

This homework is due October 24, 2017, at Noon.

1. Buoyancy

An engineer would like to deploy an autonomous communications balloon (like Project Loon’s balloons: <https://plus.google.com/+ProjectLoon/posts/PVItgyeYweY>) to provide internet connectivity to a particular geographical region. To provide reliable connectivity, the balloon must hold its position over the region it services. The balloon can control its altitude (a) by changing its buoyancy, but it doesn’t have any engines. In order to move horizontally (horizontal position p), the balloon drifts on air currents.

Consulting meteorologists, the engineer has modeled the air currents around the desired balloon position (the point (0,0)) and found the flow field shown in Figure 2.



Figure 1: Project Loon Balloon

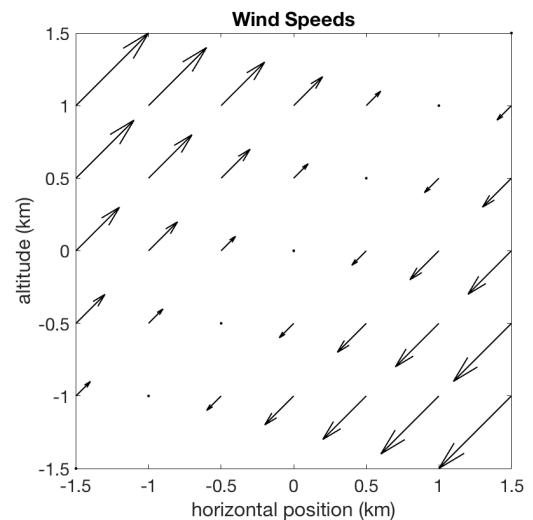


Figure 2: Wind speeds

where the wind speed at each point is described by the equations:

$$v_p = -20p + 20a$$

$$v_a = -20p + 20a$$

where the velocities are in kilometers per hour and the horizontal position and altitude are in kilometers. Putting this together with the balloon’s buoyancy control, the balloon’s dynamics are described by:

$$\begin{bmatrix} \dot{p} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} -20 & 20 \\ -20 & 20 \end{bmatrix} \begin{bmatrix} p \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$

- (a) Write the dynamics equation in controllable canonical form.
- (b) What is the matrix T for the change of variables $\vec{z} = T\vec{x}$ that transforms the original A and B matrices into controllable canonical form?
- (c) The engineer would like the balloon to converge to (0,0) with eigenvalues -1 and -1. What should be the state feedback gains \vec{K} multiplying the original state vector \vec{x} to achieve this behavior? Write the expression for u in terms of p and a .

2. Design for controllability and observability I

We are given a system

$$\begin{aligned}\dot{\vec{x}}(t) &= \begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \end{bmatrix} u(t)\end{aligned}$$

with tuneable parameter γ .

- (a) How should we tune γ to make the system controllable but not observable?
- (b) How should we tune γ to make the system observable but not controllable?

3. Design for controllability and observability II

We are given a new system

$$\dot{\vec{x}}(t) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \vec{x}(t).$$

along with only one sensor and one actuator to control and observe the system.

- (a) Which state should we control with the actuator to make the system controllable?
- (b) Which state should we measure with the sensor to make the system observable?

4. Observability

Consider the following continuous time system.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{1}$$

We want to construct an estimate z of the system state x . To do so, we construct a pretend system with the same $[A, B, C, D]$ models, the same input and the output of the last system along with an L system matrix. We do this to try and exploit the difference between the output of our pretend state and the actual output, with L being the "knob" that we can control.

$$\dot{z}(t) = Az(t) + Bu(t) + L(Cz(t) - y(t))\tag{2}$$

Define $e(t) = z(t) - x(t)$. This is the error term as a function of time.

- (a) Using the two systems defined above, construct a system of the form,

$$\frac{de}{dt}(t) = (A + LC)e(t)\tag{3}$$

(b) We want,

$$\lim_{t \rightarrow \infty} e(t) = 0$$

What does that imply about (3)?

(c) Does the initial value of the guess $z(0)$ matter in the long term?

5. CCR circuit

Consider the circuit below driven by a current source with current $u(t)$. The output $y(t)$ is the voltage across the resistor and the state variables are the capacitor voltages as marked in the circuit diagram.

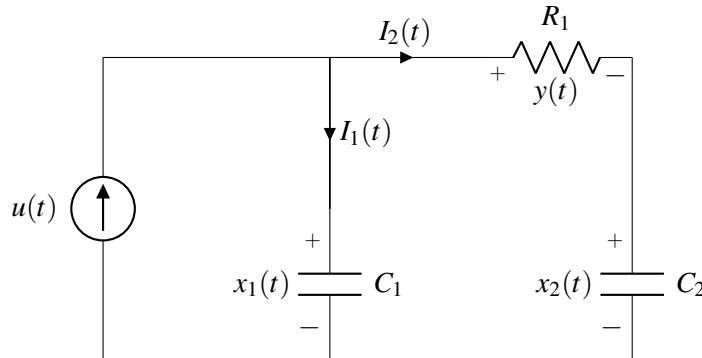


Figure 3: Two Capacitor Circuit with Current Source

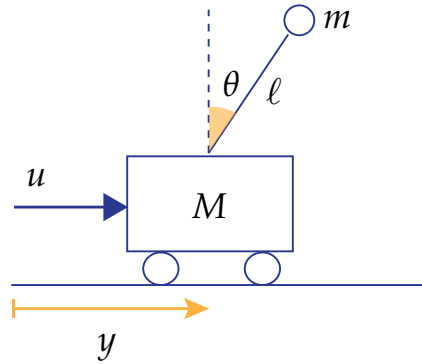
- Write a state model for this circuit.
- Find all equilibrium points when $u(t) = 0$ for all t .
- Determine if the system is controllable.
- Determine if the system is observable.
- If your answer to part (c) or (d) is no, explain the physical reason for lack of controllability or observability, whichever is applicable.

6. Inverted pendulum

Consider the inverted pendulum depicted below, whose equations of motion are

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2 \theta} \left(\frac{u}{m} + \dot{\theta}^2 \ell \sin \theta - g \sin \theta \cos \theta \right)$$

$$\ddot{\theta} = \frac{1}{\ell \left(\frac{M}{m} + \sin^2 \theta \right)} \left(-\frac{u}{m} \cos \theta - \dot{\theta}^2 \ell \cos \theta \sin \theta + \frac{M+m}{m} g \sin \theta \right).$$



- (a) Write the state model using the variables $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, and $x_3(t) = \dot{y}(t)$. We do not include $y(t)$ as a state variable because we are interested in stabilizing the point $\theta = 0$, $\dot{\theta} = 0$, $\dot{y} = 0$, and we are not concerned about the final value of the position $y(t)$.
- (b) Linearize this model at the equilibrium $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and indicate the resulting A and B matrices.
- (c) Show that the linearized model is controllable.
- (d) Suppose $M = 1$, $m = 0.1$, $l = 1$, and $g = 10$, and design a state feedback controller,

$$u(t) = k_1 \theta(t) + k_2 \dot{\theta}(t) + k_3 \dot{y}(t),$$

such that the eigenvalues of $A + BK$ (the "closed-loop eigenvalues") are $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

- (e) Suppose we set $k_2 = k_3 = 0$ and vary only k_1 ; that is, the controller uses only $\theta(t)$ for feedback. Does there exist a k_1 value such that all closed-loop eigenvalues have negative real parts?

7. Open-loop control of SIXT33N

Last time, we learned that the ideal input PWM for running a motor at a target velocity v^* is:

$$u(t) = \frac{v^* + \beta}{\theta}$$

In this problem, we will extend our analysis from one motor to a two-motor car system and evaluate how well our open-loop control scheme does.

$$\begin{aligned} v_L(t) &= d_L(t+1) - d_L(t) = \theta_L u_L(t) - \beta_L \\ v_R(t) &= d_R(t+1) - d_R(t) = \theta_R u_R(t) - \beta_R \end{aligned}$$

- (a) In reality, we need to "kickstart" electric motors with a pulse in order for them to work. That is, we can't go straight from 0 to our desired input signal for $u(t)$, since the motor needs to overcome its initial inertia in order to operate in accordance with our model.

Let us model the pulse as having a width (in timesteps) of t_p . In order to model this phenomenon, we can say that $u(t) = 255$ for $t \in [0, t_p - 1]$ ¹. In addition, the car initially (at $t = 0$) hasn't moved, so we can also say $d(0) = 0$.

Firstly, let us examine what happens to d_L and d_R at $t = t_p$, that is, after the kickstart pulse has passed. Find $d_L(t_p)$ and $d_R(t_p)$. (Hint: if it helps, try finding $d_L(1)$ and $d_R(1)$ first, and then generalizing to the t_p case.)

Note: it is very important that you distinguish θ_L and θ_R , as the motors we have are liable to vary in their parameters, just as how real resistors vary from their ideal resistance.

- (b) Let us define $\delta(t) = d_L(t) - d_R(t)$ as the difference in positions between the two wheels. If both wheels of the car are going at the same velocity, then this difference δ should remain constant, since no wheel will advance by more ticks than the other. As a result, this will be useful in our analysis and in designing our control schemes.

Find $\delta(t_p)$. For both an ideal car ($\theta_L = \theta_R$, $\beta_L = \beta_R$) where both motors are perfectly ideal and a non-ideal car ($\theta_L \neq \theta_R$, $\beta_L \neq \beta_R$), did the car turn compared to before the pulse?

(Since $d(0) = d_L(0) = d_R(0) = 0$, $\delta(0) = 0$.)

- (c) We can still declare victory, though, even if the car turns a little bit during the initial pulse (t_p will be very short in lab), so long as the car continues to go straight afterwards when we apply our control scheme; that is, as long as $\delta(t \rightarrow \infty)$ converges to a constant value (as opposed to going to $\pm\infty$ or oscillating).

Let's try applying the open-loop control scheme we learned last week to each of the motors independently, and see if our car still goes straight.

$$u_L(t) = \frac{v^* + \beta_L}{\theta_L}$$

$$u_R(t) = \frac{v^* + \beta_R}{\theta_R}$$

Let $\delta(t_p) = \delta_0$. Find $\delta(t)$ for $t \geq t_p$ in terms of δ_0 . (Hint: as in part (a), if it helps you, try finding $\delta(t_p + 1)$, $\delta(t_p + 2)$, etc and generalize to the $\delta(t)$ case.)

Does $\delta(t \rightarrow \infty)$ change from δ_0 ? Why or why not?

- (d) Unfortunately, in real life, it is hard to capture the precise parameters of the car motors like θ and β , and even if we did manage to capture them, they could vary as a function of temperature, time, wheel conditions, battery voltage, etc. In order to model this effect of **model mismatch**, we consider model mismatch terms (such as $\Delta\theta_L$) which reflects the discrepancy between the model parameters and actual parameters.

$$v_L(t) = d_L(t+1) - d_L(t) = (\theta_L + \Delta\theta_L)u_L(t) - (\beta_L + \Delta\beta_L)$$

$$v_R(t) = d_R(t+1) - d_R(t) = (\theta_R + \Delta\theta_R)u_R(t) - (\beta_R + \Delta\beta_R)$$

Let us try applying the open-loop control scheme again to this new system. Note that **no model mismatch terms appear below** - this is intentional, since we our control scheme is derived from the

¹ $x \in [a, b]$ means that x goes from a to b inclusive.

model parameters for θ , β , not from the actual $\theta + \Delta\theta$, etc.²

$$u_L(t) = \frac{v^* + \beta_L}{\theta_L}$$
$$u_R(t) = \frac{v^* + \beta_R}{\theta_R}$$

As before, let $\delta(t_p) = \delta_0$. Find $\delta(t)$ for $t \geq t_p$ in terms of δ_0 .

Does $\delta(t \rightarrow \infty)$ change from δ_0 ? Why or why not, and how is it different from the previous case of no model mismatch?

You may have noticed that open-loop control is insufficient in light of non-idealities and mismatches. Next time, we will analyze a more powerful form of control (closed-loop control) which should be more robust against these kinds of problems.

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²Why not just do a better job of capturing the parameters, one may ask? Well, as noted above, the mismatch can vary as a function of an assortment of factors including temperature, time, wheel conditions, battery voltage, and it is not realistic to try to capture the parameters under every possible environment, so it is up to the control designer to ensure that the system can tolerate a reasonable amount of mismatch.