Lecture Nov. 6th 2011

OPTIMAL FILTER DESIGN

Would like to design a filter $h[n]$ with $H(e^{j\omega})$ that approximates $H_d(e^{j\omega})$ with some optimality criteria... or stisfy specs.

(-) Least Squares: $\underset{\omega \in [-\pi,\pi]}{\text{minimize}} \; \| H(e^{j\omega}) - H_d(e^{j\omega}) \|^2$

$$\int_{-\pi}^{\pi} |H(e^{j\omega}) - H_d(e^{j\omega})|^2 \, d\omega$$

$\boxed{\text{Variation}}$ ~~minimize~~

Weighted Least-Squares:

$$\text{minimize} \int_{-\pi}^{\pi} W(\omega) |H(e^{j\omega}) - H_d(e^{j\omega})|^2 \, d\omega$$

matlab: firls

(-) Chebychev design (min max)

$$\text{minimize} \max_{\omega \in [-\pi,\pi]} |H(e^{j\omega}) - H_d(e^{j\omega})|$$

Parks-McClellan algorithm → equiripple

matlab: firpm, cfirpm

<figure (ls. vs. PM)>

Design through Convex Optimization

Idea: sample/discretize the frequency response.

$$H(e^{j\omega}) \Rightarrow H(e^{j\omega_k})$$

(•) sample points are fixed $\omega_k = k\frac{\pi}{P}$

$$0 \leq \omega_1 < \ldots < \omega_P \leq \pi$$

(•) $P \gg M+1$ (common rule $P = 15M$)

(•) yeilds a (Good) approximation of the original problem.

Example: Least-Squares

$$\underset{h}{\operatorname{argmin}} \; \| Ah - b \|^2$$

assume $h[n]$ causal $\Rightarrow h[n] = 0 \;\; \forall \;\; n<0 \;\&\; n>M$

$$A = \begin{bmatrix} 1 & e^{-j\omega_0} & \cdots & e^{-j\omega_0 M} \\ 1 & e^{-j\omega_1} & \cdots & e^{-j\omega_1 M} \\ \vdots & & \ddots & \\ 1 & e^{-j\omega_p} & \cdots & e^{-j\omega_p M} \end{bmatrix}$$

$$b = \begin{bmatrix} H_d(e^{j\omega_0}) \\ \vdots \\ H_d(e^{j\omega_p}) \end{bmatrix}$$

$$h = (A^T A)^{-1} A^T b$$

result will be generally non symmetric and complex valued.

need to do something for real and linear phase.

design of linear phase LP filter

(•) suppose M is even (odd filter)

if $h[n]$ symetric about midpoint then

$$H(e^{j\omega}) = e^{-j\frac{M}{2}\omega}\tilde{H}(e^{j\omega}) \quad \text{where } \tilde{H}(e^{j\omega}) \in \text{Real}$$

Proof: symmetric $\Rightarrow$ Real.

so, shift $h[n] \Rightarrow \tilde{h}[n] = h[n+\frac{M}{2}] \Rightarrow \tilde{H}(e^{j\omega}) \in \text{Real}$.

$$\tilde{H}(e^{j\omega}) = H(e^{j\omega})e^{+j\frac{M}{2}\omega} \Rightarrow H(e^{j\omega}) = \tilde{H}(e^{j\omega})e^{-j\frac{M}{2}\omega}$$

$$\tilde{H}(e^{j\omega}) = \tilde{h}[0] + \tilde{h}[1]e^{-j\omega} + \tilde{h}[-1]e^{+j\omega} + \tilde{h}[2]e^{-j2\omega} + \tilde{h}[-2]e^{+j2\omega} + \ldots$$

$$= \tilde{h}[0] + 2\cos(j\omega) + 2\cos(2j\omega) + \ldots$$

Zero-phase

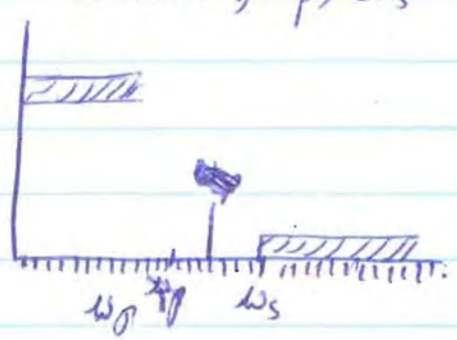
modifications

(1) Design for zero-phase first and then delay.

(2) $\tilde{h}[n]$ real $\longrightarrow$ $\tilde{H}(e^{j\omega})$ symmetric only fit to $\omega \in [0, \pi]$

(3) $\tilde{H}(e^{j\omega_k}) = \tilde{h}[0] + 2\sum_{n=1}^{\frac{M}{2}} \cos(\omega_k n)$

Task: Given M, ω_p, ω_s find best LS filter:



$$A = \begin{array}{c} \left[\begin{matrix} 1 & 2\cos(\omega_0) & 2\cos(2\omega_0) & \cdots & 2\cos(\frac{M}{2}\omega_0) \\ \vdots & & & & \vdots \\ 1 & & & & 2\cos(\frac{M}{2}\omega_p) \end{matrix}\right] \end{array} \begin{array}{l} \} P.B \\ \\ \} S.B \end{array}$$

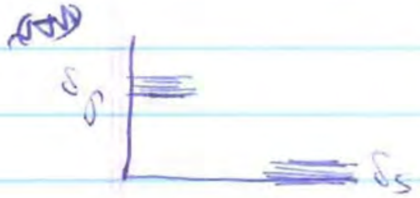
$$b = \left[\begin{matrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix}\right] \begin{array}{l} \} PB \\ \\ \\ \} SB \end{array}$$

$$\boxed{\begin{bmatrix} \tilde{h}[0] \\ \tilde{h}[1] \\ \vdots \\ \tilde{h}[\frac{M}{2}] \end{bmatrix} = (A^T A)^{-1} A^T b}$$

Extention:

(-) LS has no preference for PB or SB. Use weighting to change ratio

$$\tilde{h} = (A^T W^2 A)^{-1} W^2 A^T b$$



$$W = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \frac{\delta_p}{\delta_s} & & \\ & & & & & & \ddots & \\ & & & & & & & \frac{\delta_p}{\delta_s} \end{bmatrix}$$

(-) To minimiz transition $\omega_s - \omega_p$ use bisection