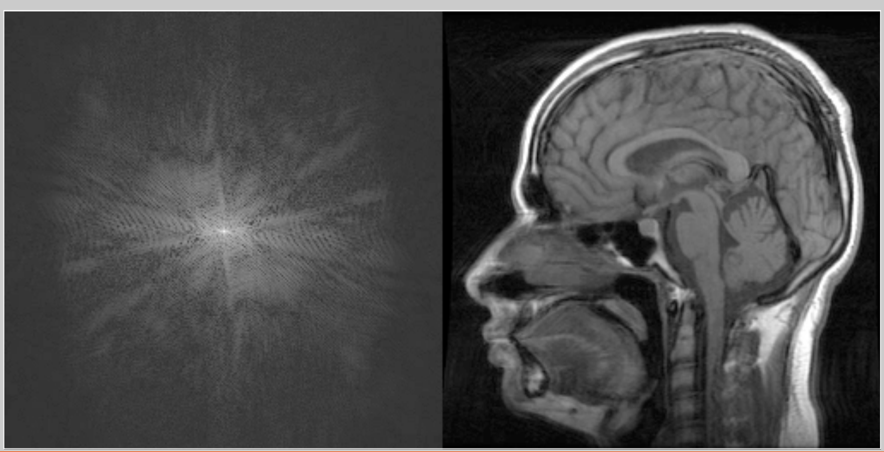
d) none of the answers

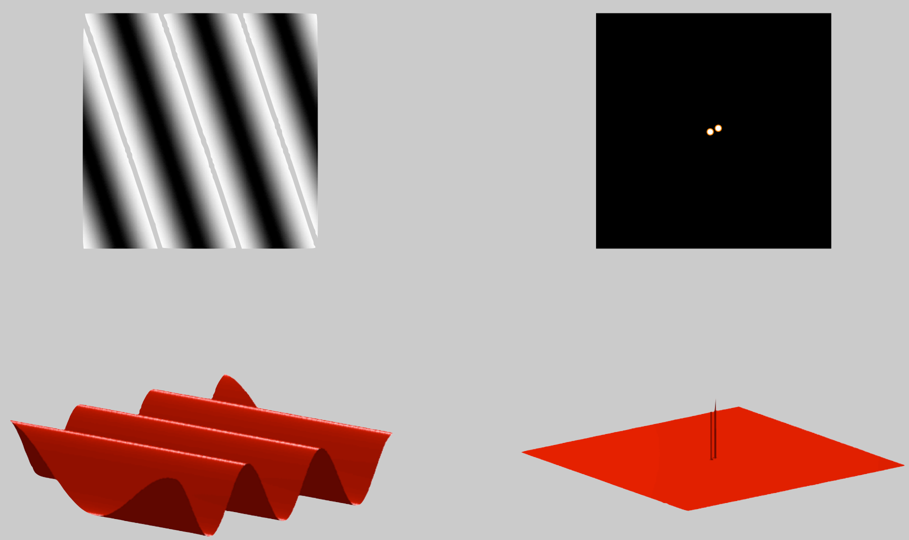
Challenge: What is the password?

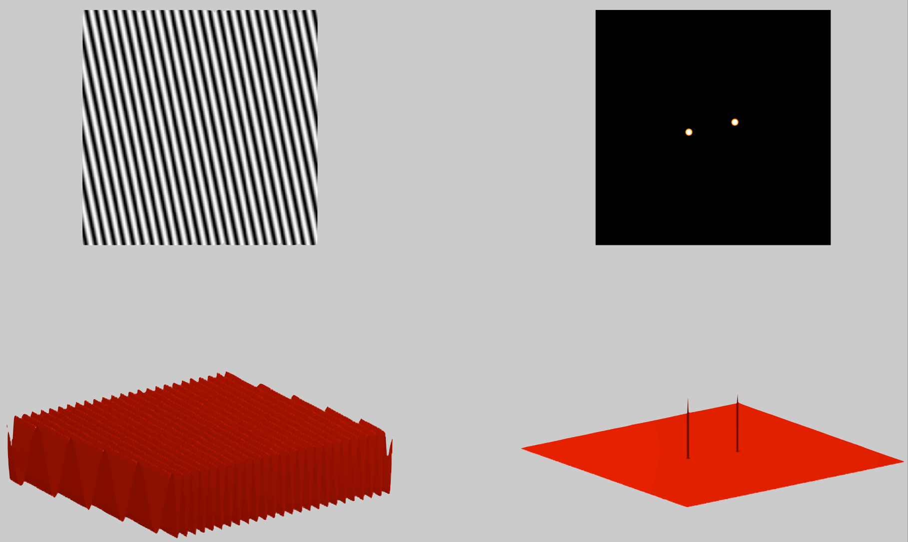


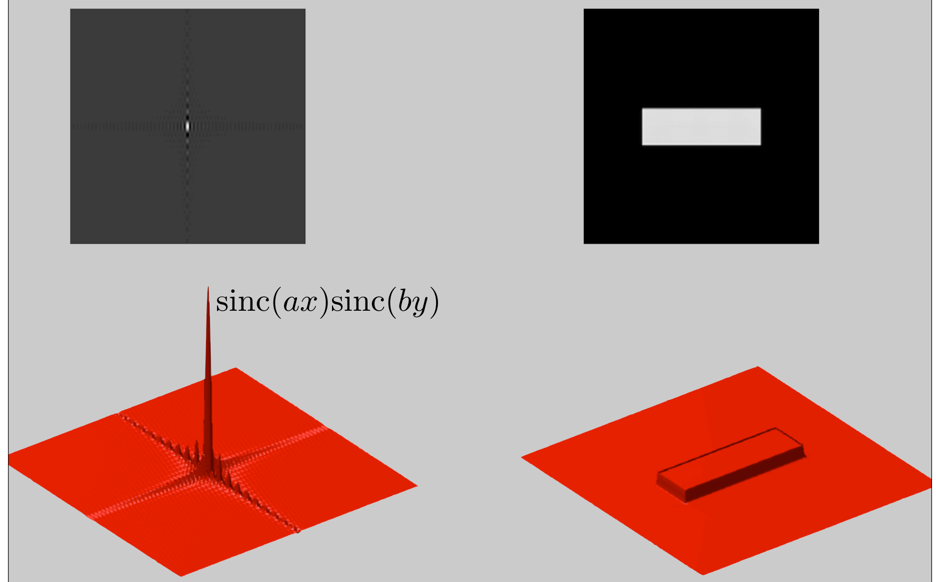
2D Fourier Transform

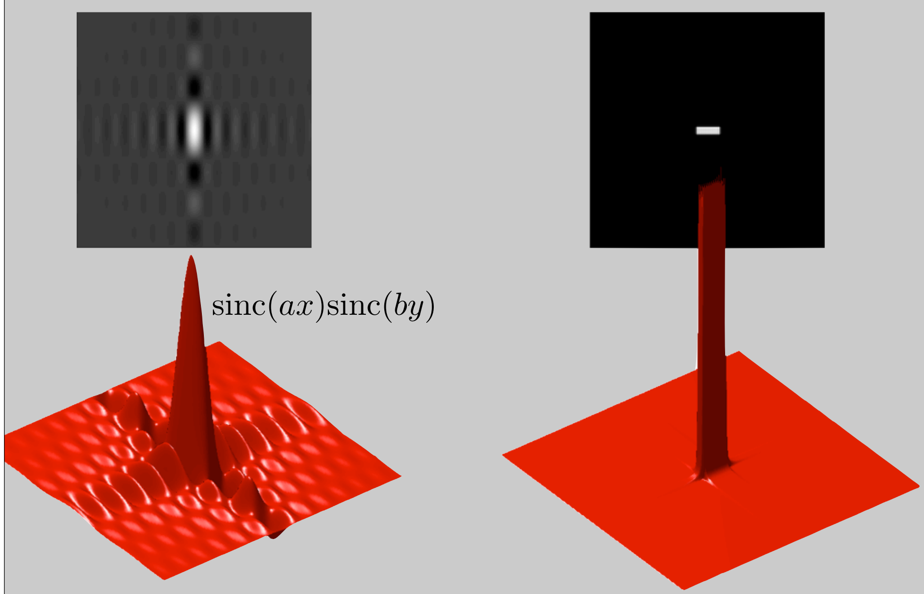
$$F(f_x, f_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(f_x x + f_y y)}dxdy$$











2D DTFT

$$F(\omega_x, \omega_y) = \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} f[n_x, n_y] e^{-j(\omega_x n_x + \omega_y n_y)}$$

$$-\pi \leq \omega_x, \omega_y \leq \pi$$

$$F(\kappa_x, \kappa_y) = \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} f[n_x, n_y] e^{-j2\pi(\kappa_x n_x + \kappa_y n_y)}$$

$$-0.5 \leq \kappa_x, \kappa_y \leq 0.5$$

- I prefer 2nd
- “Massaging” the DTFT leads to separable transforms in each axis

Separability of 2D DTFT

- white board....

2D - DFT

- Similarly to 1D:
  - Forward:

$$F[k_x, k_y] = \sum_{n_x=0}^{N-1} \sum_{n_y=0}^{M-1} f[n_x, n_y] e^{-j2\pi(n_x k_x/N + n_y k_y/M)}$$

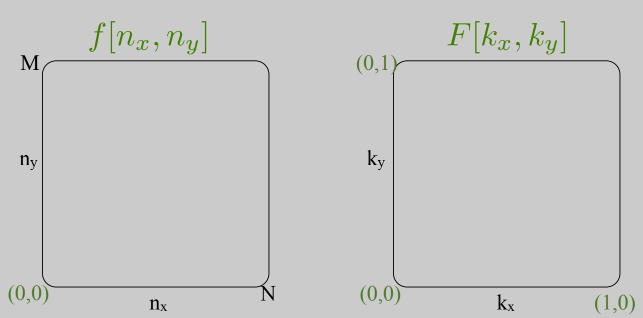
$$\kappa_x = k_x/N, \kappa_y = k_y/M$$

  - Inverse:

$$f[n_x, n_y] = \frac{1}{NM} \sum_{k_x=0}^{N-1} \sum_{k_y=0}^{M-1} F[k_x, k_y] e^{+j2\pi(n_x k_x/N + n_y k_y/M)}$$

2D - DFT

$$F[k_x, k_y] = \sum_{n_x=0}^{N-1} \sum_{n_y=0}^{M-1} f[n_x, n_y] e^{-j2\pi(n_x k_x/N + n_y k_y/M)}$$



Need to fftshift in 2D to get it to look like DTFT.

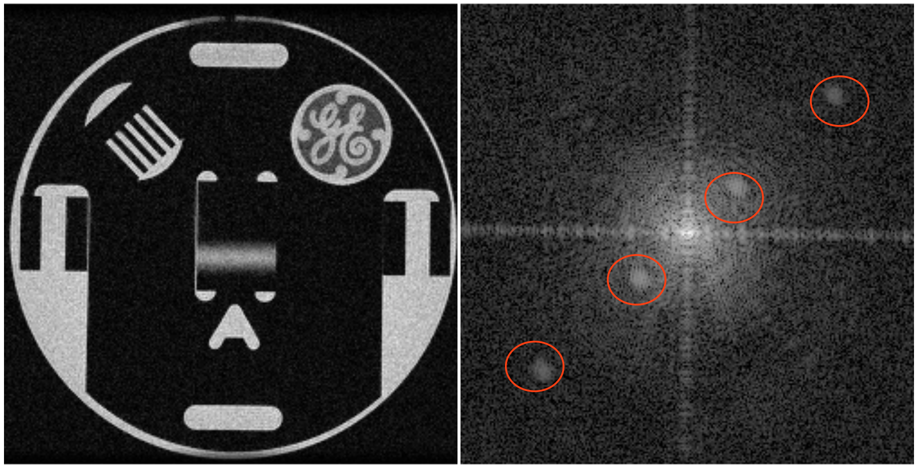
Properties of 2D DFT

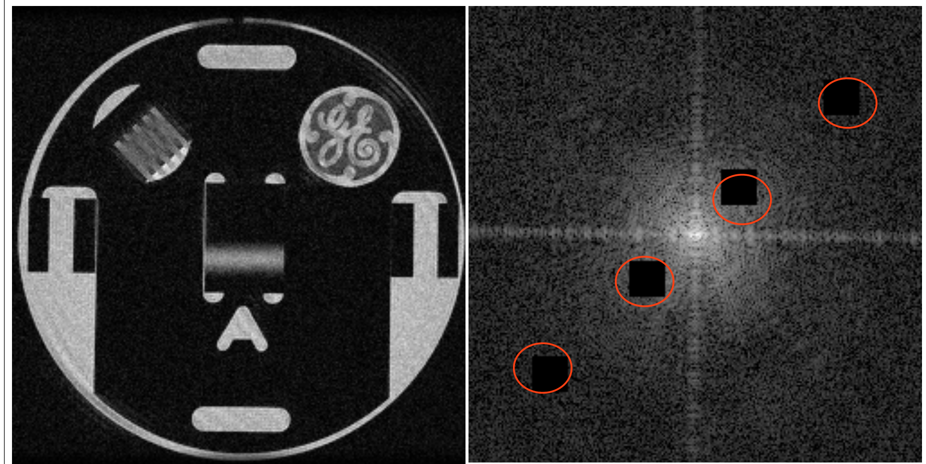
- Circular Convolution

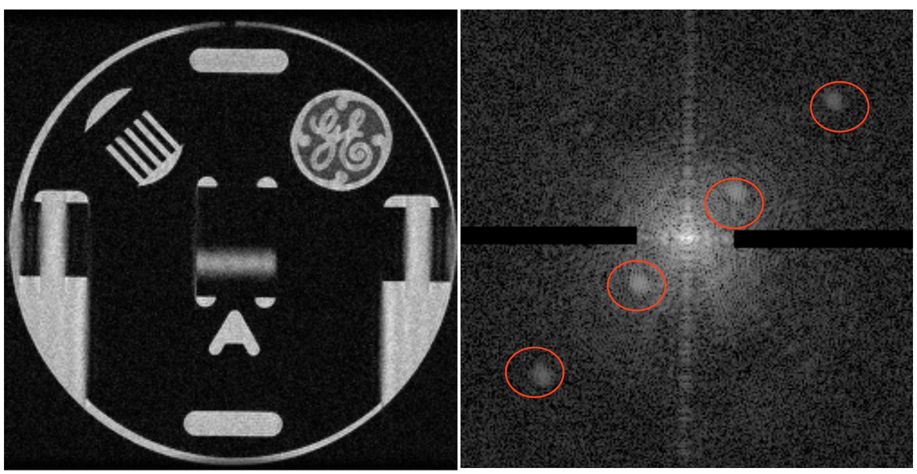
$$f[n_x, n_y] ** h[n_x, n_y] = F[k_x, k_y] H[k_x, k_y]$$

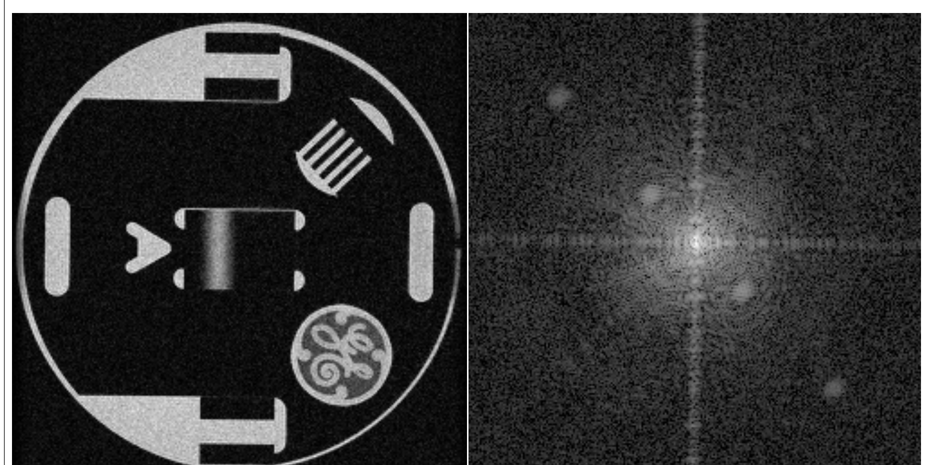
- Circular shift

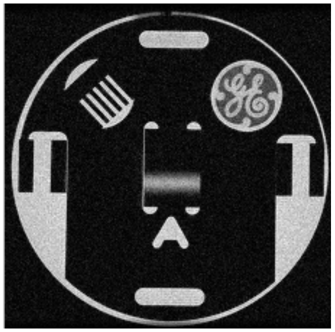
$$f[(n_x - m_x)_N, (n_y - m_y)_M] = e^{-j2\pi(k_x m_x/N + k_y m_y/M)} F[k_x, k_y]$$

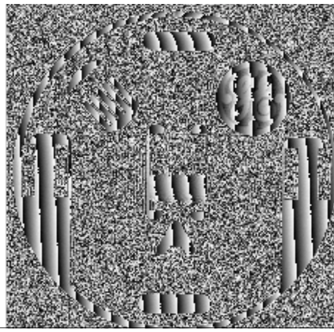


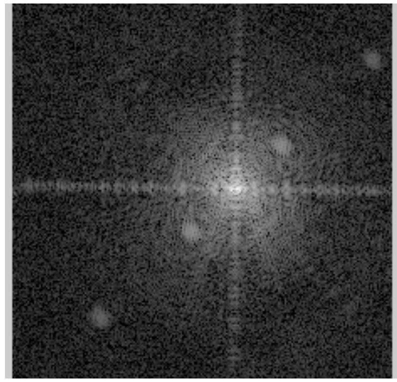


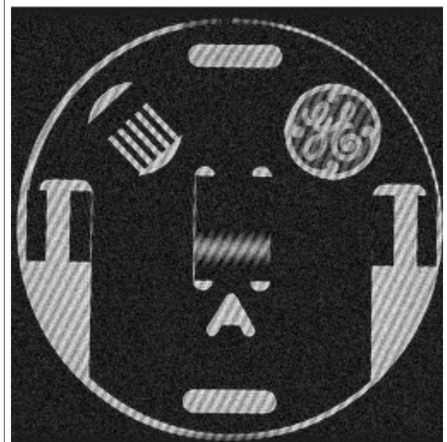


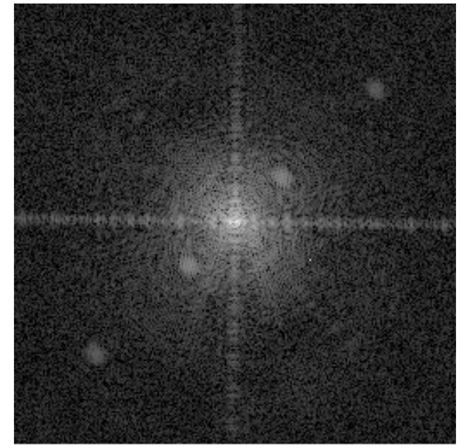




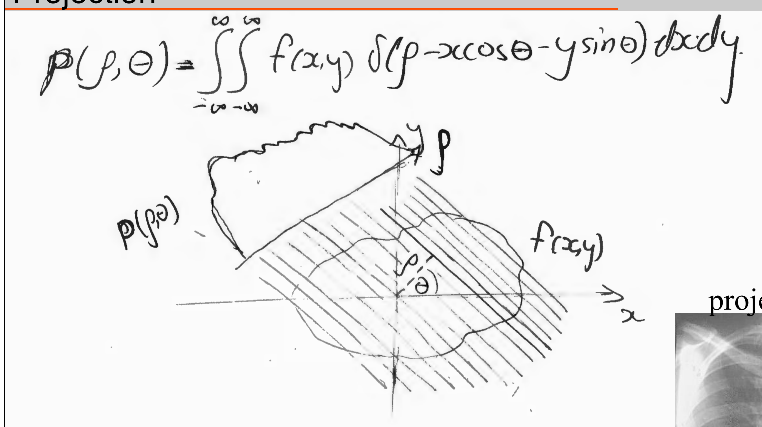




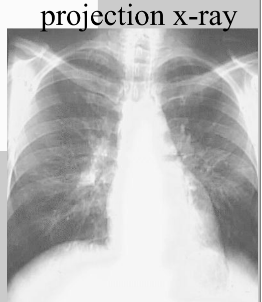




Projection

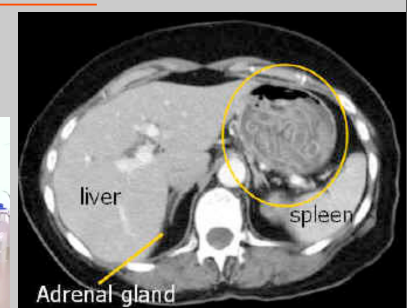


projection x-ray



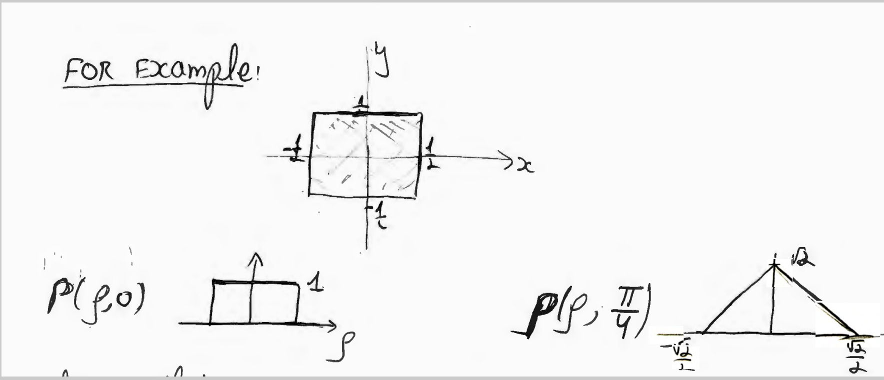
Many Projections - Tomography

http://www.youtube.com/watch?v=4gklQHM19aY&feature=related

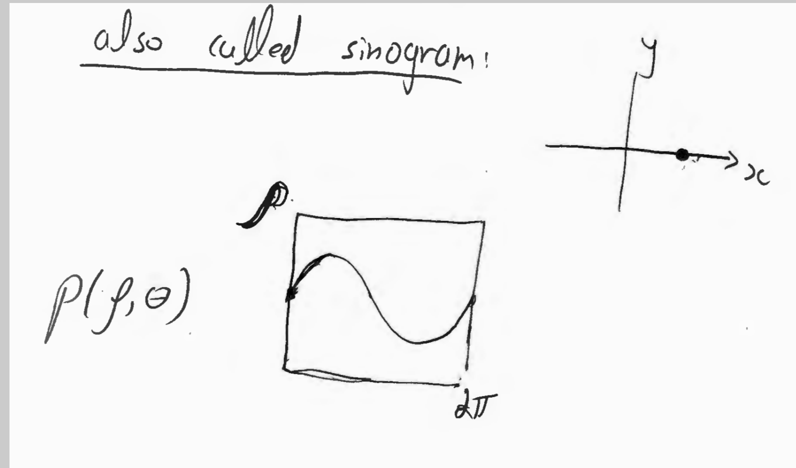


Radon Transform

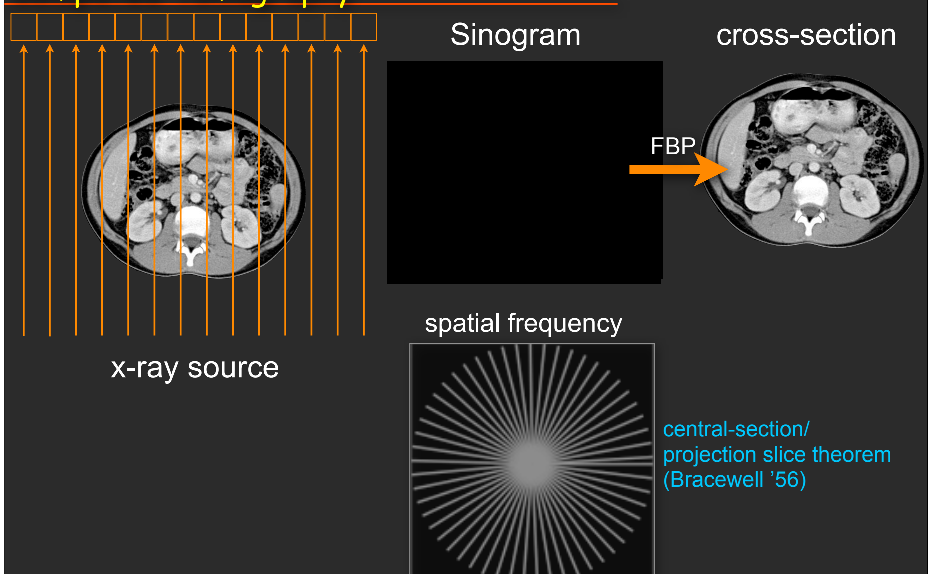
$$p(\rho,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(\rho - x\cos\theta - y\sin\theta)\,dx\,dy$$



Radon Transform: Sinogram



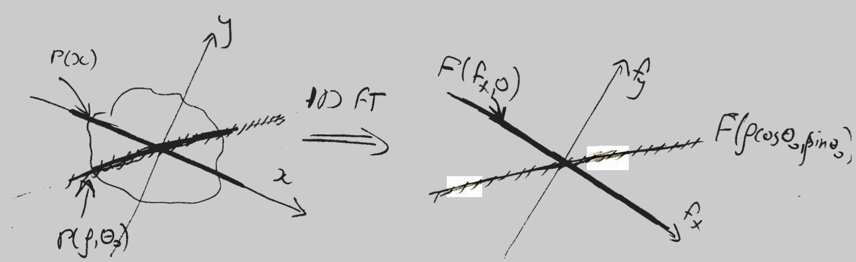
Computed Tomography



Projection Slice Theorem (Bracewell)

$$\mathcal{F}_{1D}\{p(\rho,\theta_0)\} = F(\rho\cos\theta_0, \rho\sin\theta_0)$$

↑ 2D Fourier transform.



Projection Slice Theorem (Bracewell)

Proof: show only for $\theta = 0 \Rightarrow$ generalize to arbitrary θ

$$P(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$F(f_x,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(f_x x + f_y y)}\, dx\, dy =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi f_x x}\, dx\, dy =$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y)\, dy \right] e^{-j2\pi f_x x}\, dx =$$

$$= \int_{-\infty}^{\infty} P(x)\, e^{-j2\pi f_x x}\, dx = \mathcal{F}\{P(x)\}$$

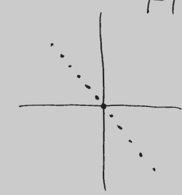
Discrete Reconstruction

DISCRETE RECONSTRUCTION

$$P_{\theta_m}[\rho_n] \xrightarrow{\text{1D DFT}} P_{\theta_m}[\rho_k]$$

n: $0 - [N-1]$, k: $[0 - N-1]$

2D DTFT $F(k_x, k_y)$



$$k_x : \frac{\omega_x}{2\pi} = \frac{\rho_k}{N}\cos\theta_m$$

$$k_y : \frac{\omega_y}{2\pi} = \frac{\rho_k}{N}\sin\theta_m$$

$$F[n,m] = \int\int F(k_x,k_y)\, e^{-2\pi j(k_x n + k_y m)}\, dk_x\, dk_y =$$

$$= \int\int F(\rho,\theta)\, e^{-2\pi j(\rho\cos\theta\, n + \rho\sin\theta\, m)}\, |\rho|\, d\rho\, d\theta$$

Discretization

DISCRETIZE:

$$\simeq \sum\sum F\left(\frac{\rho_k}{N},\theta_m\right) e^{-2\pi j\left(\frac{\rho_k}{N}\cos\theta_m\, n + \frac{\rho_k}{N}\sin\theta_m\, m\right)} \left|\frac{\rho_k}{N}\right|$$