Chapter 5. Transform Analysis of LTI Systems

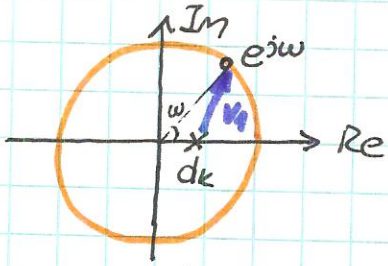
How filters modify spectrum – prep for design

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Geometric Interpretation of the Magnitude Response:

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \cdot \frac{\prod |1 - c_k e^{-j\omega}|}{\prod |1 - d_k e^{-j\omega}|}$$

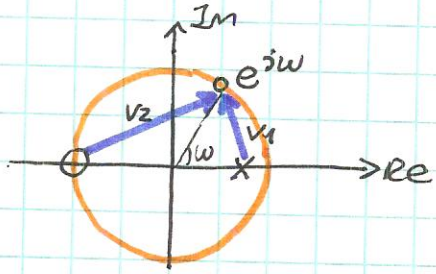
$$|1 - d_k e^{-j\omega}| = |e^{j\omega} - d_k| = |v_1|$$



Example: $H(z) = 0.05 \frac{1 + z^{-1}}{1 - 0.9 z^{-1}}$

$H(1) = 1$

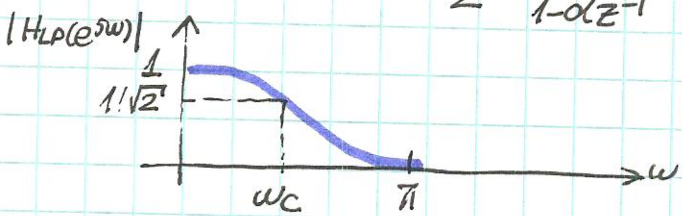
$$|H(e^{j\omega})| = 0.05 \frac{|v_2|}{|v_1|}$$



zero annihilates, pole emphasizes frequency close to it.

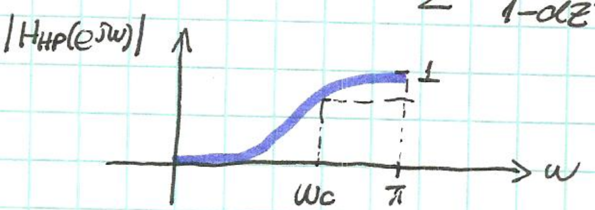
Simple Filters:

Low-Pass: $H_{LP}(z) = \frac{1-\alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}}$ $|\alpha| < 1$ for stability



3 dB cutoff frequency ω_c related to α by: $\alpha = \frac{1 - \sin\omega_c}{\cos\omega_c}$ (Show this.)

High Pass: $H_{HP}(z) = \frac{1+\alpha}{2} \frac{1 - z^{-1}}{1 - \alpha z^{-1}}$ $H(1) = 0$ $H(-1) = H(e^{j\pi}) = 1$

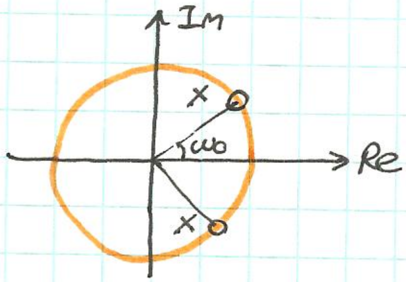


$\alpha = \frac{1 - \sin\omega_c}{\cos\omega_c}$ (same)

Band-Stop (Notch): at least second order

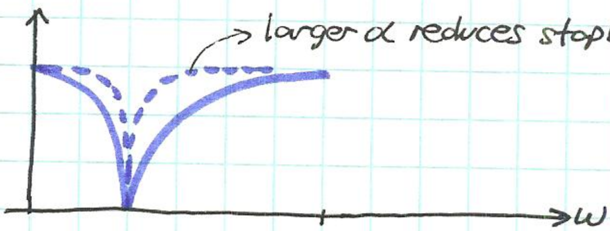
$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \qquad |\beta|<1,\ |\alpha|<1$$

Note: $1-2\beta z^{-1}+z^{-2} = (1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})$, $\cos\omega_0 = \beta$



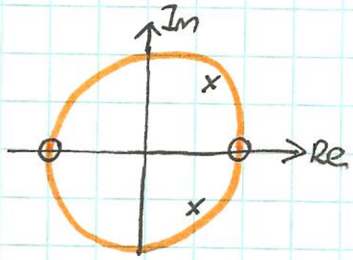
zeros on the unit circle, poles approach zeros as $\alpha \to 1$

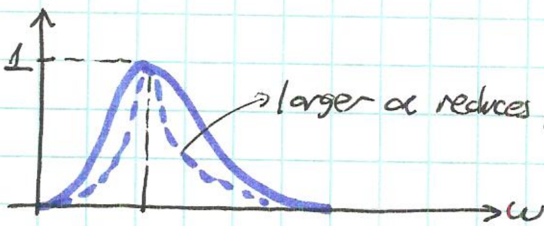
$$H_{BS}(\mp 1) = \frac{1+\alpha}{2} \frac{2 \pm 2\beta}{(1+\alpha)(1\pm\beta)} = 1$$



Band-Pass:

$$H_{BP}(z) = \frac{1-\alpha}{2} \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \qquad |\alpha|<1,\ |\beta|<1$$



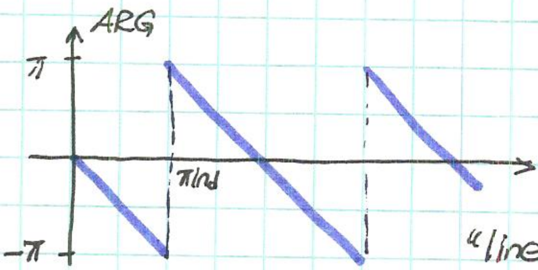


Phase Response: see Fig 5.1 for notation

Example: $H(e^{j\omega}) = e^{-j\omega n_d} \quad \leftrightarrow \quad h[n] = \delta[n-n_d]$

$|H(e^{j\omega})| = 1$

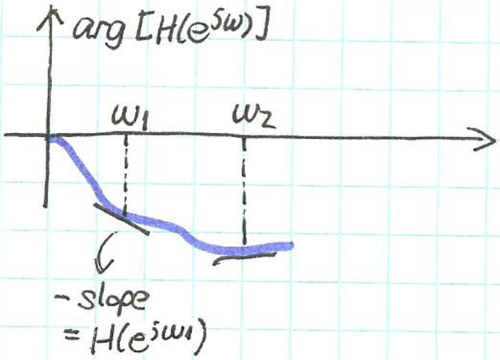
$\arg[H(e^{j\omega})] = -\omega n_d$



"linear phase"

Linear phase response is desirable: all frequency components delayed by the same amount.

<u>Group Delay:</u> $\operatorname{grd}[H(e^{j\omega})] \triangleq -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} \equiv n_d$ for linear phase



Figures 5.4, 5.5, 5.6

$$H(z) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \Rightarrow \arg[H(e^{j\omega})] = -\sum_{k=1}^{N}\arg[1-d_k e^{-j\omega}] + \sum_{k=1}^{M}\arg[1-c_k e^{-j\omega}]$$

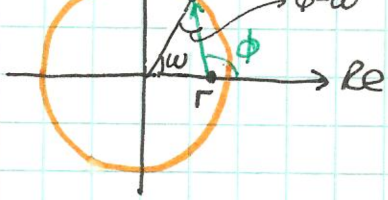
$$\operatorname{grd}[H(e^{j\omega})] = -\sum_{k=1}^{N}\operatorname{grd}[1-d_k e^{-j\omega}] + \sum_{k=1}^{M}\operatorname{grd}[1-c_k e^{-j\omega}]$$

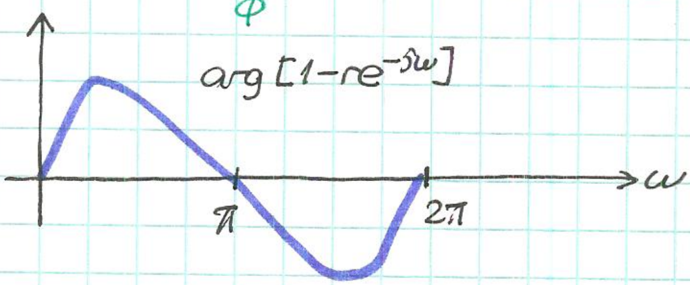
$$\arg[1-\underbrace{re^{j\theta}}_{c_k \text{ or } d_k}e^{-j\omega}] = \tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)$$

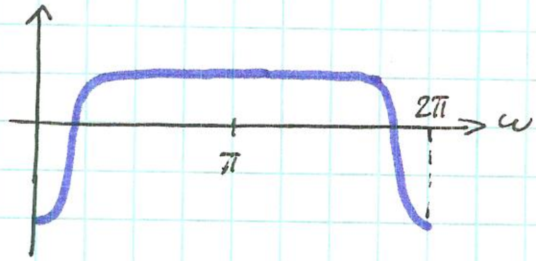
$$\operatorname{grd}[1-re^{j\theta}e^{-j\omega}] = \frac{r^2-r\cos(\omega-\theta)}{|1-re^{j\theta}e^{-j\omega}|^2}$$

<u>Geometric Interpretation (for $\theta=0$)</u>

$$\arg[1-re^{-j\omega}] = \arg[(e^{j\omega}-r)e^{-j\omega}] = \underbrace{\arg[e^{j\omega}-r]}_{\phi} - \underbrace{\arg[e^{j\omega}]}_{\omega}$$







$\theta \neq 0 \Rightarrow$ shift to the right by θ

Figure 5.11 : variation with r (θ fixed at π)

* Poles increase magnitude, but introduce phase lag and group delay.
* Zeros do the opposite.
* These effects are more marked when $r \to 1$.

Example: 2nd order IIR with complex poles

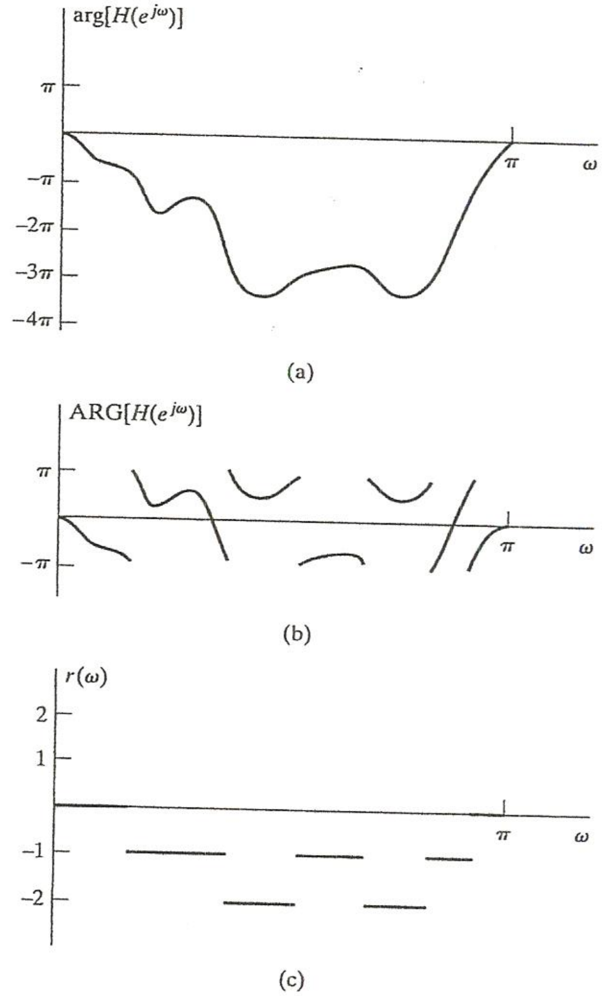
$$H(z) = \frac{1}{(1-re^{j\theta}z^{-1})(1-re^{-j\theta}z^{-1})}$$

Figure 5.13

Example: 3rd order IIR

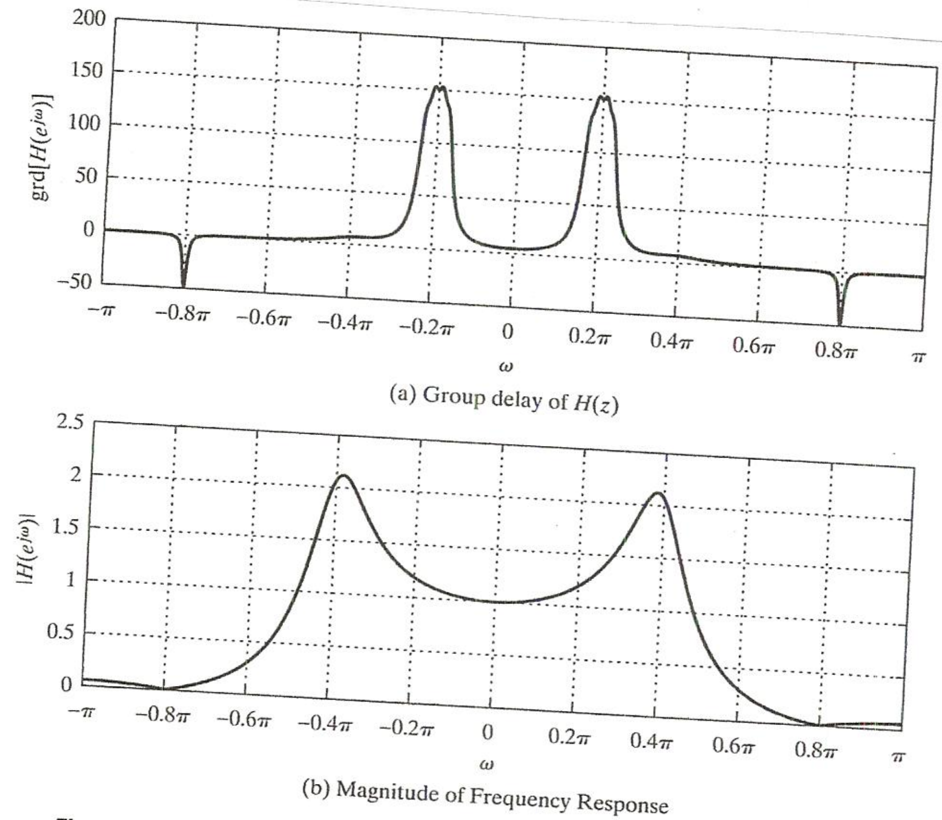
Figure 5.14 What type of filter?

Figure 5.15 Spikes in grd due to poles.



arg[$H(e^{j\omega})$]
π
$-\pi$
-2π
-3π
-4π
π
ω
(a)
ARG[$H(e^{j\omega})$]
π
$-\pi$
π
ω
(b)
$r(\omega)$
2
1
-1
-2
π
ω
(c)

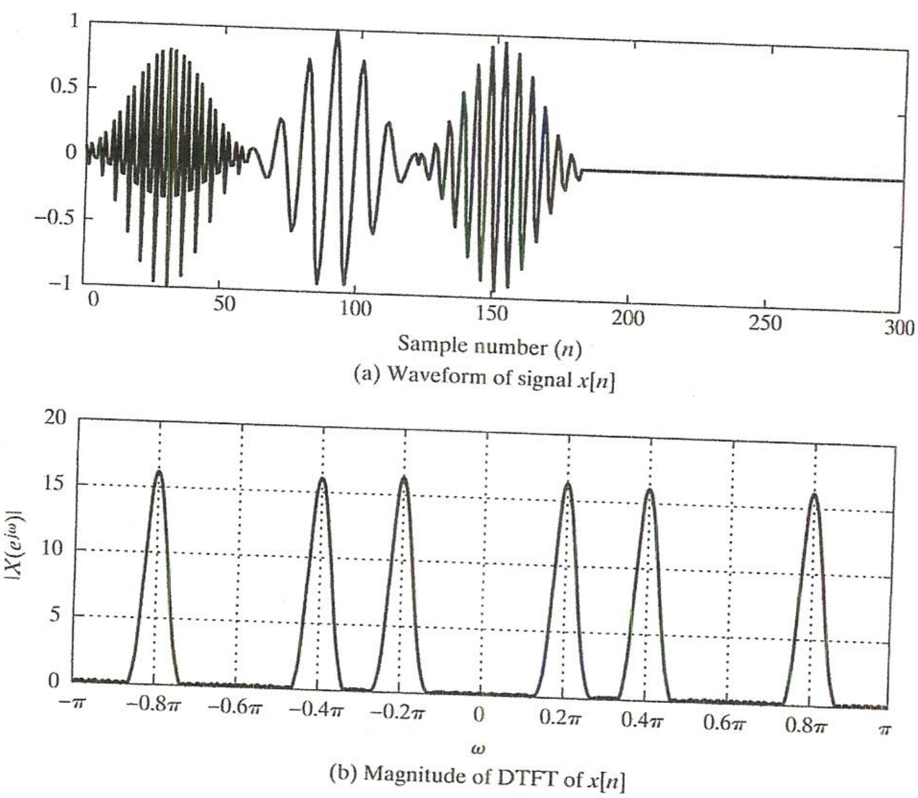
Figure 5.1.



(a) Group delay of $H(z)$

(b) Magnitude of Frequency Response

Figure 5.4 Frequency response of system in the example of Section 5.1.2; (a) Group delay function, $\text{grd}[H(e^{j\omega})]$, (b) Magnitude of frequency response, $|H(e^{j\omega})|$.



(a) Waveform of signal $x[n]$

(b) Magnitude of DTFT of $x[n]$

Figure 5.5 Input signal for example of Section 5.1.2; (a) Input signal $x[n]$, (b) Corresponding DTFT magnitude $|X(e^{j\omega})|$.

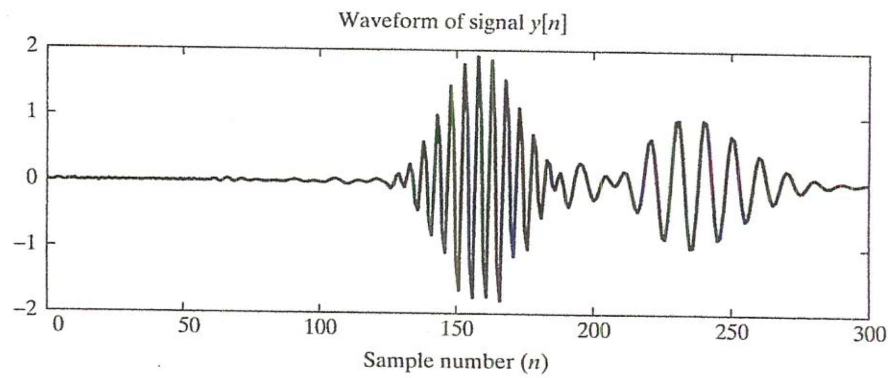


Figure 5.6 Output signal for the example of Section 5.1.2.

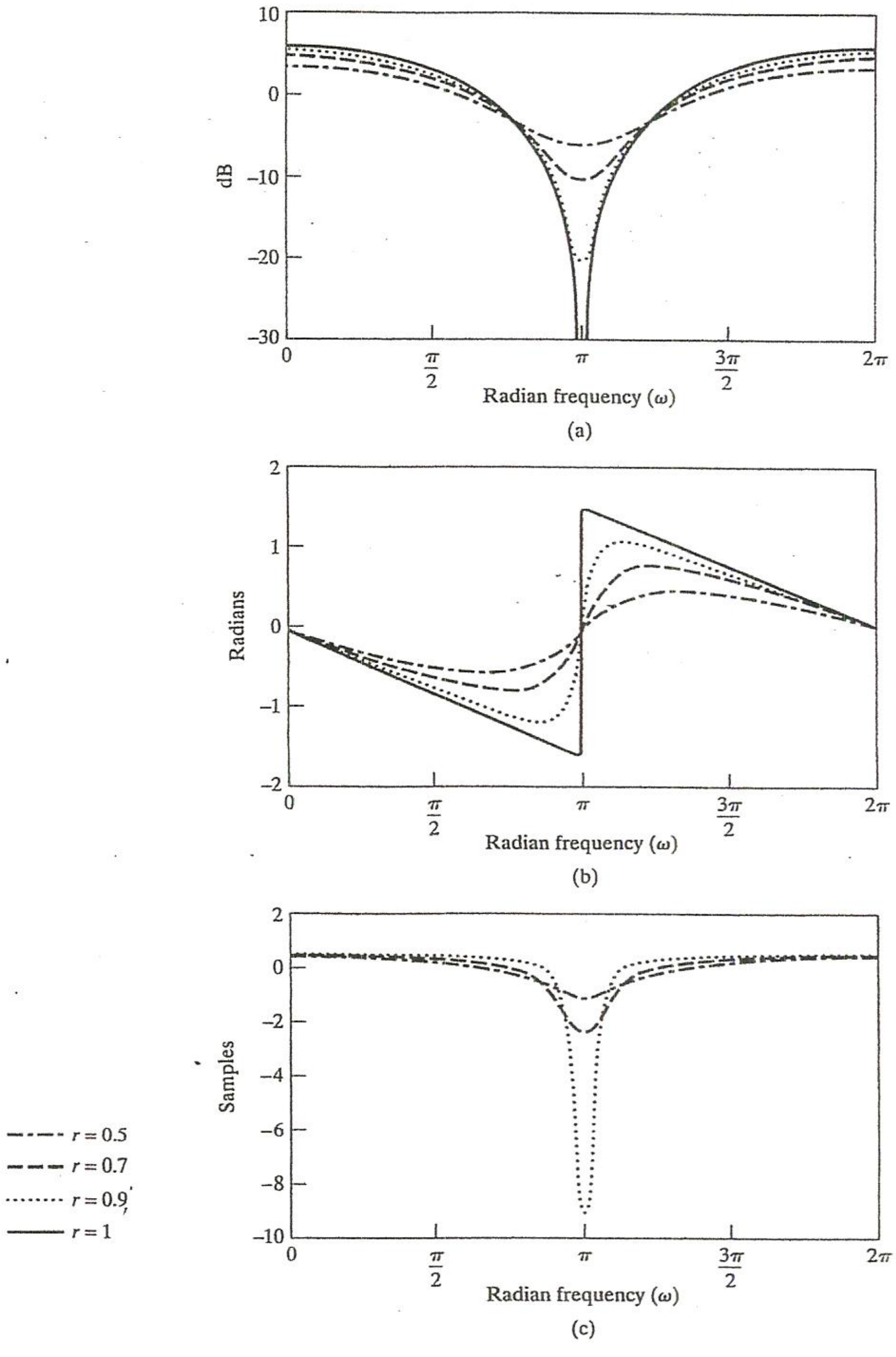


Figure 5.11 Frequency response for a single zero, with $\theta = \pi$, $r = 1$, 0.9, 0.7, and 0.5. (a) Log magnitude. (b) Phase. (c) Group delay for $r = 0.9$, 0.7, and 0.5.

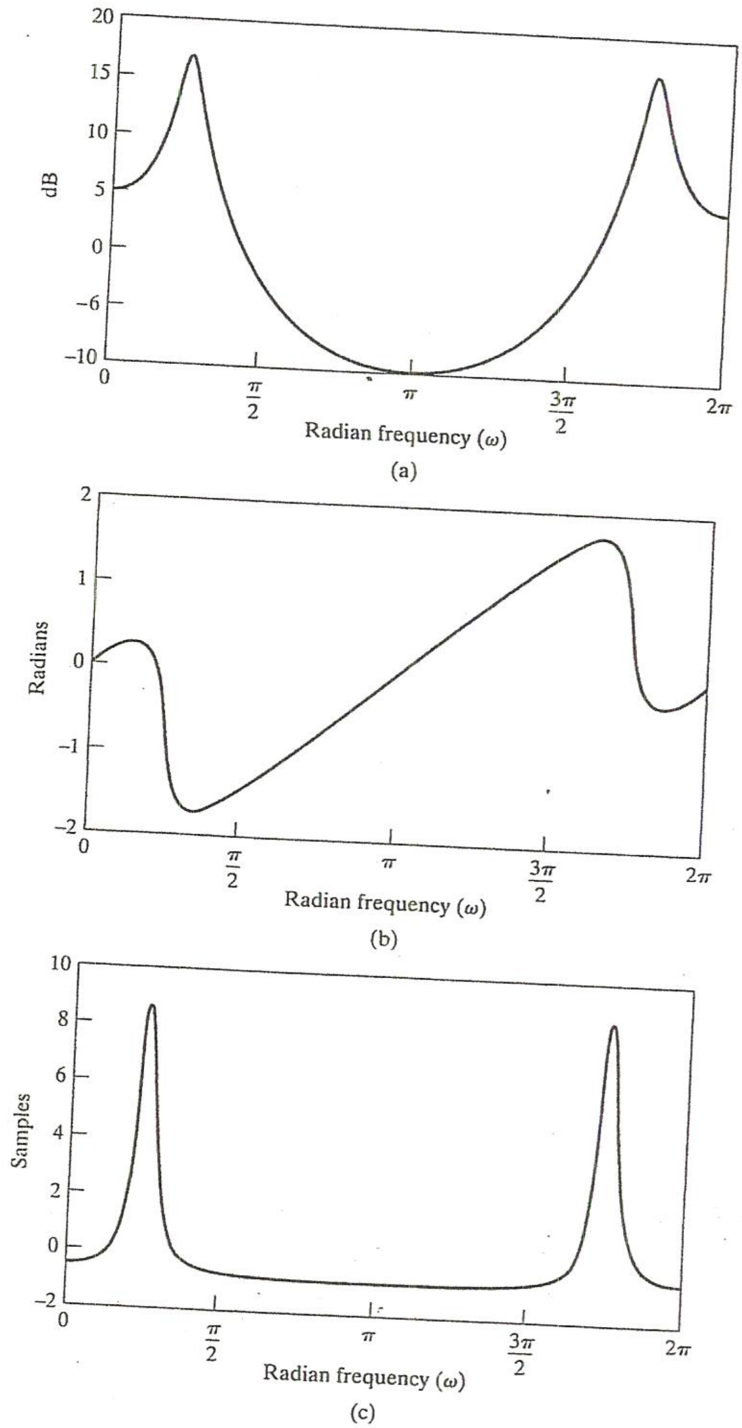

20
15
10
5
0
−6
−10
dB
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
Radian frequency (ω)
(a)
2
1
0
−1
−2
Radians
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
Radian frequency (ω)
(b)
10
8
6
4
2
0
−2
Samples
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
Radian frequency (ω)
(c)

Figure 5.13

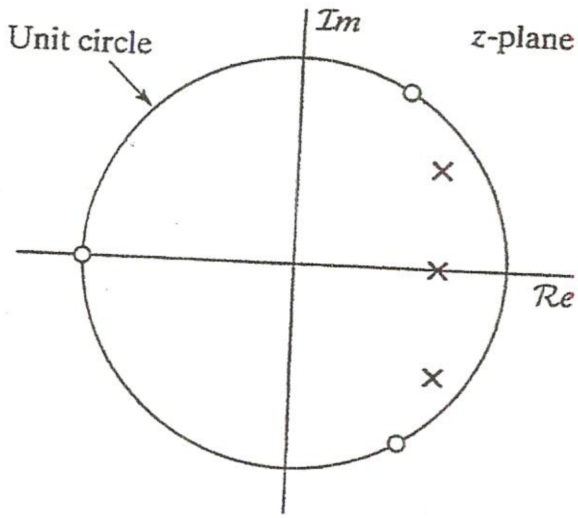
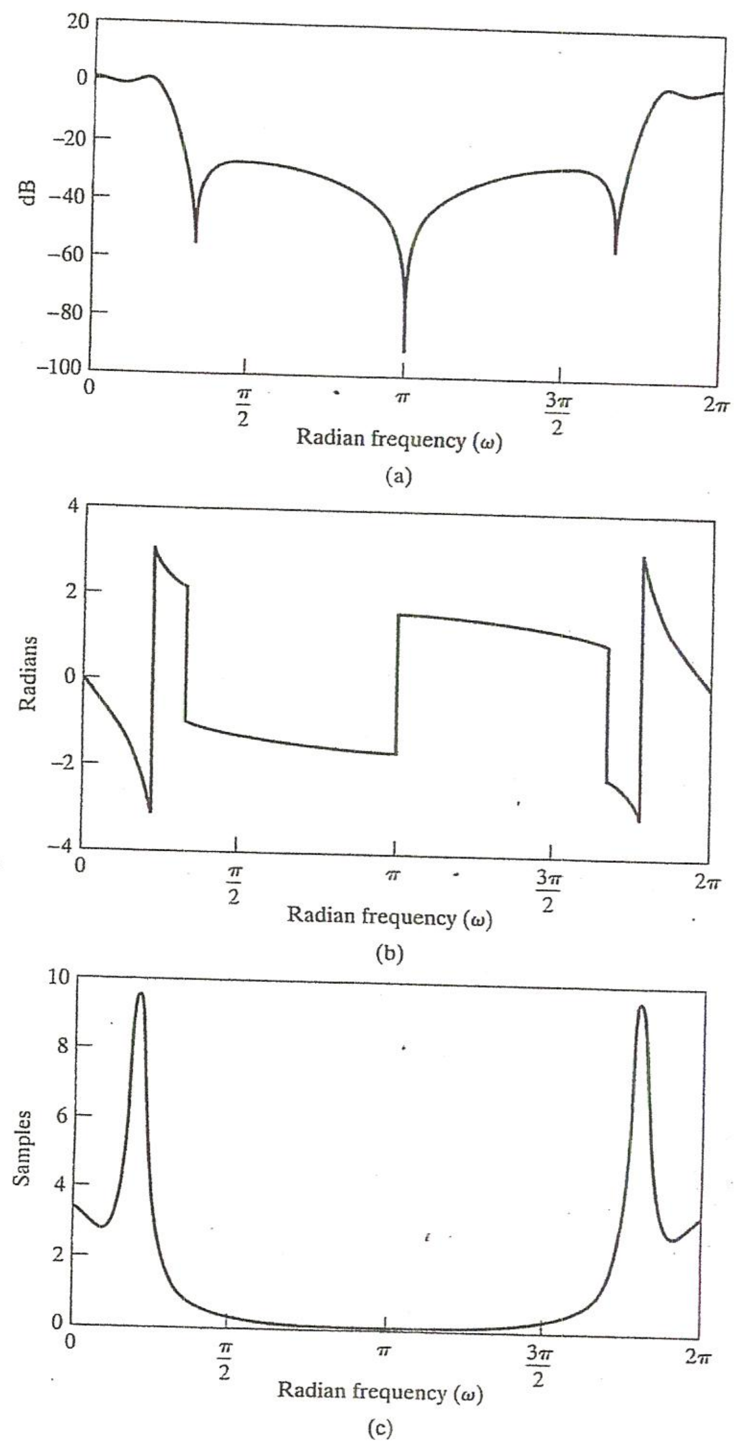


Figure 5.14 Pole–zero plot for the lowpass filter of Example 5



dB
20
0
−20
−40
−60
−80
−100
0
π/2
π
3π/2
2π
Radian frequency (ω)
(a)
Radians
4
2
0
−2
−4
0
π/2
π
3π/2
2π
Radian frequency (ω)
(b)
Samples
10
8
6
4
2
0
0
π/2
π
3π/2
2π
Radian frequency (ω)
(c)

Figure 5.15