EECS 16B Fall 2017 — Designing Information Devices and Systems II — Miki Lustig and Michel Maharbiz

Homework 11

This homework is due November 21, 2017, at noon.

1. Aliasing

The concept of "aliasing" is intuitively about having a signal of interest whose samples look identical to a different signal of interest — creating an ambiguity as to which signal is actually present.

While the concept of aliasing is quite general, it is easiest to understand in the context of sinusoidal signals.

(a) Consider two signals,

$$x_1(t) = a\cos(2\pi f_0 t + \phi)$$

and

$$x_2(t) = a\cos(2\pi(-f_0 + mf_s)t - \phi)$$

where $f_s = 1/T_s$. Are these two signals the same or different when viewed as functions of continuous time t?

(b) Consider the two signals from the previous part. These will both be sampled with the sampling interval T_s. What will be the corresponding discrete-time signals $x_{d,1}[n]$ and $x_{d,2}[n]$? (The $[n]$ refers to the nth sample taken — this is the sample taken at real time nT_s.) Are they the same or different?

(c) What is the sinusoid $a\cos(\omega t + \phi)$ that has the smallest $\omega \geq 0$ but still agrees at all of its samples (taken every T_s seconds) with $x_1(t)$ above?

(d) Watch the following video: `https://www.youtube.com/watch?v=jQDjJRYmeWg`.

Assume the video camera running at 30 frames per second. That is to say, the camera takes 30 photos within a second, with the time between photos being constant.

Given that the main rotor has 5 blades, list *all* the possible rates at which the main rotor is spinning in revolutions per second assuming no physical limitations.

Hint: Your answer should depend on k where k can be any integer.

2. The vector space of polynomials

A polynomial of degree at most n on a single variable can be written as

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots p_n x^n$$

where we assume that the coefficients $p_0, p_1, \ldots, p_n$ are real. Let P_n be the vector space of all polynomials of degree at most n.

(a) Consider the representation of $p \in P_n$ as the vector of its coefficients in $\mathbb{R}^{n+1}$.

$$\vec{p} = \begin{bmatrix} p_0 & p_1 & \ldots & p_n \end{bmatrix}^T$$

Show that the set $\mathscr{B}_n = \{1, x, x^2, \ldots, x^n\}$ forms a basis of P_n, by showing the following.

- Every element of P_n can be expressed as a linear combination of elements in $\mathscr{B}_n$.
- No element in $\mathscr{B}_n$ can be expressed as a linear combination of the other elements of $\mathscr{B}_n$.
(*Hint*: Use the aspect of the fundamental theorem of algebra which says that a nonzero polynomial of degree n has at most n roots, and use a proof by contradiction.)

(b) Suppose that the coefficients $p_0, \ldots, p_n$ of p are unknown. To determine the coefficients, we evaluate p on $n+1$ points, $x_0, \ldots, x_n$. Suppose that $p(x_i) = y_i$ for $0 \leq i \leq n$. Write the matrix V in terms of the x_i, such that

$$V \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

(c) For the case where $n = 2$, compute the determinant of V and show that it is equal to

$$\det(V) = \prod_{0 \leq i < j \leq n} (x_j - x_i).$$

Under what conditions is this matrix invertible?

(d) We can define an inner product on P_n by setting

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

Show that this satisfies the following properties of a real inner product. (We would need a complex conjugate if we wanted a complex inner product).

- $\langle p, p \rangle \geq 0$, with equality if and only if $p = 0$.
- For all $a \in \mathbb{R}$, $\langle ap, q \rangle = a\langle p, q \rangle$.
- $\langle p, q \rangle = \langle q, p \rangle$.

(e) (optional) An alternative inner-product could be placed upon real polynomials if we simply represented them by a sequence of their evaluations at $0, 1, \ldots, n$ and adopted the standard Euclidean inner product on sequences of real numbers. Can you give an example of an orthonormal basis with this alternative inner product?

3. Sinc functions

The sinc function is defined as,

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

(a) Verify that,

$$\lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} = 1$$

Hint. Look into L'Hospital's Rule.

(b) Verify that,

$$\frac{1}{\pi} \int_0^{\pi} \cos(\omega x) d\omega = \operatorname{sinc}(x)$$

(c) If we think of the above integral as a sum of oscillatory functions of different frequencies, the sinc function has continuous frequency components in the range $[0, \pi]$. What is the range of frequencies of the function,

$$f(x) = \text{sinc}\left(\frac{x}{T}\right)?$$

Hint. Try to use substitution and the above integral.

(d) Continuing the idea of integration as the continuous version of sums, we define the continuous inner product between two real functions as,

$$\langle p, q \rangle = \int_{-\infty}^{\infty} p(x)q(x)dx$$

similarly to question 2 (note that we integrate from $-\infty$ to ∞ instead of -1 to 1). Given that the length squared of a vector $\vec{v}$ of length n is,

$$\|v\|^2 = \langle v, v \rangle = \sum_{i=1}^{n} v_i \overline{v_i},$$

what is the analogous length squared of a function, f, using the inner product definition above? This is a one-line answer.

We only consider interpolating functions where $\|f\| < \infty$, so the inner product is well defined. It turns out that the set of shifted sinc functions are orthonormal with the above inner product! Let,

$$\phi_k(x) = \text{sinc}(x - k) \text{ where } k \text{ is an integer.}$$

Then,

$$\|\phi_k\| = 1 \text{ and } \langle \phi_m, \phi_n \rangle = \begin{cases} 1, & m = n \\ 0, & \text{otherwise} \end{cases}$$

4. LTI Low Pass Filters

Given a sequence of discrete samples with high frequency noise, we can de-noise our signal with a discrete low-pass filter. Two examples are given below:

$$y[n] = 0.5y[n-1] + x[n] \tag{1}$$
$$y[n] = 0.25x[n] + 0.25x[n-1] + 0.25x[n-2] + 0.25x[n-3] \tag{2}$$

(a) Write the impulse responses $h[n]$ for (1) and (2). Are they IIR or FIR?

(b) Are either of these filters causal? Are either of these filters stable?

(c) Give the output y for each filter given the input sequence $x[n] = cos(\pi n) + n$ from $n = 0$ to $n = 7$. Assume that $y[n] = 0$ for $n < 0$.

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