1. Mechanical Projection

In $\mathbb{R}^n$, the projection of vector $\vec{a}$ onto vector $\vec{b}$ is defined as:

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\left\| \vec{b} \right\|^2} \vec{b} = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\left\| \vec{b} \right\|} \hat{b}$$

where $\hat{b}$ is the normalized $\vec{b}$, i.e., a unit vector with the same direction as $\vec{b}$.

(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ – that is, onto the x-axis. Graph these two vectors and the projection.

(b) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ – that is, onto the y-axis. Graph these two vectors and the projection.

(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Graph these two vectors and the projection.

(d) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.

(e) Project $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the span of the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ – that is, onto the x-y plane in $\mathbb{R}^3$.

(f) Project $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the plane described by $x + y + z = 1$.

(g) What is the geometric/physical interpretation of projection? Justify using the previous parts.

(h) For the first 4 parts, we looked at two different projections for each vector. For those cases, using only the projected vectors and the vectors we projected onto, do we have enough information to reconstruct the original vector?

(i) Given information about n projections of a vector in $\mathbb{R}^n$, when do we have enough information to reconstruct the original vector? Always? Never?

2. Least Squares: A Toy Example

Let's start off by solving a little example of least squares.

We're given the following system of equations:

$$\begin{bmatrix} 1 & 4 \\ 3 & 8 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) Why can we not solve for $\vec{x}$ exactly?

(b) Find $\vec{\hat{x}}$, the *least squares estimate* of $\vec{x}$, using the formula we derived in lecture.

(c) Now, let's try to find $\vec{\hat{x}}$ in a different (geometric) way. How might you do it?

3. Linearizing Different Problems

Notice that least squares can only be applied to linear systems. Suppose that we have a vector $\vec{x}$ and a vector $\vec{y}$, and $\vec{y}[n] = f(\vec{x}[n])$. We would like to approximate f using least squares, where f is not necessarily a linear function.

(a) Let's begin with a linear approximation. We want to find some a such that $y = ax$. Set this up as a least squares problem. What are the elements in the matrix $\mathbf{A}$?

(b) Let's add a DC offset to the problem. Suppose that $y = ax + b$. Set this up as a least squares problem. What are the elements in the matrix $\mathbf{A}$?

(c) Suppose that $y = ax^2 + bx + c$. Set this up as a least squares problem. What are the elements in the matrix $\mathbf{A}$?

(d) Suppose that $y = ae^{bx}$. Set this up as a least squares problem. What are the elements in the matrix $\mathbf{A}$?