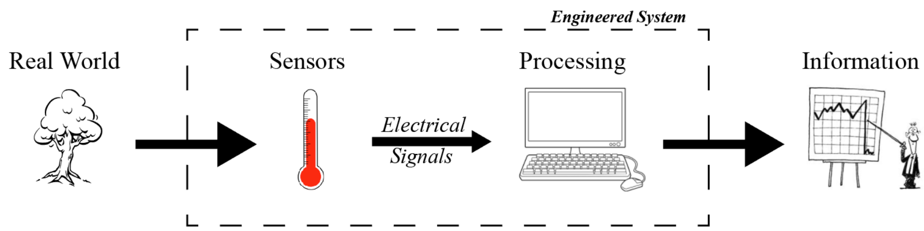
12.1 Context



Usually, our ultimate goal is to design systems that solve people's problems. To do so, it's critical to understand how we go from real-world events all the way to useful information that the system might then act upon. The most common way an engineered system interfaces with the real world is by using sensors and/or actuators that are often composed of electronic circuits; these communicate via electrical signals to processing units, which are *also* composed today entirely electronic circuits. In order to fully understand and *design* a useful system, we will need to first understand **Electrical Circuit Analysis**.

In this note, we will be intentionally ignoring the underlying physics of electrical circuits, and will instead focus on a standard procedure and set of rules that will allow us to systemically "solve" such circuits, (i.e., given a circuit diagram, solve for all of the releveant electrical quantities in that circuit). The reason for this approach is that one does not need to understand any real physics in order to be able to analyze electrical circuits, and in fact, impartial or incorrect physical intuition often leads to errors and then confusion. By abstracting the physics away during the analysis step, we hope to emphasize that the analysis in and of itself is simply a matter of taking a visual diagram (representing an electrical circuit) and applying a set of rules to it that will convert the diagram in to a set of (linear) equations that can then be solved using the techniques we have developed in the first module.

12.2 Basic Circuit Quantities

Let's start with some definitions for basic quantities present in an electrical circuit:

Quantity	Symbol	Units
Voltage	V	Volts (V)
Current	I	Amperes (A)
Resistance	R	Ohms (Ω)

We use these quantities in a **Circuit Diagram**, a collection of elements where each has some voltage *across* it and some current *through* it. This is an important distinction; voltage, or electric potential, is only defined *relative* to another point. A simple analogy could be described using elevation. A mountain's summit could be 9000ft above sea level (where sea level would be the reference of an elevation of 0 ft). Instead, if we choose the bottom of the ocean as the reference point (elevation of 0 ft), then there will still be 9000ft between sea level and the summit of the mountain, but the elevation at the mountain referenced to the bottom of the ocean would not be 9000ft (it would be much greater). Voltage can be thought of in the same way as it is also a relative value. As a shorthand, we will sometimes (later in this class) refer to a voltage at a particular point, but if we do so, we are implicitly using some other known point as a reference against which the voltage at that single point is being measured. Often times, this reference point is called **ground**.

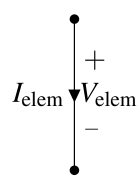
12.3 Basic Circuit Elements

A list of some of the most common circuit elements follows. For each, their common schematic representation, the defining relationship between the voltage across the element and the current through it, and a graphical example of that relationship is given.

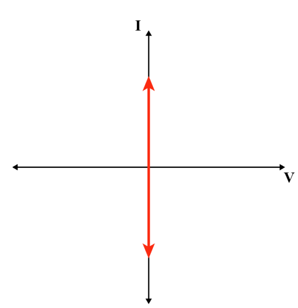
Wire: The most common element in a schematic is the wire, drawn as a solid line.

$V_{elem} =$ 0 A wire is an ideal connection with zero voltage across it.
$I_{elem} =$? The current through a wire can take any value, and is determined by the rest of the circuit.

Symbol



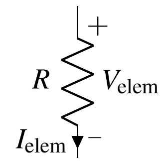
IV Curve



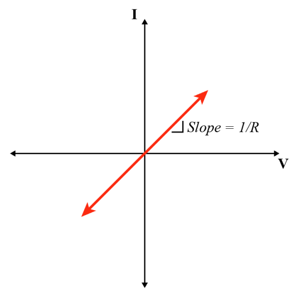
Resistor:

$V_{elem} = IR$ The voltage across a resistor can be determined using Ohm's Law.
$I_{elem} = \frac{V}{R}$ The current through a resistor can be determined using Ohm's Law.

Symbol



IV Curve

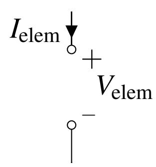


Open Circuit: This element is the dual of the wire.

$V_{elem} =$? The voltage across an open circuit can take any value, and is determined by the rest of the circuit.

$I_{elem} =$ 0 No current is allowed to flow through an open circuit.

Symbol



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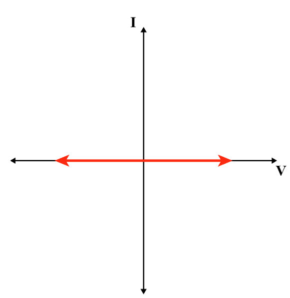
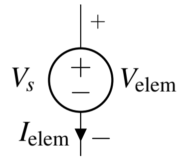


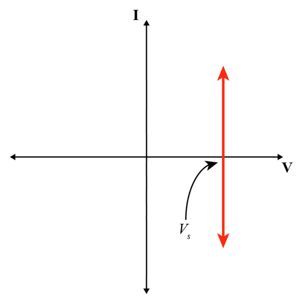
Voltage Source: A voltage source is a component that forces a specific voltage across its terminals. The $+$ and $-$ sign indicates which direction the voltage is pointing. The voltage difference between the "+" terminal and the "−" terminal is always equal to V_s, no matter what else is happening in the circuit.

$V_{elem} = \quad V_s \quad$ The voltage across the voltage source is always equal to the source value.
$I_{elem} = \quad ? \quad$ The current through a voltage source is determined by the rest of the circuit.

Symbol

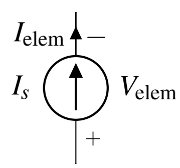


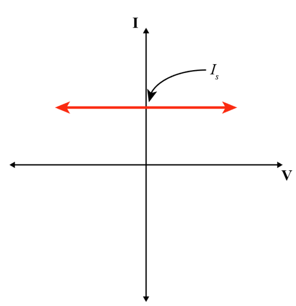
IV Curve



Current Source: A current source forces current in the direction specified by the arrow indicated on the schematic symbol. The current flowing through a current source is always equal to I_s, no matter what else is happening in the circuit. Note the symmetry between this element and the voltage source.

$V_{elem} = \quad ? \quad$ The voltage across a current source is determined by the rest of the circuit.
$I_{elem} = \quad I_s \quad$ The current through a current source is always equal to the source value.



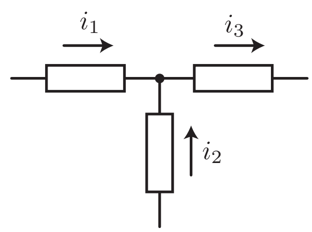


12.4 Rules for Circuit Analysis

The following rules govern the behavior of the circuit. They can be used to set up the relationships among the various circuit variables that we will use when solving the circuit (i.e., finding all of the voltages and currents in the circuit).

12.4.1 Kirchhoff's Current Law (KCL)

A place in a circuit where two or more of the above circuit elements meet is called a *junction*. KCL states that the net current flowing out of (or equivalently, into) any junction of a circuit is identically zero. To put this more simply, the current flowing into a junction must equal the current flowing out of that junction.



For example, consider the circuit shown above. From the left branch, there is i_1 current flowing into the junction, so there is $-i_1$ current flowing out of the junction. Similarly, there is $-i_2$ current flowing out of the junction and i_3 current flowing out of the junction. Therefore, the currents must satisfy

$$(-i_1)+(-i_2)+i_3 = 0 \text{ or } i_1+i_2 = i_3 \tag{1}$$

12.4.2 Kirchoff's Voltage Law (KVL)

Kirchoff's Voltage Law states that the sum of voltages across the elements connected in a loop must be equal to zero. As we will see later, this is equivalent to saying "what goes up must come down". We actually will not use KVL in the analysis procedure we outline next, but we include it here both for completeness, and so that you have an additional rule available to double check your final answers.

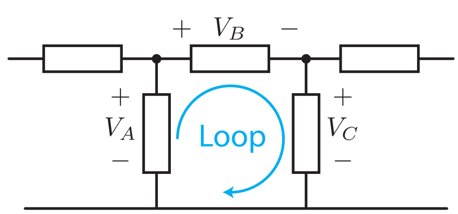


Figure 1: KVL illustration

Mathematically, KVL states that:

$$\sum_{\text{Loop}} V_k = 0. \tag{2}$$

When adding the voltage “drops” around the loop we must follow a convention. If the arrow corresponding to the loop goes into the ”+” of an element we subtract the voltage across that element. Conversely, if the arrow goes into the ”-” of an element, we add the votage across that element. Following this convention for the example in Figure 1 we find:

$$V_A - V_B - V_C = 0. \tag{3}$$

Note that if we had defined the loop in the opposite direction,

$$-V_A + V_C + V_B = 0. \tag{4}$$

12.4.3 Ohm’s Law and Resistors

As already described when we introduced resistors as an element, for these elements, the voltage across them is directly proportional to the current that flows through them, where the proportionality constant is the "resistance" (R) of the device. This relationship is known as **Ohm’s Law**.

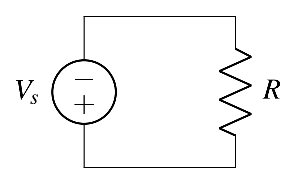
$$V_{\text{elem}} = I_{\text{elem}} R. \tag{5}$$

The unit of R is Volts/Amperes, or more commonly “Ohms” (Ω).

12.5 Circuit Analysis Algorithm

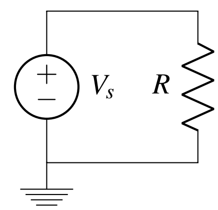
Within this module, we will learn how to take a real world system and build a circuit diagram that models the behavior of that system (so that we can eventually design our own systems). In this note however, we will assume that we have already found or been given an accurate model (i.e., a circuit diagram), and will be focusing on how to analyze that model. Specifically, once you have or are given an accurate circuit diagram, you should *always* be able to follow the given analysis algorithm to “solve” the circuit (i.e., compute all of the voltages and currents) correctly. Circuits are deterministic; if you follow the steps accurately and completely, the math will work.

As an example, we will use the following diagram, that consists of a voltage source, a resistor and two wires. Since there are four elements, each with two terminals (i.e., two sides), and since these elements must be connected in a closed loop, there are four junctions in this circuit (one at each corner of the diagram below).

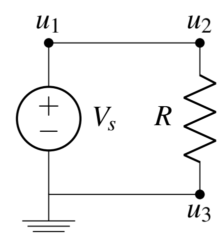


For the sake of clarity, after each step of the analysis algorithm you will be shown what the current circuit diagram would look like. When you perform the algorithm on your own however, you do not need to redraw the circuit each time; instead you can simply label/annotate a single diagram.

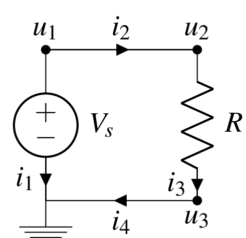
- **Step 1:** Pick a junction and label it as $u = 0$ (“ground”), meaning that we will measure all of the voltages in the rest of the circuit relative to this point.



- **Step 2:** Label all remaining junctions as some "u_i", representing the voltage at each junction relative to the zero junction/ground.

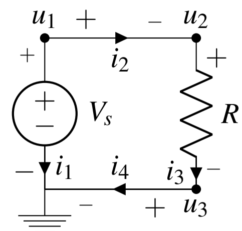


- **Step 3:** Label the current through every element in the circuit "i_n". Every element in the circuit that was listed above should have a current label, including ideal wires. (As we will describe later, once you have some familiarity with the procedure there are simplifications we can make to avoid the need to label the current in every single wire, but we describe here the most complete version of the algorithm so that you can always return to this if you ever have any doubt about whether a certain simplification is valid or not.) The direction of the arrow indidates which direction of current flow you are considering to be positive. At this stage of the algrithm, you can pick the direction of all of the current arrows *arbitrarily* - as long as you are consistent with this choice and follow the rules described in the rest of this algorithm, the math will work out correctly.



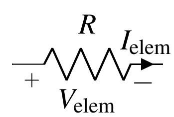
  It may be worth noting at this point that we only label the current once for each element because KCL also hold within the element itself - i.e., the current that enters an element must be equal to the current that exits that same element.

- **Step 4:** Add $+/-$ labels on each element that indicate the direction with which voltage will be measured across that element while following the "Passive Sign Convention" (discussed below)



12.5.1 Passive sign convention

As mentioned previously, the $+$ and $-$ signs we drew across each of our elements indicate the manner in which we intend to measure the voltage across that element. (In other words, similar to what we did with the current arrow, the $+$ and $-$ signs here define the direction that we will be measuring the voltage across that element. If in this circuit the voltage rises when going from the $-$ sign to the $+$ sign, the voltage we measure will be a positive value; if on the other hand in this circuit the voltage drops when going from the $-$ to the $+$ sign, the voltage we measure will be negative.) For reasons that we will explain later, the passive sign convention tells us what the relationship between the way in which we measure the voltage and the direction in which we assume the current flows must be. Specifically, the passive sign convention dictates that positive current should *enter* the positive terminal and *exit* the negative terminal of an element. Below is an example for a resistor:



As long as this convention is followed consistently, it does not matter which direction you arbitrarily assigned each element current to; the voltage referencing will work out to determine the correct final sign.

- **Step 5:** Set up the relationship $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x}$ is comprised of the unknown circuit variables we want to solve for. $\mathbf{A}$ will be an $n \times n$ matrix where n is equal to the number of unknown variables. For the circuit above, we have 3 unknown potentials (u) and 4 unknown currents (i), therefore we form a 7×7 matrix.

$$\begin{bmatrix} ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

- **Step 6:** Use KCL to fill in as many **Linearly Indpendent** rows of $\mathbf{A}$ and $\vec{b}$ as possible.

  Let's begin by writing KCL equations for every junction in the circuit.

$$\begin{aligned} i_1 + i_2 &= 0 \\ -i_2 + i_3 &= 0 \\ -i_3 + i_4 &= 0 \\ -i_4 - i_1 &= 0 \end{aligned}$$

Notice the last equation we get is linearly dependent on the first three - you can see this by adding all three of the first equations to each other and multiplying the entire result by -1. In order to end up with a square and invertible **A** matrix, we will therefore omit this equation. Note that in general, if you use KCL at every junction, you will get one linearly dependent equation, and so you can typically simply skip one junction; skipping the junction that has been labeled as ground is a common choice.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

- **Step 7:** Use the IV relationships of each of the elements to fill in the remaining equations (rows of **A** and values of $\vec{b}$).

  In this example, we can use IV relations to find four more equations. We know that the difference in potentials across the voltage source must be the voltage on the voltage source. We also know that the voltage across the resistor is equal to the current times the resistance, from Ohm's Law. For the wires, we know the difference in potential is 0. Thus, we have the following equations.

$$\begin{aligned} u_1 - 0 &= V_s \\ u_1 - u_2 &= 0 \\ u_2 - u_3 &= Ri_3 \\ u_3 - 0 &= 0 \end{aligned}$$

  Our matrix is then:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -R & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_{V_s} \\ i_1 \\ i_2 \\ i_3 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

  It is worth pausing at this point to highlight why the procedure always works, and in particular, why we will always have as many equations as we do unknowns. If a circuit has m elements in it and n junctions, there will be $(n-1)$ u's (since we have defined one of them as ground/zero), and m currents (one for each element). Since each element has a defining I-V relationship, Step 7 will provide us with m equations. Similarly, with n junctions, we will get $(n-1)$ linearly independent KCL equations from Step 6. As a side note, the order of Steps 6 and 7 can be interchanged - in fact, you may want to do Step 7 (IV relations) first since when you then come back to Step 6 (KCL) you just have to fill in as many equations using KCL as needed to make the **A** matrix square.

  At this point the analysis procedure is effectively complete - all that's left to do is solve the system of linear equations (by applying Gaussian Elimination, inverting **A**, etc.) to find the values for the u's and i's.

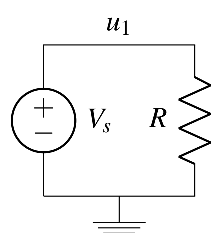
12.6 Simplifying the Circuit Analysis Procedure

While the analysis procedure we described in the previous section will always work, and introducing the procedure at this level of comprehensiveness is necessary to ensure that one can always follow it successfully, as is most likely clear, even for very simple circuits the procedure will quickly involve a large number of variables and hence large matrices. Fortunately, once we make some simple observations about the behavior of wires (specifically, that there is no voltage drop across them) and the implications of KCL on junctions that involve only two elements - and particulatly junctions between an element and a wire (i.e., that the current in the element must then equal the current in the wire), we can substantially reduce the number of variables and hence the size of the matrices.

12.6.1 Labeling Nodes Instead of Junctions

Since wires always have zero voltage drop across them, there is no specific need for us to keep track of the voltage (relative to ground) on the two sides of a wire separately. In other words, every junction within a set of junctions that are connected to each other only by wires can be labeled with a single voltage variable u. A set of such junctions connected to each other only via wires is defined as a *node*. (Formally, a node is defined as a region of the circuit that is "equipotential" - i.e., that has no voltage drop across it - but it should hopefully be clear that our earlier definition meets exactly this criteria.)

As an example, let's consider the circuit we were analyzing, but return to Steps 1 and 2. As shown below, the junctions between the bottm wire and the voltage source as well as between the bottom wire and the resistor are all actually a single node, and we can label that entire node as being ground. Similarly, the junctions between the top wire and the voltage source as well as between the top wire and the resistor are all actually a single node, and hence we can label that entire node as u_1.



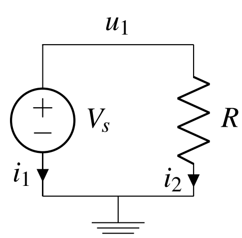
When we followed the original analysis procedure (where the voltage across the wire was treated as an unknown quantity to be solved) on this circuit, we ended up with three unknown u's; by labeling only the nodes, we have brought this down to a single unknown u (u_1). In general, since wires are abundant in circuit diagrams, labeling only the nodes (instead of the junctions) will indeed substantially reduce the number of variables.

12.6.2 Trival Junctions

Let's consider for a moment all junctions that involve only two elements (which is what we are refering to when we titled this subsection as "trivial" junctions). In this case, KCL dictates that the current entering that junction must be equal to the current exiting that junction; since no other elements are involved, this implies that the currents flowing through the two elements must be equal to each other (as long as we chose the direction of current flow to be the same of course - if not, the currents will simply be opposite in sign).

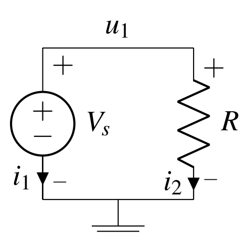
Given this fact, another simplification to be made in our analysis procedure is to label the currents only in the **non-wire elements** in our circuit. We should then keep track of the fact that a wire connected via a trivial junction to an elements must carry the same current as the element. In other words, we should then only use KCL on either (1) non-trivial junctions involving more than 2 components or (2) trivial junctions between an element and a wire where the current carried on the wire has already been assumed to equal that of a different element.

Returning to our example, if we repeat Step 3 (and assume we make use of the simplification made in the previous subsection), we would now label only the current through the voltage source and through the resistor.



With this simplified approach, when we return to Step 6 (KCL), we would apply this at e.g. the junction in the upper right hand corner of the diagram (between the top wire and the top of the resistor), which would result in the equation:

$$i_1 = -i_2 \text{ or } -i_1 - i_2 = 0 \tag{6}$$



12.6.3 Summary of Simplified Procedure

- **Step 1:** Pick a **node** and label it as $u = 0$ ("ground"), meaning that we will measure all of the voltages in the rest of the circuit relative to this point.
- **Step 2:** Label all remaining **nodes** as some "u_i", representing the voltage at each node relative to the ground node.
- **Step 3:** Label the current through every **non-wire** element in the circuit "i_n".

- **Step 4:** Add $+/-$ labels (indicating direction of voltage measurement) on each **non-wire** element by following the passive sign convention.
- **Step 5:** Set up the relationship $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x}$ is comprised of the u_i's and i_n's defined in the previous steps.
- **Step 6:** If there are m **non-wire** elements in the circuit and n **nodes** (including the ground node), use KCL on $(n-1)$ junctions to fill in $(n-1)$ rows of $\mathbf{A}$ and $\vec{b}$.
- **Step 7:** Use the IV relationships of each of the m **non-wire** elements to fill in the remaining m equations (rows of $\mathbf{A}$ and values of $\vec{b}$).
- **Solve!**