1. **Proofs**

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(b) Prove that if a matrix's columns are linearly dependent, there will be either infinite or no solutions to $\mathbf{A}\vec{x} = \vec{b}$. What is the physical interpretation of this statement?

(c) Suppose we have an experiment where we have n measurements of linear combinations of n unknowns. We want to show that if at least one of the experiment's measurements can be predicted from the other measurements, then there will be either infinite or no solutions. Reword this statement into a proof problem and, as practice, complete the proof.

(d) **Practice Problem:** Now suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

(e) **Challenging Practice Problem:** Prove that for a $m \times n$ matrix, the number of linearly independent vectors (both column and row) is at most $\min(m,n)$.

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

Part 1: Rotation Matrices as Rotations

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by 15° and 30°, respectively. Design a procedure to rotate the unit square by 45° using only $\mathbf{T}_1$ and $\mathbf{T}_2$, and plot the result in the IPython notebook. How would you rotate the square by 60°?

(b) Try to rotate the unit square by 60° using only one matrix. What does this matrix look like?

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. (*Hint: Use your trigonometric identities!*)

(d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? *Don't use inverses!*

(e) Use part (d) to obtain the "inverse" rotation matrix for a matrix that rotates a vector by θ. Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Follow your TA to obtain the answers to the following questions!

(a) Let's see what happens to the unit square when we rotate the matrix by 60° and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect it along the y-axis and then rotate the matrix by 60°.

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?