# EE123 Digital Signal Processing

Lecture 13



## Announcements

- Midterm this Friday in class
  - Open everything
- All Homework solutions posted
- No homework this week
- Old exams posted



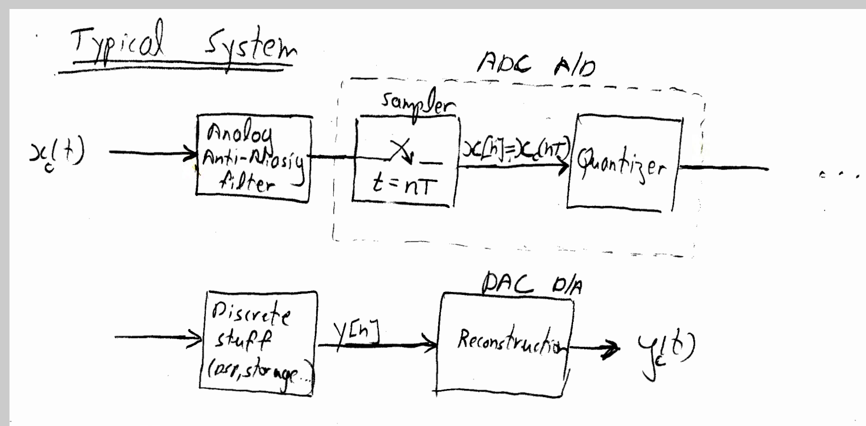
## Sampling of Continuous Time Signals (Ch.4)

- Sampling:
  - Conversion from C.T (not quantized) into D.T (usually quantized)
- Reconstruction
  - D.T (quantized) to C.T
- Why?
  - Digital storage (audio, images, videos)
  - Digital communications (fiber optics, cellular...)
  - DSP (compression, correction, restoration)
  - Digital synthesis (speech, graphics)

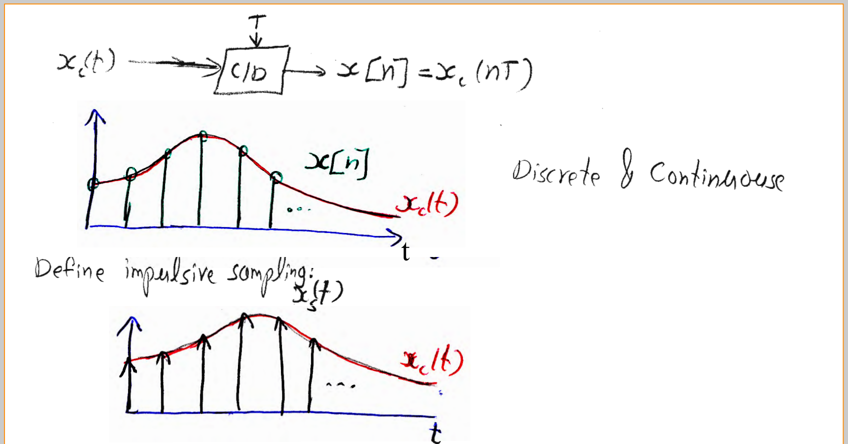


## Sampling of C.T. Signals

- Typical System:



## Ideal Sampling Model




$$x_s(t) = \cdots + x_c(0)\delta(t) + x_c(T)\delta(t-T) + \cdots$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$



## Ideal Sampling Model

$$x_s(t) = x_c \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

- Not physical: used for modeling & derivations

$$x[n] \leftrightarrow x_s(t) \leftrightarrow x_c(t)$$

- How is x[n] related to $x_s(t)$ in freq. domain?



## Frequency Domain Analysis

How is $x[n]$ related to $x_s(t)$ in Freq. domain?

$x_s(t)$: C.T. $\quad X_s(j\Omega) = \sum_n x_c(nT) e^{-j\Omega nT}$

$x[n]$: discrete $\quad X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} \quad \updownarrow \omega = \Omega T$

$$X(e^{j\omega}) = X_s(j\Omega)\Big|_{\Omega = \frac{\omega}{T}} \qquad X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega = \Omega T}$$



## Frequency Domain Analysis

- How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \underbrace{\sum_n \delta(t-nT)}_{\triangleq s(t)}$$

## Frequency Domain Analysis

- How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \underbrace{\sum_n \delta(t - nT)}_{\triangleq s(t)}$$

notation Break

recall $Ш(t) = \sum_n \delta(t-n)$

$$s(t) = \sum_n \delta(t-nT) = \sum_n \delta\left(T\left(\frac{t}{T}-n\right)\right) =$$

recall $\delta(at) = \frac{1}{|a|}\delta(t)$

$$= \frac{1}{T}\sum_n \delta\left(\frac{t}{T}-n\right) = \frac{1}{T} Ш\left(\frac{t}{T}\right)$$



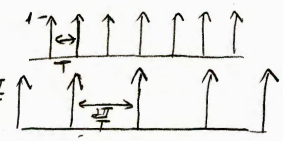
## Frequency Domain Analysis

- How is $x_s(t)$ related to $x_c(t)$?

$$x_s(t) = x_c(t) \underbrace{\sum_n \delta(t - nT)}_{\triangleq s(t)}$$

$$s(t) \Longleftrightarrow S(j\Omega)$$

$$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi}{T}k\right), \quad \frac{2\pi}{T} = \Omega_s$$





## Frequency Domain Analysis

since

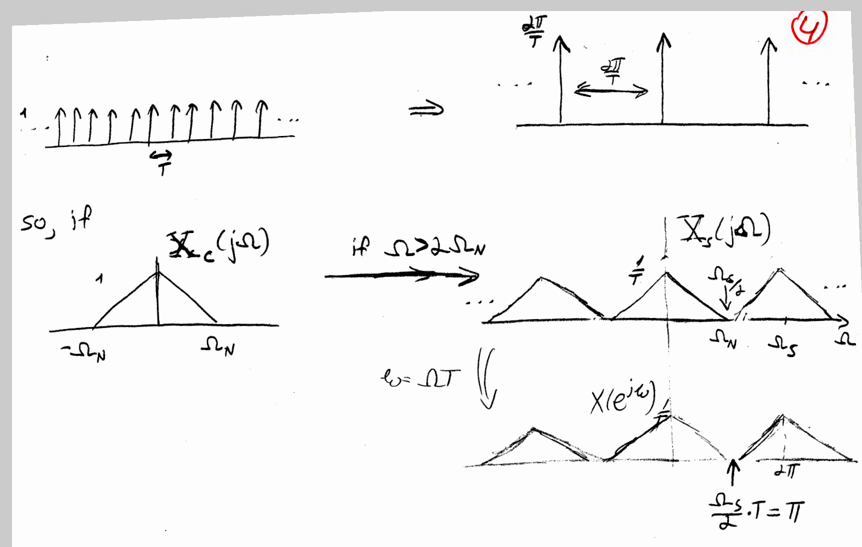
$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega) =$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \Big| \quad \Omega_s = \frac{2\pi}{T}$$

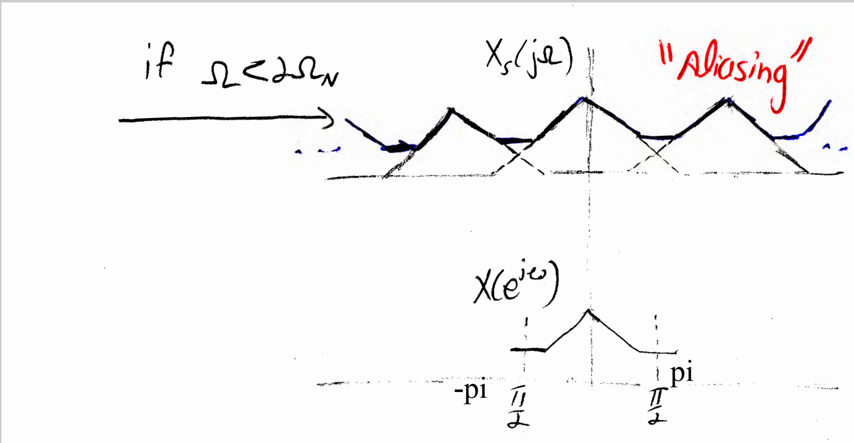
- Replication of $X_c$ !



## Frequency Domain Analysis



## Aliasing



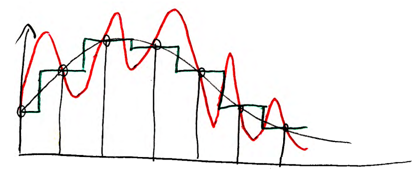



## Reconstruction of Bandlimited Signals

Nyquist Sampling Thm: Suppose $x_c(t)$ is bandlimited, $X_c(j\Omega) = 0 \;\; \forall \; |\Omega| \geq \Omega_N$.

If $\Omega_s \geq 2\Omega_N$, then $x_c(t)$ can be uniquely determined from its samples $x[n] = x_c(nT)$

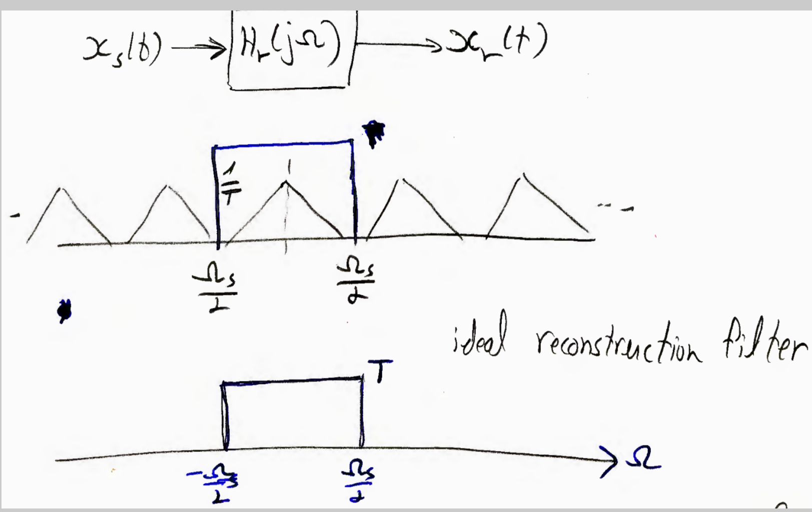
band limitedness is key to uniqueness



multiple signals can yield the same samples but only one is bandlimited by $\Omega_s/2$



## Reconstruction in Frequency Domain





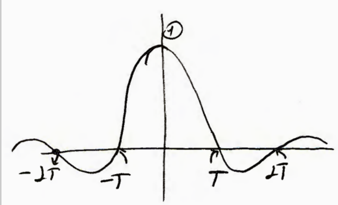

## Reconstruction in Time Domain

$$h_r(t) = \frac{1}{2\pi}\int_{-\frac{\Omega_s}{2}}^{\frac{\Omega_s}{2}} T e^{j\Omega t}\, d\Omega = \frac{T}{2\pi}\frac{1}{jt} e^{j\Omega t}\Big|_{-\Omega_s/2}^{\Omega_s/2} =$$

$$= \frac{T}{\pi t}\,\frac{e^{j\frac{\Omega_s}{2}t} - e^{-j\frac{\Omega_s}{2}t}}{2j} =$$

$$= \frac{T}{\pi t}\sin\left(\frac{\Omega_s}{2}t\right) = \frac{T}{\pi t}\sin\left(\frac{\pi}{T}t\right)$$

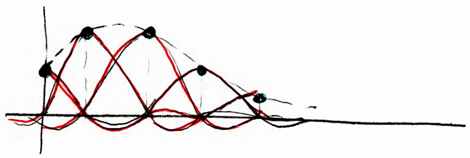
$$= \mathrm{sinc}\left(\frac{t}{T}\right)$$



## Reconstruction in Time Domain

$$x_r(t) = x_s(t) * h_r(t) = \left(\sum_n x[n]\delta(t-nT)\right) * h_r(t) =$$

$$= \sum_n x[n] h_r(t-nT)$$

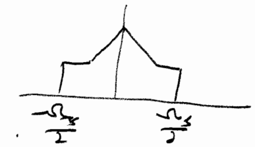


The sum of sinc's gives $x_r(t)$ → Unique signal bandlimited by $\Omega_s/2$



## Aliasing

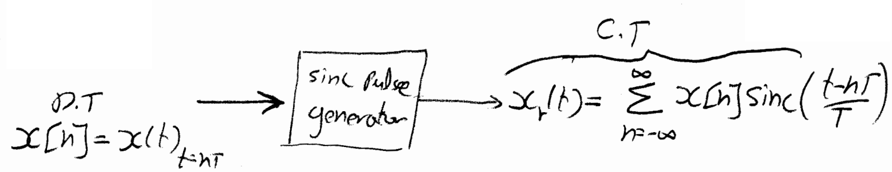
If $\Omega_N > \Omega_s/2$, $x_r(t)$ is an aliased version of $x_c(t)$




$$X_r(j\Omega) = \begin{cases} T X_s(j\Omega) & \text{if } |\Omega| < \Omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$



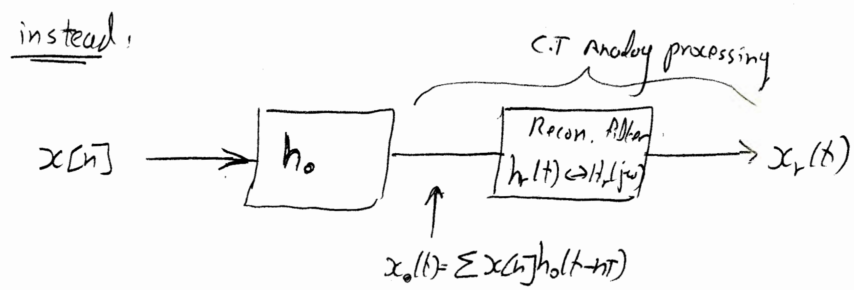
## Practical ADC (Ch. 4.8.4)




- Scaled train of sinc pulses
- Difficult to generate sinc ⇒ Too long!

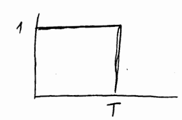


## Practical ADC




- $h_0$ is finite length pulse ⇒ easy to implement

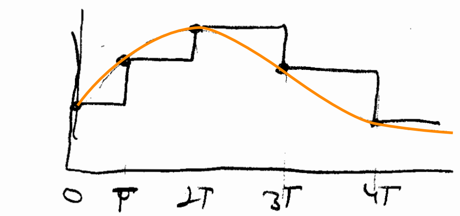
for example:




$$H_0(j\Omega) = T e^{-j\pi\frac{\Omega}{\Omega_s}} \text{sinc}\left(\frac{\Omega}{\Omega_s}\right)$$

## Practical ADC

Output zero-order-hold




$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]\, h_0(t-nT) = h_0(t) * x_s(t)$$

taking FT:

$$X_0(j\Omega) = H_0(j\Omega) \cdot X_s(j\Omega) =$$

$$= H_0(j\Omega) \cdot \frac{1}{T} \sum_k X(j(\Omega - k\Omega_s))$$



## Practical ADC

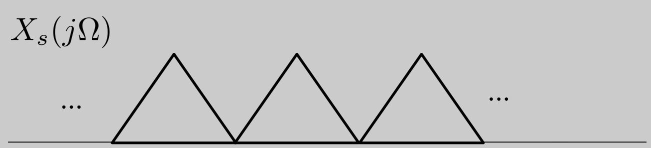
Output of recon filter:

$$X_r(j\Omega) = H_r(j\Omega) \cdot H_0(j\Omega) \cdot X_s(j\Omega) =$$

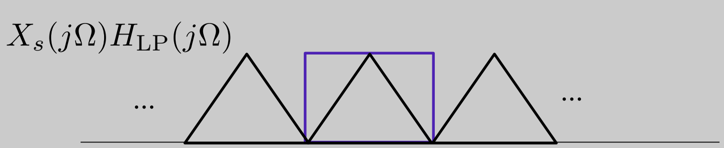
$$= \underbrace{H_r(j\Omega)}_{\text{recon filter}} \cdot \underbrace{T e^{-j\pi\frac{\Omega}{\Omega_s}} \operatorname{sinc}\left(\frac{\Omega}{\Omega_s}\right)}_{\text{from zero order hold}} \cdot \underbrace{\frac{1}{T}\sum X(j(\Omega - k\Omega_s))}_{\text{shifted copies from sampling}}$$



## Practical ADC

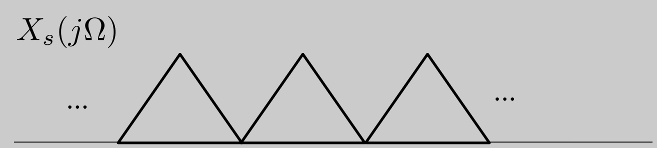



Ideally:

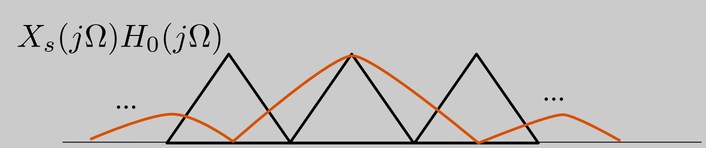




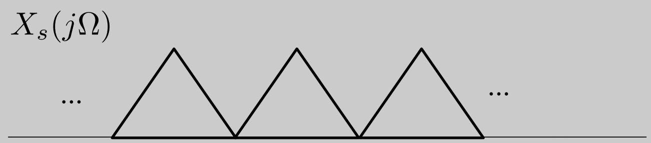

## Practical ADC



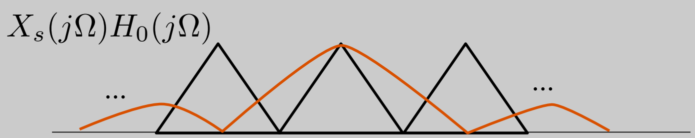

Practically:

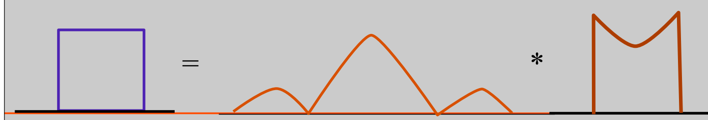


## Practical ADC



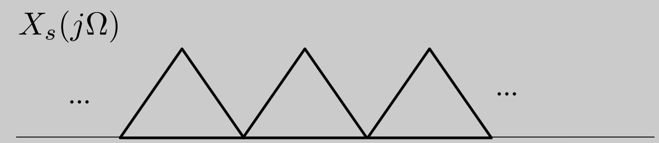

Practically:



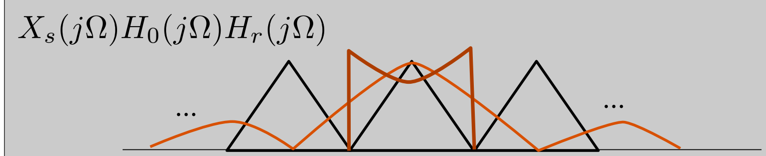






## Practical ADC

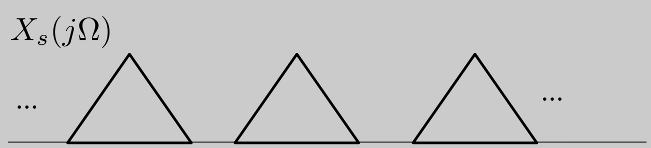



Practically:

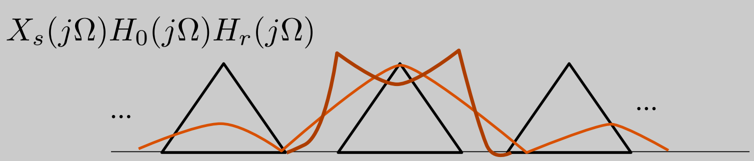





## Easier Implementation with Digital upsampling




Practically:






## Easier Implementation with Digital upsampling