

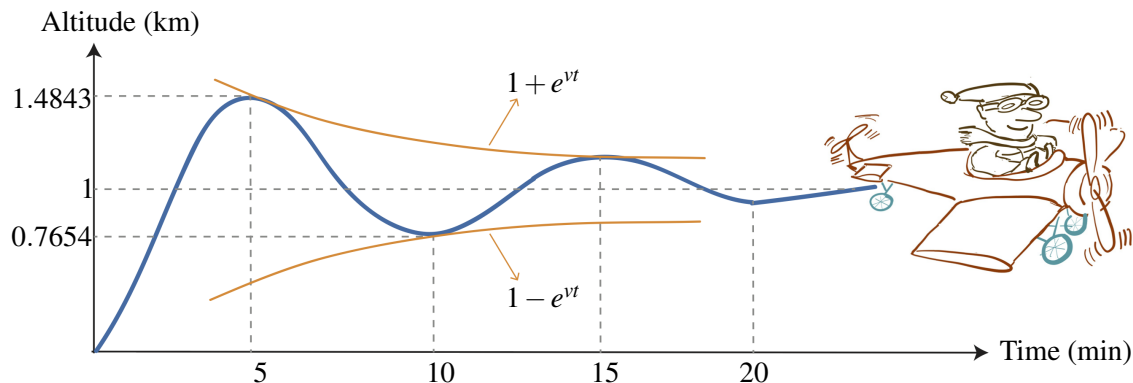
This homework is due October 17th, 2017, at noon.

1. Otto the Pilot

Otto has devised a control algorithm so that his plane climbs to the desired altitude by itself. However he is having oscillatory transients as shown in the figure. Prof. Arcaik told him that if his system has complex eigenvalues

$$\lambda_{1,2} = \nu \mp j\omega$$

then his altitude would indeed oscillate with frequency ω about the steady state value, 1 km, and that the time trace of his altitude would be tangent to the curves $1 + e^{\nu t}$ and $1 - e^{\nu t}$, near its maxima and minima respectively.



- (a) Find the real part ν and the imaginary part ω from the altitude plot.
- (b) Let the dynamical model for the altitude be

$$\frac{d}{dt} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and a_1 and a_2 are constants. Using your answer to part (a), find out what a_1 and a_2 are.

- (c) Otto can change a_2 by turning a knob. Tell him what value he should pick so that he has a “critically damped” ascent with two negative, real eigenvalues at the same location.

2. LED Strip

I have an LED strip with 5 red LEDs whose brightnesses I want to set. These LEDs are addressed as a queue: at each time step, I can push a new brightness command between 0 and 255 to the left-most LED. Each of the following LEDs will then take on the brightness previously displayed by the LED immediately to its left.

- (a) What should we use for our state vector? What does it mean that this is a state vector? What is our input?
- (b) Is our system linear? If it is linear, write out the state equations in matrix form. Please choose a reasonable order for the state variables in the state vector.
- (c) Is this system controllable? Explain intuitively what this system's controllability means in terms of LED brightnesses.
- (d) Is this system stable?
- (e) Starting from the pattern of brightnesses (from left to right) $[0, 127, 0, 255, 0]$, can we maintain this pattern for all future time steps? Can we display any fixed pattern of brightnesses for all time?

3. Controllability and discretization

In this problem, we will use the car model

$$\begin{aligned}\frac{d}{dt}p(t) &= v(t) \\ \frac{d}{dt}v(t) &= u(t)\end{aligned}$$

that was discussed in class.

- (a) Assuming that the input $u(t)$ can be varied continuously, is this system controllable?
- (b) Now assume that we can only change our control input every T seconds. Derive a discrete-time state space model for the state updates, assuming that the input is held constant between times t and $t + T$.
- (c) Is the discrete-time system controllable?

4. Controllability in circuits

Consider the circuit in Figure 1, where V_s is an input we can control:

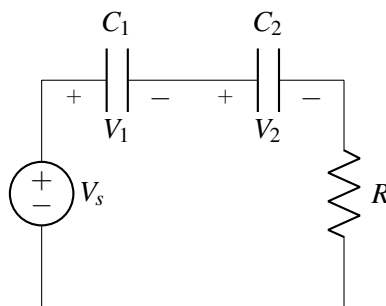


Figure 1: Controllability in circuits

- (a) Write the state space model for this circuit.
- (b) Show that this system is not controllable.
- (c) Explain, in terms of circuit currents and voltages, why this system isn't controllable. (Hint: think about what currents/voltages of the circuit we are controlling with V_s)
- (d) Draw an equivalent circuit of this system that is controllable. What quantity can you control in this system?

5. Controllability in 2D

Consider the control of some two-dimensional linear discrete-time system

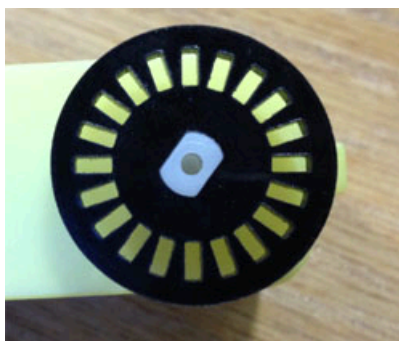
$$\vec{x}(k+1) = A\vec{x}(k) + Bu(k)$$

where A is a 2×2 real matrix and B is a 2×1 real vector.

- Let $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ with $a, c, d \neq 0$, and $B = \begin{bmatrix} f \\ g \end{bmatrix}$. Find a B such that the system is controllable no matter what nonzero values a, c, d take on, and a B for which it is not controllable no matter what nonzero values are given for a, c, d . You can use the controllability rank test, but please explain your intuition as well.
- Let $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $a, d \neq 0$. and $B = \begin{bmatrix} f \\ g \end{bmatrix}$ with $f, g \neq 0$. Is this system always controllable? If not, find configurations of nonzero a, d, f, g that make the system uncontrollable.
- We want to see if controllability is preserved under changes of coordinates. To begin with, let $\vec{z}(k) = V^{-1}\vec{x}(k)$, please write out the system equation with respect to \vec{z} .
- Now show that controllability is preserved under change of coordinates. (Hint: use the fact that $\text{rank}(MA) = \text{rank}(A)$ for any invertible matrix M .)

6. Understanding the SIXT33N car control model

As we continue along the process of making the SIXT33N cars be awesome, we'd like to better understand the car model which we will be using to develop a control scheme. As a wheel on the car turns, there is an encoder disc (see below) which also turns as the wheel turns. The encoder shines a light through the encoder disc, and as the wheel turns, the light is continually blocked and unblocked, allowing the encoder to detect rotations of the wheel per 'tick' of the encoder disc.



The following model applies separately to each motor of the car:

$$v(t) = d(t+1) - d(t) = \theta u(t) - \beta$$

Meet the variables at play in this model:

- t - The current timestep of the model. Since this is a discrete system, this will advance by 1 on every new sample in the system.
- $d(t)$ - The current number of ticks advanced by this wheel.

- $v(t)$ - The discrete-time velocity (in units of ticks/timestep) of the wheel, measured by finding the difference between two adjacent tick counts ($d(t+1) - d(t)$).
- $u(t)$ - The input to the system, in terms of "PWMs". As we will observe in lab, the circuit driving the model controls the amount of power delivered to the wheel in units of PWM intensity. This is a number between 0 and 255, where 0 means no power is delivered to the motor and 255 means maximum capable power is delivered to the motor.
- θ - In units of "ticks/PWM", this models how much more the motor turns for every increase in PWM. This is empirically measured from the car.
- β - In units of ticks/timestep, this models the effect of friction on the car (if no power is applied to the motors, we'd expect the velocities of the motors to decrease). This is empirically measured from the car.

In this problem, we will assume that the motor conforms perfectly to this model to get an intuition of how the model works.

- If we wanted to make the motor drive at a certain target velocity v^* , with what PWM $u(t)$ should we feed the motor?
- What signs should θ and β have? Should they be positive or negative? Note that applying PWM to the motor driver circuit can only ever deliver power in a way so as to cause the motor to move forwards and never backwards, and there are no braking mechanisms on the motor.
- Even if the motor conforms perfectly to the model, our inputs still limit the range of velocities of the motor. Given that $0 \leq u(t) \leq 255^1$, determine the maximum and minimum velocities possible with the motor. What does this tell us about the braking of the car?
- Our intuition tells us that a motor on a car should eventually stop turning if we stop applying any power to it. Find $v(t)$ as $t \rightarrow \infty$, assuming $v(0) = v_0$ (say $v_0 > 0$) and $u(t) = 0$. Does our model obey our intuition? What does that tell us about our model?

Stay tuned for closed-loop control of the SIXT33N motors!

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¹See <https://www.arduino.cc/en/uploads/Tutorial/pwm1.gif> for an example of how PWM works and why this is the case.