1. **Orthogonal Matching Pursuit Lecture**

Orthogonal Matching Pursuit (OMP) algorithm:

Inputs:

- A set of m songs, each of length n: $\mathbf{S} = \{\vec{S}_0, \vec{S}_1, ..., \vec{S}_{m-1}\}$
- An n-dimensional received signal vector: $\vec{r}$
- The sparsity level k of the signal
- Some threshold, th. When the norm of the signal is below this value, the signal contains only noise.

Outputs:

- A set of songs that were identified, F, which will contain at most k elements
- A vector $\vec{x}$ containing song messages $(a_1, a_2, \ldots)$, which will be of length k or less
- An n-dimensional residual $\vec{y}$

Procedure:

- Initialize the following values: $\vec{y} = \vec{r}$, $j = 1$, k, $\mathbf{A} = \begin{bmatrix} \quad \end{bmatrix}$, $F = \{\emptyset\}$
- while $((j \leq k)$ and $(\| \vec{y} \| \geq th))$:
  (a) Cross-correlate $\vec{y}$ with the shifted versions of all songs. Find the song index i and the shifted version of the song, $\vec{S}_i^N$, with which the received signal has the highest correlation value.
  (b) Add i to the set of song indices F.
  (c) Column concatenate matrix $\mathbf{A}$ with the correctly shifted version of the song: $\mathbf{A} = \begin{bmatrix} \mathbf{A} & | & \vec{S}_i^N \end{bmatrix}$
  (d) Use least squares to obtain the message value: $\vec{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \vec{r}$
  (e) Update the residual value $\vec{y}$ by subtracting: $\vec{y} = \vec{r} - \mathbf{A}\vec{x}$
  (f) Update the counter: $j = j + 1$

2. Orthogonal Matching Pursuit

Let's work through an example of the OMP algorithm. Suppose that we have a vector $\vec{x} \in \mathbb{R}^4$. We take 3 measurements of it, $b_1 = \vec{m}_1^T \vec{x} = 4$, $b_2 = \vec{m}_2^T \vec{x} = 6$, and $b_3 = \vec{m}_3^T \vec{x} = 3$, where $\vec{m}_1$, $\vec{m}_2$ and $\vec{m}_3$ are some measurement vectors. We are given that $\vec{x}$ is sparse and only has 2 non-zero entries. In particular,

$$\mathbf{M}\vec{x} \approx \vec{b}$$

$$\begin{bmatrix} - & \vec{m}_1^T & - \\ - & \vec{m}_2^T & - \\ - & \vec{m}_3^T & - \end{bmatrix} \vec{x} \approx \vec{b}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

where exactly 2 of x_1 to x_4 are non-zero. Use Orthogonal Matching Pursuit to estimate x_1 to x_4.

(a) Why can we not solve for $\vec{x}$ directly?

(b) Why can we not apply the least squares process to obtain $\vec{x}$?

(c) Compute the inner product of every column with the $\vec{b}$ vector. Which column has the largest inner product? This will be the first column of the matrix $\mathbf{A}$. Why are we using the inner product instead of the correlation? Does it make sense to shift the columns of $\mathbf{A}$?

(d) Now find the projection of $\vec{b}$ onto the columns of $\mathbf{A}$. Use this to update the residual.

(e) Now compute the inner product of every column with the new residual vector. Which column has the largest inner product? This will be the second column of $\mathbf{A}$.

(f) Project $\vec{b}$ onto the columns of $\mathbf{A}$ to find $\vec{x}$.

3. **One Magical Procedure (Fall 2015 Final)**

Suppose that we have a vector $\vec{x} \in \mathbb{R}^5$ and an $N \times 5$ measurement matrix $\mathbf{M}$ defined by column vectors $\vec{c}_1, \ldots, \vec{c}_5$, such that:

$$\mathbf{M}\vec{x} = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_5 \\ | & & | \end{bmatrix} \vec{x} \approx \vec{b}$$

We can treat the vector $\vec{b} \in \mathbb{R}^N$ as a noisy measurement of the vector $\vec{x}$, with measurement matrix $\mathbf{M}$ and some additional noise in it as well.

You also know that the true $\vec{x}$ is sparse – it only has two non-zero entries and all the rest of the entries are zero in reality. Our goal is to recover this original $\vec{x}$ as best we can.

However, your intern has managed to lose not only the measurements $\vec{b}$ but the entire measurement matrix $\mathbf{M}$ as well!

Fortunately, you have found a backup in which you have all the pairwise inner products $\langle \vec{c}_i, \vec{c}_j \rangle$ between the columns of $\mathbf{M}$ and each other as well as all the inner products $\left\langle \vec{c}_i, \vec{b} \right\rangle$ between the columns of $\mathbf{M}$ and the vector $\vec{b}$. Finally, you also know the inner product $\left\langle \vec{b}, \vec{b} \right\rangle$ of $\vec{b}$ with itself.

All the information you have is captured in the following table of inner products. (These are not the vectors themselves.)

$\langle \cdot, \cdot \rangle$	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$	$\vec{c}_4$	$\vec{c}_5$	$\vec{b}$
$\vec{c}_1$	2	0	1	-1	1	1
$\vec{c}_2$		2	1	-1	-1	-5
$\vec{c}_3$			2	0	-1	2
$\vec{c}_4$				2	-1	6
$\vec{c}_5$					2	-1
$\vec{b}$						29

(So, for example, if you read this table, you will see that the inner product $\langle \vec{c}_2, \vec{c}_3 \rangle = 1$, that the inner product $\left\langle \vec{c}_3, \vec{b} \right\rangle = 2$, and that the inner product $\left\langle \vec{b}, \vec{b} \right\rangle = 29$. By symmetry of the real inner product, $\langle \vec{c}_3, \vec{c}_2 \rangle = 1$ as well.)

Your goal is to find which entries of $\vec{x}$ are non-zero and what their values are.

(a) Use the information in the table above to answer which of the $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with $\vec{b}$.

(b) Let the vector with the largest magnitude inner product with $\vec{b}$ be $\vec{c}_a$. Let $\vec{b}_p$ be the projection of $\vec{b}$ onto $\vec{c}_a$. Write $\vec{b}_p$ symbolically as an expression only involving $\vec{c}_a$, $\vec{b}$, and their inner products with themselves and each other.

(c) Use the information in the table above to find which of the column vectors $\vec{c}_1, \ldots, \vec{c}_5$ has the largest magnitude inner product with the residue $\vec{b} - \vec{b}_p$.
Hint: The linearity of inner products might prove useful.

(d) Suppose that the vectors we found in parts (a) and (c) are $\vec{c}_a$ and $\vec{c}_c$. These correspond to the components of $\vec{x}$ that are non-zero, that is, $\vec{b} \approx x_a \vec{c}_a + x_c \vec{c}_c$. However, there might be noise in the measurements $\vec{b}$, so we want to find the linear least squares estimates $\widehat{x}_a$ and $\widehat{x}_c$. Write a matrix expression for $\begin{bmatrix} \widehat{x}_a \\ \widehat{x}_c \end{bmatrix}$ in terms of appropriate matrices filled with the inner products of $\vec{c}_a$, $\vec{c}_c$, $\vec{b}$.

(e) Compute the numerical values of $\widehat{x}_a$ and $\widehat{x}_c$ using the information in the table.