

**This homework is due December 1, 2017, at noon.**

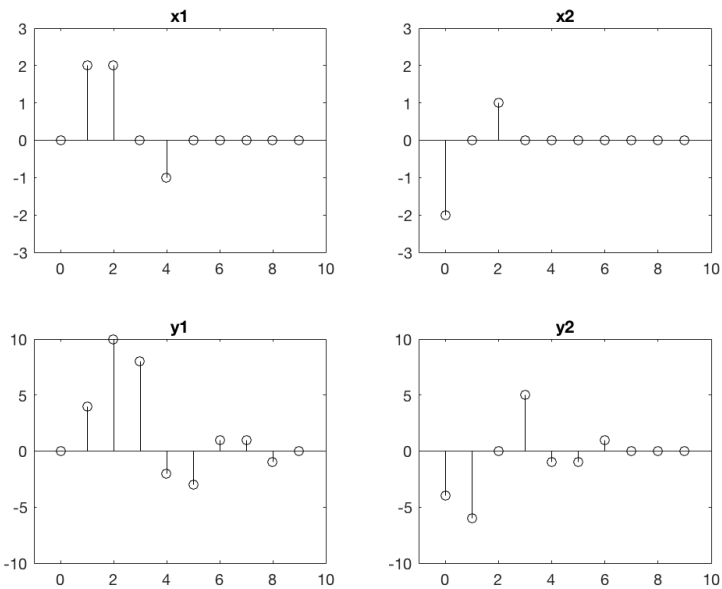
**1. LTI Inputs**

We have an LTI system whose exact characteristics we do not know. However, we know that it has a finite impulse response that isn't longer than 5 samples. We also observed two sequences,  $x_1$  and  $x_2$ , pass through the system and observed the system's responses  $y_1$  and  $y_2$ .

$x_1$	0	2	2	0	-1	0	0	0	0	0
$y_1$	0	4	10	8	-2	-3	1	1	-1	0

$x_1$	-2	0	1	0	0	0	0	0	0	0
$y_1$	-4	-6	0	5	-1	-1	1	0	0	0



(a) Given the above sequences, what would be the output for the input:

$x_3$	0	0	-2	0	1	0	0	0	0	0
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(b) What is the output of the system for the input  $x_1 - x_2$ ?

(c) Given the above information, how could you find the impulse response of this system? What is the impulse response?

(d) What is the output of this system for the following input:

$x_4$	1	1	1	0	0	0	0	0	0	0
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## 2. DFT Projection

We can think about discrete sequences  $x[n]$  with  $N$  samples as  $N$ -dimensional vectors. In this  $N$ -dimensional vector space, the DFT is a coordinate transformation, and the DFT is computed by projecting these vectors onto a set of DFT basis vectors. The meaning of the coefficients on each of these basis vectors is the “frequency content” of the original signal  $x[n]$ .

- The DFT basis vectors  $\vec{u}_k$  are given by  $u_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N}n}$  for  $k = 0, 1, \dots, N-1$  and  $n = 0, 1, \dots, N-1$ . Write the basis vectors for sequences of length 2. Use these vectors to write the change of basis matrix  $U$  that fulfills the change of basis equation  $\vec{X} = U^{-1}\vec{x}$  where  $\vec{X}$  is the representation of  $\vec{x}$  in the frequency domain.
- Plot these basis vectors as discrete sequences of length 2. Plot these basis vectors as vectors in the cartesian plane  $[x[0], x[1]]^\top$ .
- Find the basis vectors for sequences of length 3 and 4. Plot these basis vectors as discrete sequences (you can plot the imaginary component lighter, or as stems topped with an x instead of a dot). What is the length (magnitude) of the basis vectors? What component of a discrete sequence does the first basis vector correspond to in every case?
- Does  $U$  have any special properties? How can we use this property to compute  $U^{-1}$ ? Write  $U^{-1}$  in terms of the basis vectors  $\vec{u}_k$ .

## 3. Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem we explore some properties of the roots of unity. An  $N$ th root of unity is a complex number  $z$  satisfying the equation  $z^N = 1$  (or equivalently  $z^N - 1 = 0$ ).

- Show that  $z^N - 1$  factors as

$$z^N - 1 = (z - 1) \left( \sum_{k=0}^{N-1} z^k \right).$$

- Show that any complex number of the form  $W_N^k = e^{j\frac{2\pi}{N}k}$  for  $k \in \mathbb{Z}$  is an  $N$ -th root of unity.
- Draw the fifth roots of unity in the complex plane. How many unique fifth roots of unity are there?
- Let  $W_5 = e^{j\frac{2\pi}{5}}$ . What is another expression for  $W_5^{42}$ ?
- What is the complex conjugate of  $W_5$ ? What is the complex conjugate of  $W_5^{42}$ ? What is the complex conjugate of  $W_5^4$ ?
- Write the expression for the  $N$  basis vectors  $\vec{u}_k$  of size  $N$  for the DFT of a signal of length  $N$  in terms of  $W_N^k$ .

## 4. Denoising signals using the DFT

Professor Maharbiz is sad. He just managed to create a beautiful audio clip consisting of a couple pure tones with beats and he wants Professor Lustig to listen to it. He calls Professor Lustig on a noisy phone and plays the message through the phone. Professor Lustig then tells him that the audio is very noisy and that he is unable to truly appreciate the music. Unfortunately, Professor Maharbiz has no other means of letting Professor Lustig listen to the message. Luckily, they have you! You propose to implement a denoiser at Professor Lustig’s end.

- (a) In the IPython notebook, listen to the noisy message. Plot the time signal and comment on visible structure, if any.
- (b) Take the DFT of the signal and plot the magnitude. In a few sentences, describe what the spikes you see in the spectrum are. *Hint: take a look at the documentation for `numpy.fft.fft`. Note the `norm="ortho"` option.*
- (c) There is a simple method to denoise this signal: Simply threshold in the DFT domain! Threshold the DFT spectrum by keeping the coefficients whose absolute values lie above a certain value. Then take the inverse DFT and listen to the audio. You will be given a range of possible values to test. Write the threshold value you think works best.

Yay, Professor Maharbiz is no longer sad!

**Contributors:**

- Justin Yim.
- John Maidens.
- Siddharth Iyer.
- Brian Kilberg.