EECS 16A Designing Information Devices and Systems I
Fall 2017 Discussion 3B

Reference Definitions

Vector spaces: A *vector space* V is a set of elements that is closed under vector addition and scalar multiplication and contains a zero vector.

That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

Subspaces: A subset W of a *vector space* V is a *subspace* of V if the above three conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace W.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

Basis: A *basis* for a vector space or subspace is an *ordered set of linearly independent vectors* that *spans the vector space or subspace*.

Therefore, if we want to check whether a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ forms a basis for a vector space V, we check for two important properties:

(a) $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

(b) $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} = V$

As we move along, we'll learn how to identify and construct a basis, and we'll also learn some interesting properties of bases.

Dimension: The *dimension* of a vector space is the *minimum number* of vectors needed to span the entire vector space. That is, the dimension of a vector space equals the number of vectors in a basis for this vector space.

1. Identifying a Basis

Does each of these sets of vectors describe a basis for some vector space?

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

2. Constructing a Basis

Let's consider a subspace of $\mathbb{R}^3$, V, that has the following property: for every vector in V, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, $a_1 = 2(a_2 + a_3)$.

Find a basis for V. What is the dimension of V?

3. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space $\mathbb{R}^m$ and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$.

(a) For the first part of the problem, let $m > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

(b) Let $m = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

(c) Now, let $m < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? What vector space could they form a basis for?

Hint: Think about whether the vectors can be linearly independent.

4. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

5. Exploring Column Spaces and Null Spaces

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following five matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?

ii. What is the null space of $\mathbf{A}$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?

iv. Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^2$ (or $\mathbb{R}^3$ for part (b))? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 0 & -1 & -2 \end{bmatrix}$