EE123
Digital Signal Processing

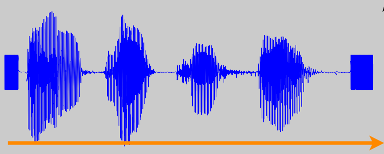
Lecture 12

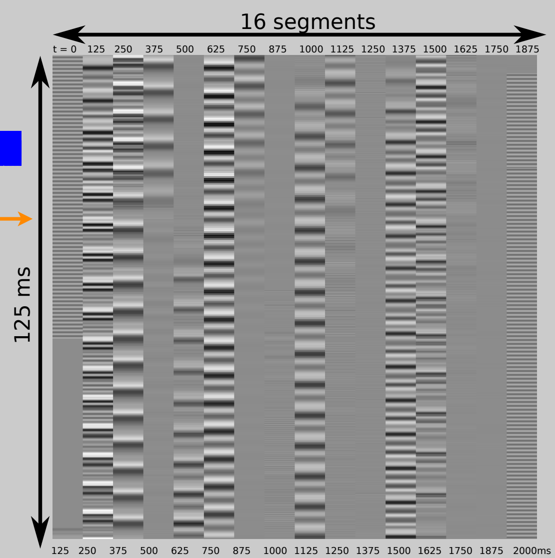
Announcements

- Midterm next Friday
  - –Open everything
  - –Prepare a sheet - not cheat!

- δ(x) : Impulse line (vertical or horizontal?)
- $\delta(x,y) = {}^2\delta(x,y) = \delta(x)\delta(y)$
- $\int\int {}^2\delta(x)dxdy \to \infty$

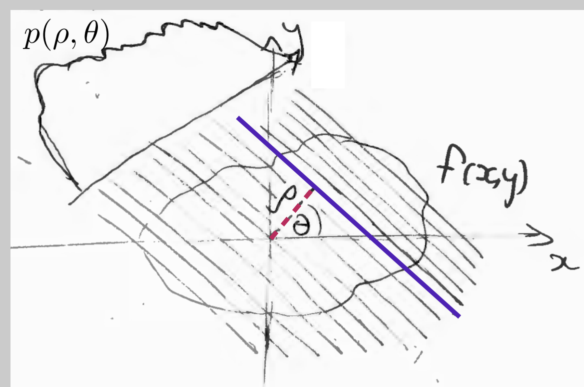
Challenge: What is the password?





Projection

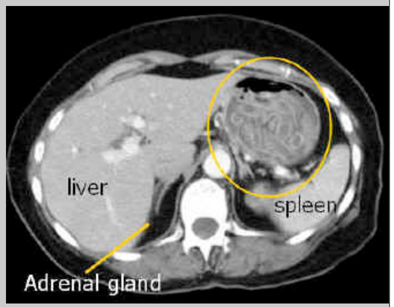
$$p(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(\rho - x\cos\theta - y\sin\theta)dxdy$$



projection x-ray

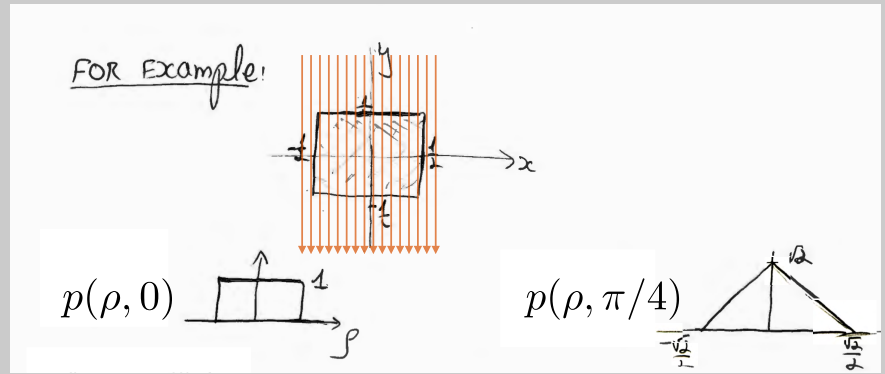
Many Projections - Tomography

http://www.youtube.com/watch?v=4gklQHM19aY&feature=related



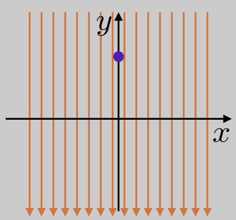
Radon Transform

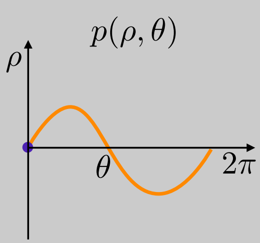
$$p(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(\rho - x\cos\theta - y\sin\theta)dxdy$$



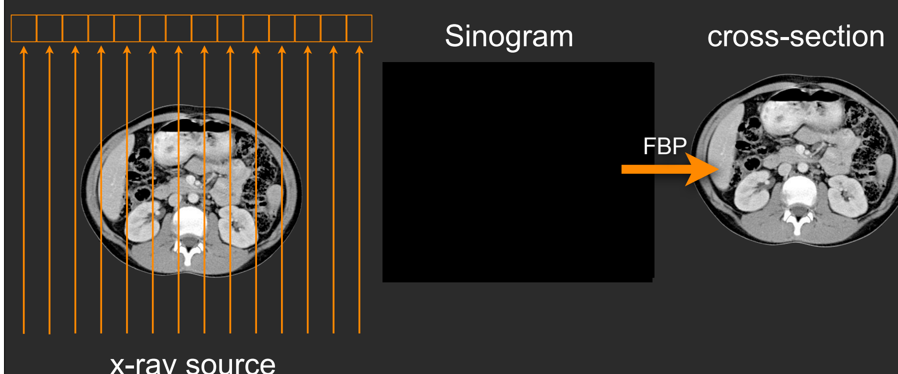
Radon Transform: Sinogram

- Also called Sinogram
- Impulse ⇒ Sinusoid



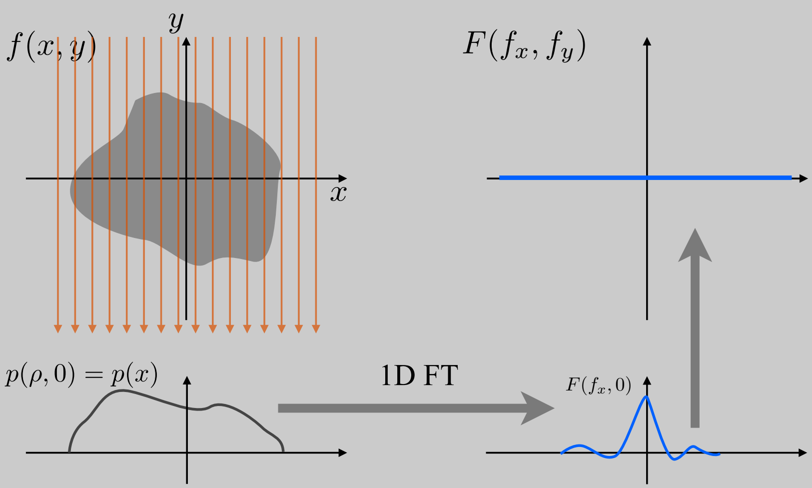


Computed Tomography



Projection Slice Theorem (Bracewell)

$$\mathcal{F}_{1\mathrm{D}}\{p(\rho,\theta)\} = F(\rho\cos\theta, \rho\cos\theta)$$



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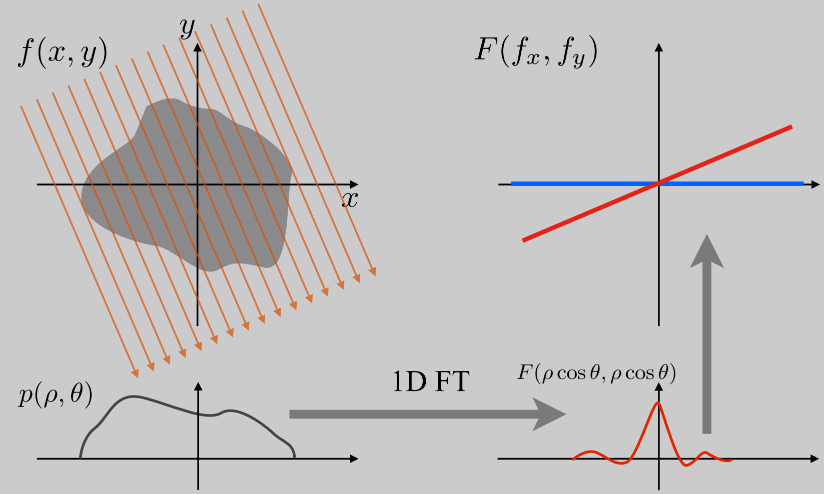

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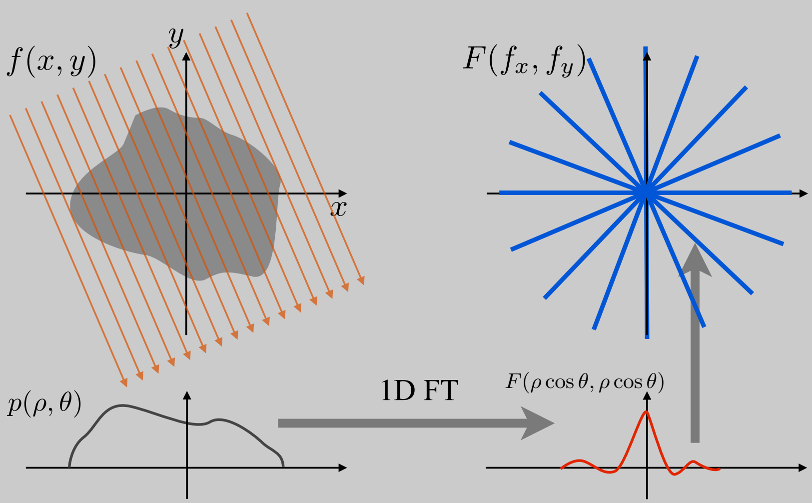


Projection Slice Theorem (Bracewell)

Proof: show only for $\theta = 0$ $\Rightarrow$ generalize to arbitrary θ

$$p(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$F(f_x, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(f_x x + f_y y)}\,dx\,dy =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi f_x x}\,dx\,dy =$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right] e^{-j2\pi f_x x}\,dx =$$

$$= \int_{-\infty}^{\infty} p(x)\, e^{-j2\pi f_x x}\,dx = \mathcal{F}\{p(x)\}$$

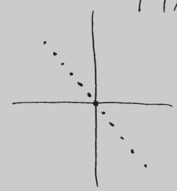
Discrete Reconstruction

DISCRETE RECONSTRUCTION

$$P_{\theta_m}[\rho_n] \xrightarrow{\text{1D DFT}} P_{\theta_m}[\rho_k]$$

$0-[N-1]$ $\quad$ $[0-N-1]$

2D DTFT $F(k_x, k_y)$



$$k_x = \frac{\omega_x}{2\pi} = \frac{\rho_k}{N}\cos\theta_m$$

$$k_y = \frac{\omega_y}{2\pi} = \frac{\rho_k}{N}\sin\theta_m$$

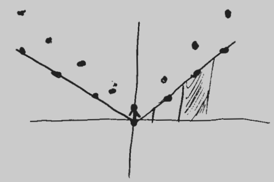
$$F[n,m] = \iint F(k_x,k_y)\, e^{-2\pi j(k_x n + k_y m)}\, dk_x\, dk_y =$$

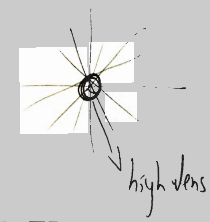
$$= \iint F(\rho,\theta)\, e^{-2\pi j(\rho\cos\theta n + \rho\sin\theta m)}\, |\rho|\, d\rho\, d\theta$$

Discretization

DISCRETIZE:

$$\simeq \sum\sum F\left(\frac{\rho_k}{N},\theta_m\right) e^{-2\pi j\left(\frac{\rho_k}{N}\cos\theta_m n + \frac{\rho_k}{N}\sin\theta_m m\right)} \left|\frac{\rho_k}{N}\right|$$



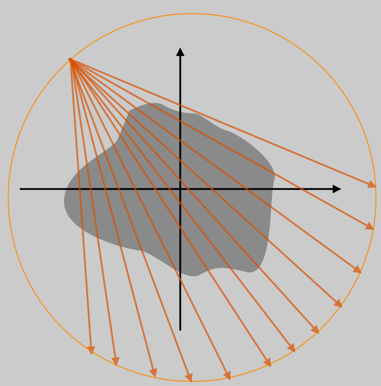


Called a "rho" filter

Fan Beam CT

- Single Source
- Many detectors

- How to reconstruct?

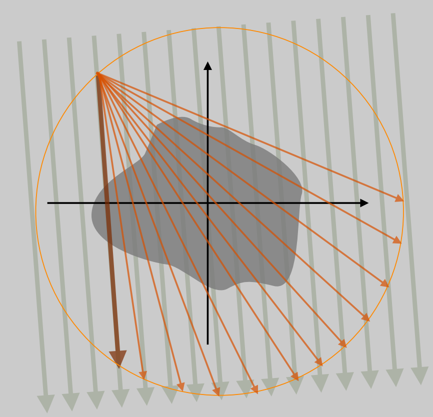


Fan Beam CT

- Single Source
- Many detectors

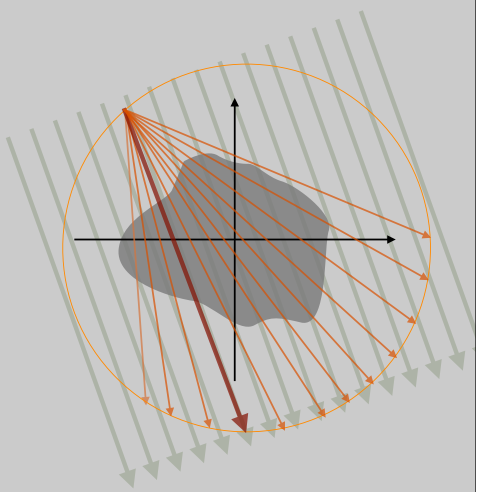
- How to reconstruct?

- Re-binning!



Fan Beam CT

- Single Source
- Many detectors
- How to reconstruct?
- Re-binning!

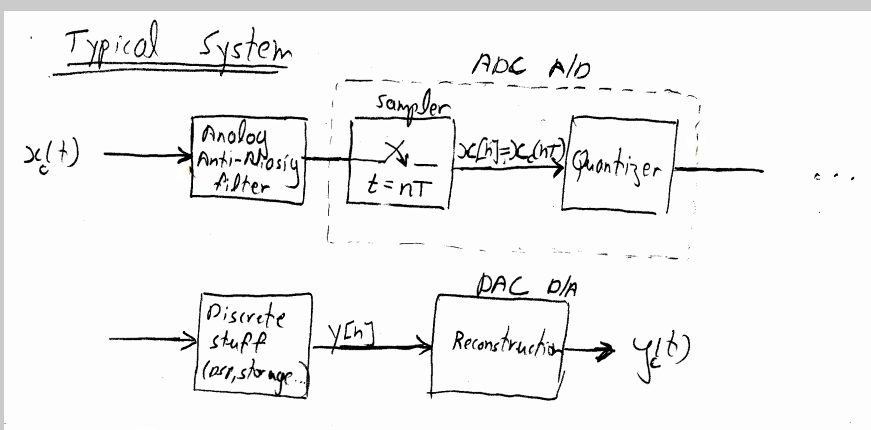


Sampling of Continuous Time Signals (Ch.4)

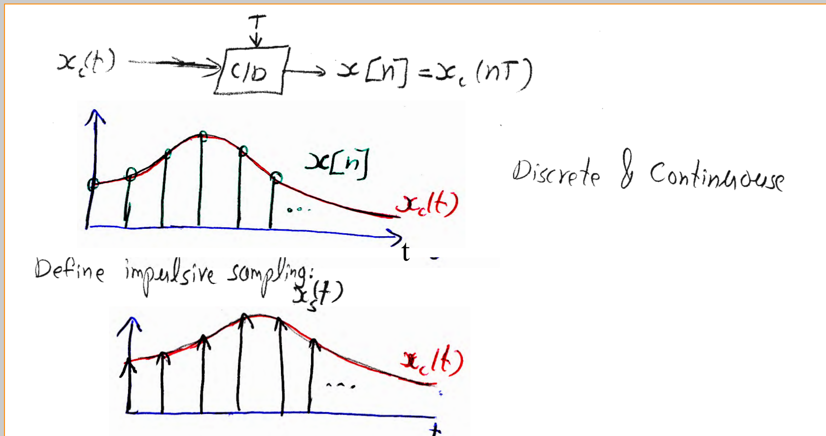
- Sampling:
  - –Conversion from C.T (not quantized) into D.T (usually quantized)
- Reconstruction
  - –D.T (quantized) to C.T
- Why?
  - –Digital storage (audio, images, videos)
  - –Digital communications (fiber optics, cellular...)
  - –DSP (compression, correction, restoration)
  - –Digital synthesis (speech, graphics)

Sampling of C.T. Signals

- Typical System:



Ideal Sampling Model



$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Not physical: used for modeling & derivations

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- How is x[n] related to xs(t) in freq. domain?

Frequency Domain Analysis

How is $x[n]$ related to $x_s(t)$ in freq. domain?

$x_s(t)$: C.T. $\qquad X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$

$x[n]$: discrete $\qquad X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \quad \updownarrow \ \omega = \Omega T$

$$\boxed{X(e^{j\omega}) = X_s(j\Omega)\Big|_{\Omega = \frac{\omega}{T}}} \qquad \boxed{X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega = \Omega T}}$$

Frequency Domain Analysis

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Frequency Domain Analysis

- How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \underbrace{\sum_n \delta(t - nT)}_{\triangleq s(t)}$$

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notation Break

recall $Ш(t) = \sum_n \delta(t-n)$

$$s(t) = \sum_n \delta(t - nT) = \sum_n \delta\left(T\left(\frac{t}{T} - n\right)\right) =$$

recall $\delta(at) = \frac{1}{|a|}\delta(t)$

$$= \frac{1}{T} \sum_n \delta\left(\frac{t}{T} - n\right) = \frac{1}{T} Ш\left(\frac{t}{T}\right)$$

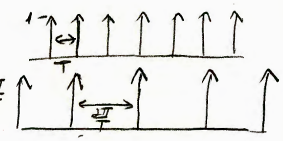
Frequency Domain Analysis

- How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \underbrace{\sum_n \delta(t - nT)}_{\triangleq s(t)}$$

$$s(t) \Longleftrightarrow S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\Big(\Omega - \underbrace{\tfrac{2\pi}{T}}_{=\Omega_s} k\Big)$$



Frequency Domain Analysis

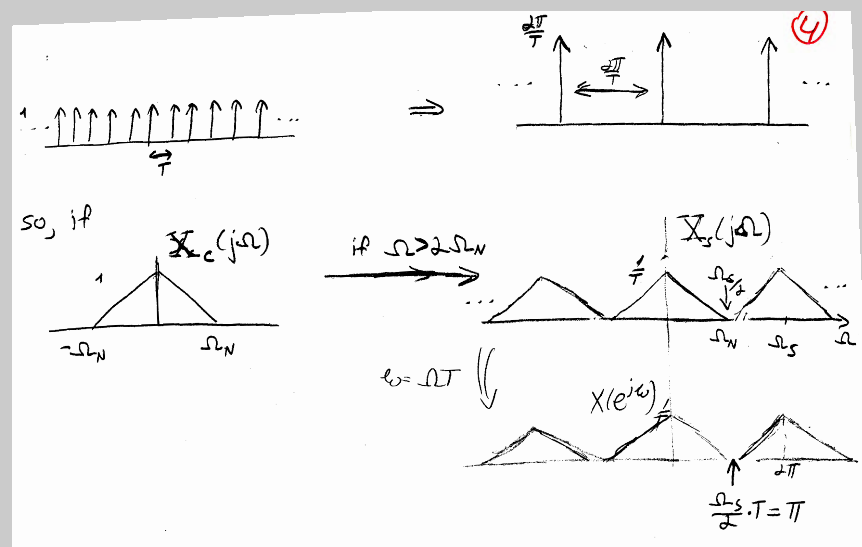
since

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\omega) =$$

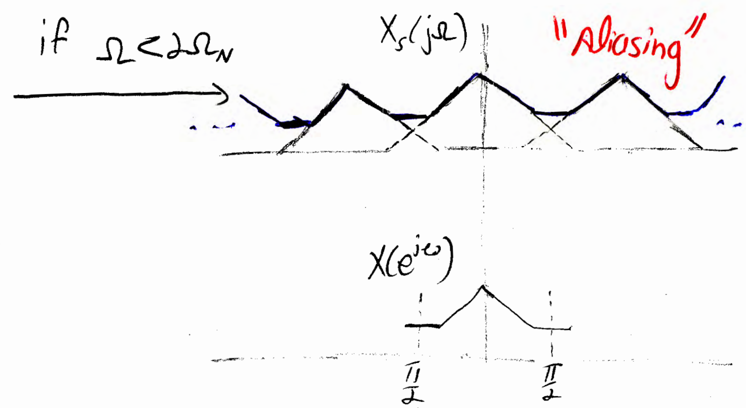
$$\boxed{X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}}$$

- Replication of X_c !

Frequency Domain Analysis



Aliasing



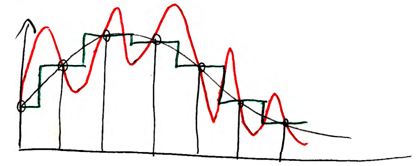
Reconstruction of Bandlimited Signals

Nyquist Sampling Thm: Suppose $x_c(t)$ is bandlimited,

$X_c(j\Omega) = 0 \quad \forall \ |\Omega| \geq \Omega_N$.

If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$

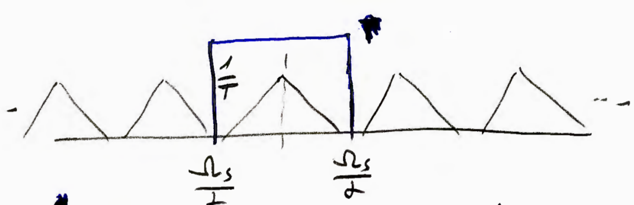
band limitedness is key to uniqueness



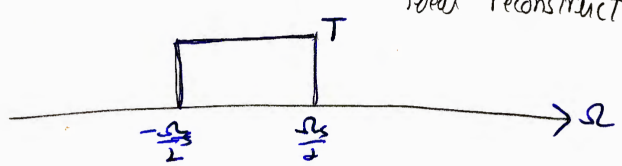
multiple signals can yield the same samples but only one is bandlimited by $\Omega_s/2$

Reconstruction in Frequency Domain

$x_s(t) \rightarrow \boxed{H_r(j\Omega)} \rightarrow x_r(t)$



ideal reconstruction filter



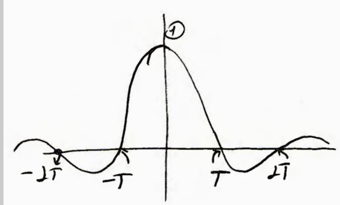
Reconstruction in Time Domain

$$h_r(t) = \frac{1}{2\pi}\int_{-\frac{\Omega_s}{2}}^{\frac{\Omega_s}{2}} T e^{j\Omega t}\, d\Omega = \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} =$$

$$= \frac{T}{\pi t}\,\frac{e^{j\frac{\Omega_s t}{2}} - e^{-j\frac{\Omega_s}{2}t}}{2j} =$$

$$= \frac{T}{\pi t}\sin\left(\frac{\Omega_s}{2}t\right) = \frac{T}{\pi t}\sin\left(\frac{\pi}{T}t\right)$$

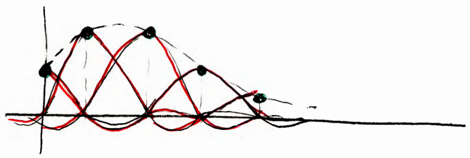
$$= \mathrm{sinc}\left(\frac{t}{T}\right)$$



Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t) =$$

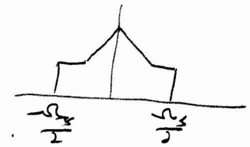
$$= \sum_n x[n] h_r(t-nT)$$



The sum of sinc's gives $x_r(t) \longrightarrow$ Unique signal bandlimited by $\Omega_s/2$

Aliasing

If $\Omega_N > \frac{\Omega_s}{2}$, $x_r(t)$ is an aliased version of $x_c(t)$



$$X_r(j\Omega) = \begin{cases} T X_s(j\Omega) & \text{if } |\Omega| < \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$