EECS 16A Designing Information Devices and Systems I
Fall 2017 Official Lecture Notes Note 27(Draft)

27.1 Listening to Songs From Many Devices

Now that you have learned about cross correlation and least squares techniques, we can put these ideas together to implement something useful! Let's imagine you have a home system with a single receiver and many different transmitters. The transmitters could be things like temperature sensors, light-meters, humidity sensors, or something that measures how loudly your cat is purring. These all want to send information to the receiver, but how will the receiver distinguish between the different signals? Recall from the correlation note and locationing lab that these beacons work best when each one has a different *signature code* - let's call them 'songs' for ease from now on - and when these songs are (mostly) orthogonal to one another and (mostly) orthogonal to all the shifted versions of itself. In other words, they have high auto-correlation values and the cross-correlation values with other codes are very low. The songs are streams of ± 1 and are known by the sensors and the receiver. The songs are multiplied with a *message signal* to get the transmission signal. In the case of a temperature sensor, this message signal would be some real number corresponding to the temperature.

Now imagine we have 2,000 sensors, each with it's own song: $\vec{S}_0, \vec{S}_1, ..., \vec{S}_{1999}$

Each song is length 400, and therefore the circulant matrix for each song has 400 rows, each corresponding to a delay or shift of 0-399.

$$C_{\vec{S}_1} = \begin{bmatrix} \overrightarrow{S_1^{(0)}}^T \\ \overrightarrow{S_1^{(1)}}^T \\ \vdots \\ \overrightarrow{S_1^{(399)}}^T \end{bmatrix}$$

Note: for the following notation $\overrightarrow{S_1^{(0)}}$, the subscript is the index of the song, and the superscript in parentheses is the index of the shift.

Each message is a real number that scales the song: $a_0, a_1, ..., a_{1999}$.

The delay of each message is given by: $N0, N1, ..., N1999$.

Therefore- the signal received ($\vec{r}$) is a linear combination of shifted songs from all transmitting users given by:

$$\vec{r} = a_0 \overrightarrow{S_0^{(N0)}} + a_1 \overrightarrow{S_1^{(N1)}} + ... + a_{1999} \overrightarrow{S_{1999}^{(N1999)}}$$

In the simplest case, there will be only one sensor transmitting data to the receiver. If sensor 130 is sending a message of value 1 with a delay of 10, the received signal will be:

$$\vec{r} = 1 * \overrightarrow{S_{130}^{(10)}}$$

Now if we perform cross correlation with only $\overrightarrow{S_{130}}$, the results will have a peak in the 10th bin, with a normalized amplitude of 1. This is shown in Fig. 1.

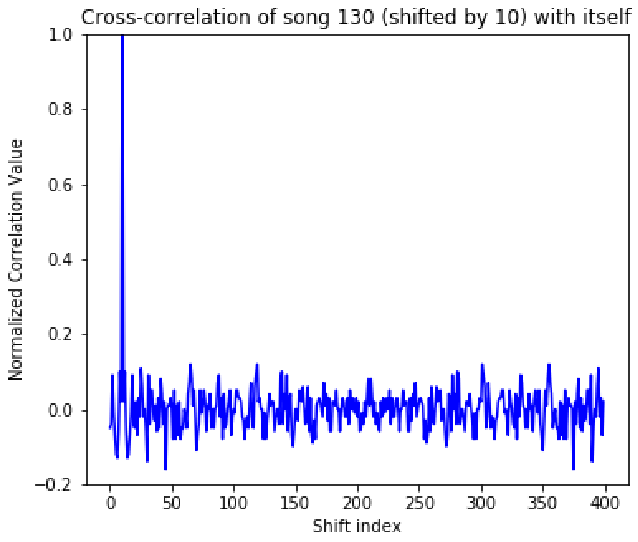


Figure 1: Maximum correlation of received signal with circular shifts of $\vec{S_{130}}$

If we correlate with all songs and plot the maximum from each one, we get the results in Fig. 2.

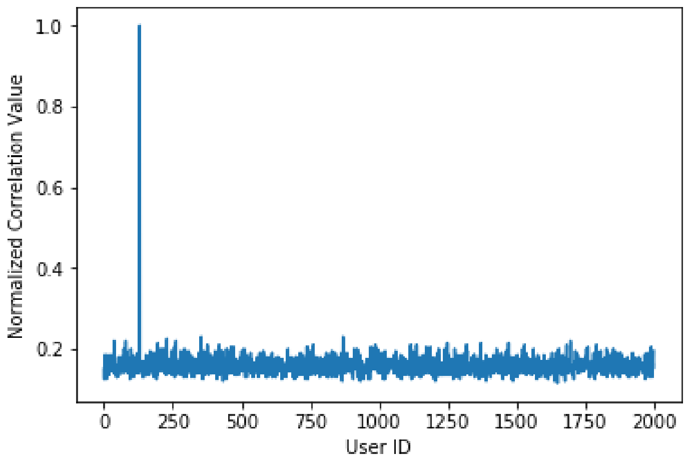


Figure 2: Maximum correlation of received signal with circular shifts of all the songs

Now imagine devices 40, 100, and 312 are transmitting with delays of 13, 20, and 45 (respectively) and message values of 10, 10, and 8 (respectively). Our received signal is now:

$$\vec{r} = 10\overrightarrow{S_{40}^{(13)}} + 10\overrightarrow{S_{100}^{(20)}} + 8\overrightarrow{S_{312}^{(45)}}$$

If we cross-correlate the received signal with every song, find the peak of each cross-correlation, and plot them together (normalizing by the length of the song), we get the plot in Fig. 3.

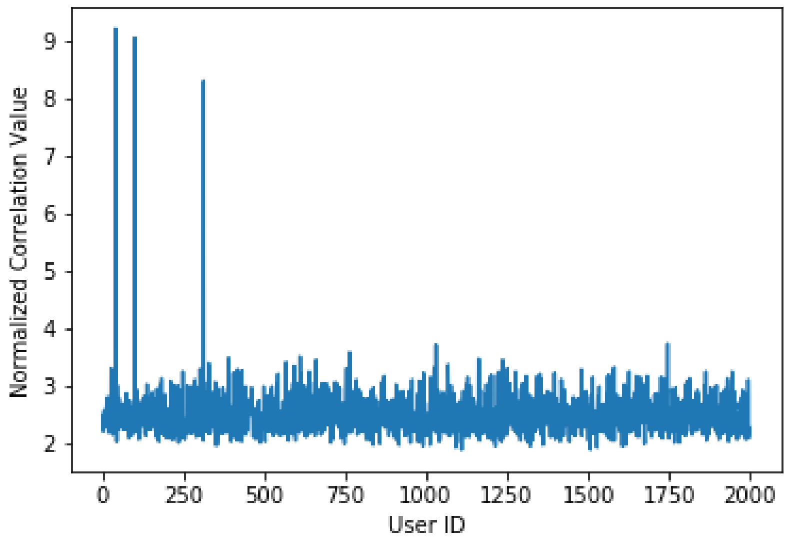


Figure 3: Maximum correlation of received signal with circular shifts of all the songs

Let us look at one more example in which 4 users (user 40, 100, 312 and 350) transmit and their corresponding messages are 100, 10, 8 and 0.02. The result of the correlation with different signatures is shown in Fig. 4.

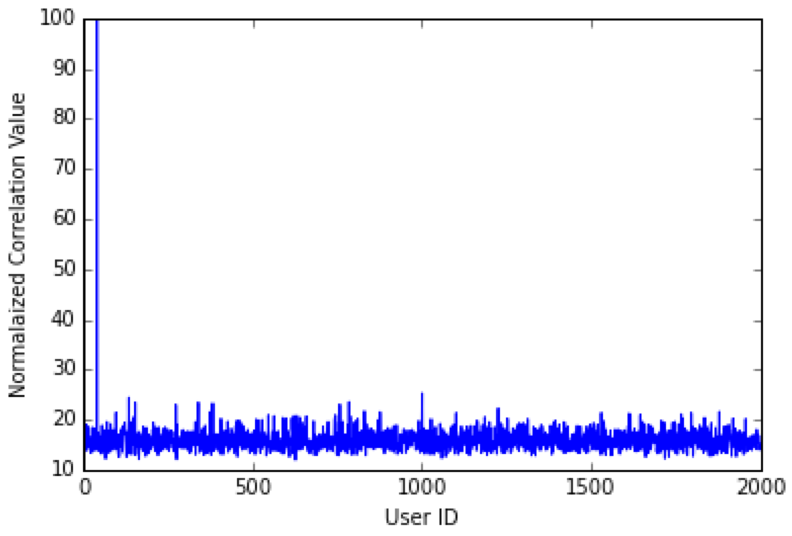
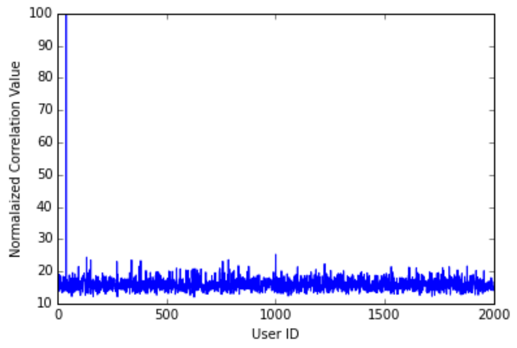
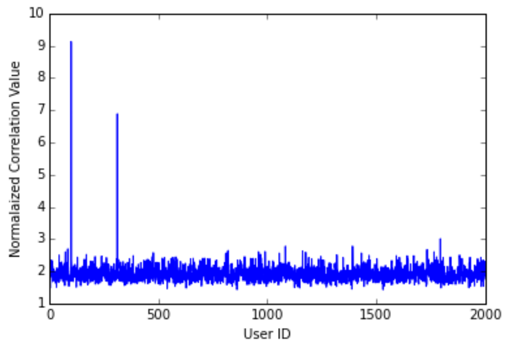


Figure 4: Maximum correlation of received signal with circular shifts of all the signatures

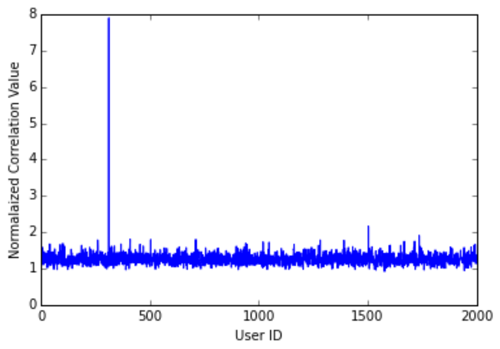
The peaks corresponding to users 100, 312 and 350 don't seem to appear at all! What could possibly cause this? Perhaps the very high value of user 40's made it so that we cannot see the songs from users 100, 312 and 350.



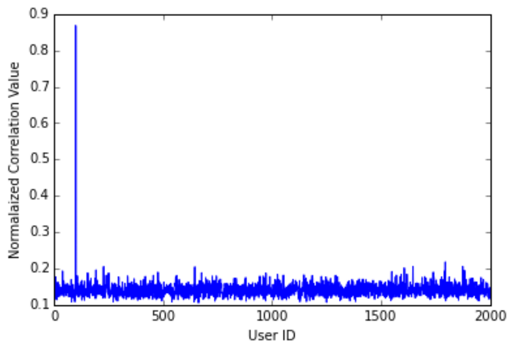
(a) Step 1: Correlation of signatures with the received signal



(b) Step 2: Correlation with residue after removal of user 40's song



(c) Step 3: Correlation with residue after removal of user 100's song



(d) Step 4: Correlation with residue after removal of user 312's song

Figure 5: Maximum correlation of received signal with circular shifts of all the signatures

Maybe if we are able to identify the dominant song and subtract it from the received signal and cross-correlate again, we will be able to see the other songs. We can easily see that the dominant song here is $\overrightarrow{S_{40}}$. After removing the song corresponding to user 40, the result of the correlation with the signatures is shown in Fig. 5b

We now see the peaks corresponding to users 100 and 312, but not 350. If we repeat the same steps as before, removing the dominant song and correlating again, we see the results of the subsequent steps in Fig. 5c and Fig. 5d

In Fig. 5d we would have expected to find a peak at user 350 but surprisingly we don't. Why is that the case? Maybe, the way we extracted the signals out introduced more noise? We need a new technique!

27.2 Orthogonal Matching Pursuit

There is a better way to determine the values of all the songs in the received signal and this algorithm is called Orthogonal Matching Pursuit, or OMP.

Recall again that the received signal looks something like this:

$$\vec{r} = a_0\overrightarrow{S_0^{(N0)}} + a_1\overrightarrow{S_1^{(N1)}} + ... + a_{1999}\overrightarrow{S_{1999}^{(N1999)}}$$

and that simple cross-correlation is very good at giving us the ID and shift of the songs we are hearing, but not the actual value, ie we can determine $\overrightarrow{S_1^{(N1)}}$ well, but not a_1 well.

Note that this technique works best for a sparse spectrum, meaning that the number of devices transmitting at any given time is much less than the number of devices. In our case, we have 2000 devices, but let's operate under the assumption that at most $k = 10$ at a time are singing their songs.

What if we re-write this equation in a form that looks like $A\vec{x} = \vec{b}$ and use least squares to solve for the song values?

$$\begin{array}{ccc} A & \vec{x} & = \vec{b} \\ \begin{bmatrix} | & | & & | \\ \overrightarrow{S_0^{(N0)}} & \overrightarrow{S_1^{(N1)}} & \dots & \overrightarrow{S_k^{(Nk)}} \\ | & | & & | \end{bmatrix} & \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix} & = \vec{r} \end{array}$$

Recall that the lest squares solution is:

$$\vec{x} = (A^TA)^{-1}A^T\vec{b}$$

And now we have a better way of solving for $a_0, a_1, ..., a_k$!

Here is a description of OMP with an example received signal of : $\vec{r} = a_{40}\overrightarrow{S_{40}^{(13)}} + a_{100}\overrightarrow{S_{100}^{(8)}}$ where $a_{40} = 100$ and $a_{100} = 10$.

1. Find the vector $\overrightarrow{S_*^{(N*)}}$ with the highest correlation with $\vec{r}$. First, what does it mean for a vector $\overrightarrow{S_*^{(N*)}}$ to have the highest correlation with $\vec{r}$? The highest correlation between two vectors means the error vector between them is the lowest. We find this vector via a simple cross-correlation with all the circular shifts of all the songs, the same first step we have always done. This gives us $\overrightarrow{S_{40}^{(13)}}$.

2. Use least squares to solve for $\vec{x}$ in the eqn $A\vec{x} = \vec{b}$ where $\vec{b}$ is the received signal $\vec{r}$ and A is $\begin{bmatrix} | \\ \overrightarrow{S_{40}^{(13)}} \\ | \end{bmatrix}$:

   $\vec{x} = (A^TA)^{-1}A^T\vec{r}$. In this case, $\vec{x} = 100 = a_{40}$. The 'orthogonal projection of the least squares solution onto the subspace spanned by A' (aka our 'ideal message') is given by: $A\vec{x}$.

3. Now find the residue of $\vec{r}$ left over by subtracting our ideal message from the received signal: $\vec{r} - A\vec{x}$.

4. Repeat the above steps using the updated value of $\vec{r}$. A new correlation would find that the next strongest song is: $\overrightarrow{S_{100}^{(8)}}$. We then update our matrix A to be $\begin{bmatrix} | & | \\ \overrightarrow{S_{40}^{(13)}} & \overrightarrow{S_{100}^{(8)}} \\ | & | \end{bmatrix}$ and now solve for $\vec{x}$ via least squares, finding that $\vec{x} = \begin{bmatrix} 100 \\ 10 \end{bmatrix}$. We have now recovered both of our message values!
5. We stop iterating when we have gone 10 steps, the known the sparsity of the signal, OR until the norm of the residual is below some threshold value, meaning the residual contains only noise.

The following section precisely describes the implementation of OMP.

Setup:
Let the number of possible unique songs (and users) be $m = 2000$ and the unique songs be represented by $\vec{S_i}$ (of length $n = 400$). Each signature can potentially carry a message a_i along with it. Then the measurement at the receiver is

$$\vec{r} = a_0\overrightarrow{S_0^{(N0)}} + a_1\overrightarrow{S_1^{(N1)}} + \ldots + a_{1999}\overrightarrow{S_{1999}^{(N1999)}}$$

We will assume that the number of users talking at the same time is very small *i.e.,* there are at most $k = 10$ non-zero a_is. The OMP algorithm is described below.

Inputs:

- A set of m songs, each of length n: $\mathbf{S} = \{\vec{S_0}, \vec{S_1}, ..., \overrightarrow{S_{m-1}}\}$
- An n-dimensional received signal vector: $\vec{r}$
- The sparsity level k of the signal
- Some threshold, th. When the norm of the signal is below this value, the signal contains only noise.

Outputs

- A set of songs that were identified, F, which will contain at most k elements.
- A vector, $\vec{x}$ containing song messages (a_1, etc.), which will be of length k or less.
- An n-dimensional residual $\vec{y}$

Procedure:

- Initialize the following values: $\vec{y} = \vec{r}$, $j = 1$, $k = 10$, $A = [\ ]$, $F = \{\emptyset\}$
- while $((j \leq k)$ & $(\| \vec{y} \| \geq th))$:
  1. Cross correlate $\vec{y}$ with the shifted versions of all songs. Find the song index, i, and the shifted version of the song, $\overrightarrow{S_i^{(Ni)}}$, with which the received signal has the highest correlation value.
  2. Add i to the set of song indices, F.
  3. Column concatenate matrix A with the correct shifted version of the song: $A = [A \mid \overrightarrow{S_i^{(Ni)}}]$
  4. Use least squares to obtain the message value: $\vec{x} = (A^T A)^{-1} A^T \vec{r}$
  5. Update the residual value $\vec{y}$ by subtracting: $\vec{y} = \vec{r} - A\vec{x}$
  6. Update the counter: $j = j + 1$