EE16B
Designing Information Devices and Systems II

Lecture 12B
Finite Sequences
Complex Inner Products, and Basis Transformation

Properties of D.T. Systems

$$y[n] = F\{x[n]\}$$

- Time Invariance: Shifted input ⇒ shifted output

$$y[n - n_0] = F\{x[n - n_0]\}$$

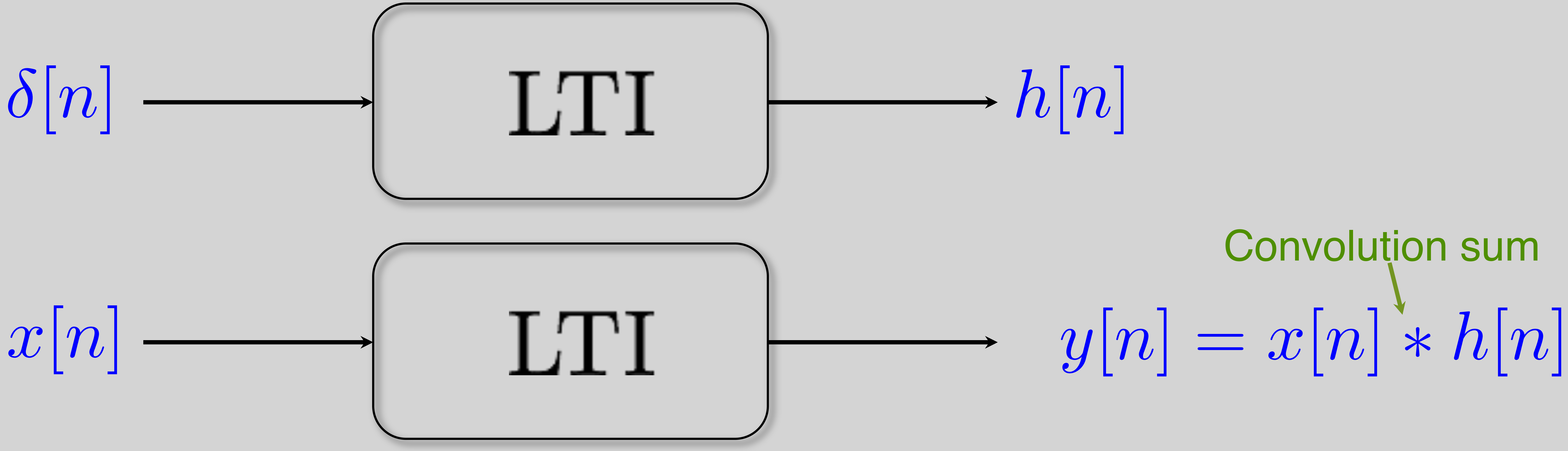
Time Invariant?

$$\vec{x}[n+1] = A\vec{x}[n] + Bu[n]$$
$$y[n] = C\vec{x}[n]$$

$$y[n] = CA^n\vec{x}[0] + CBu[n-1] + CABu[n-2] + \cdots + CA^{n-1}Bu[0]$$

Linear Time Invariant Systems

- Linear Time Invariant (LTI) systems are completely characterized by their impulse response h[n]



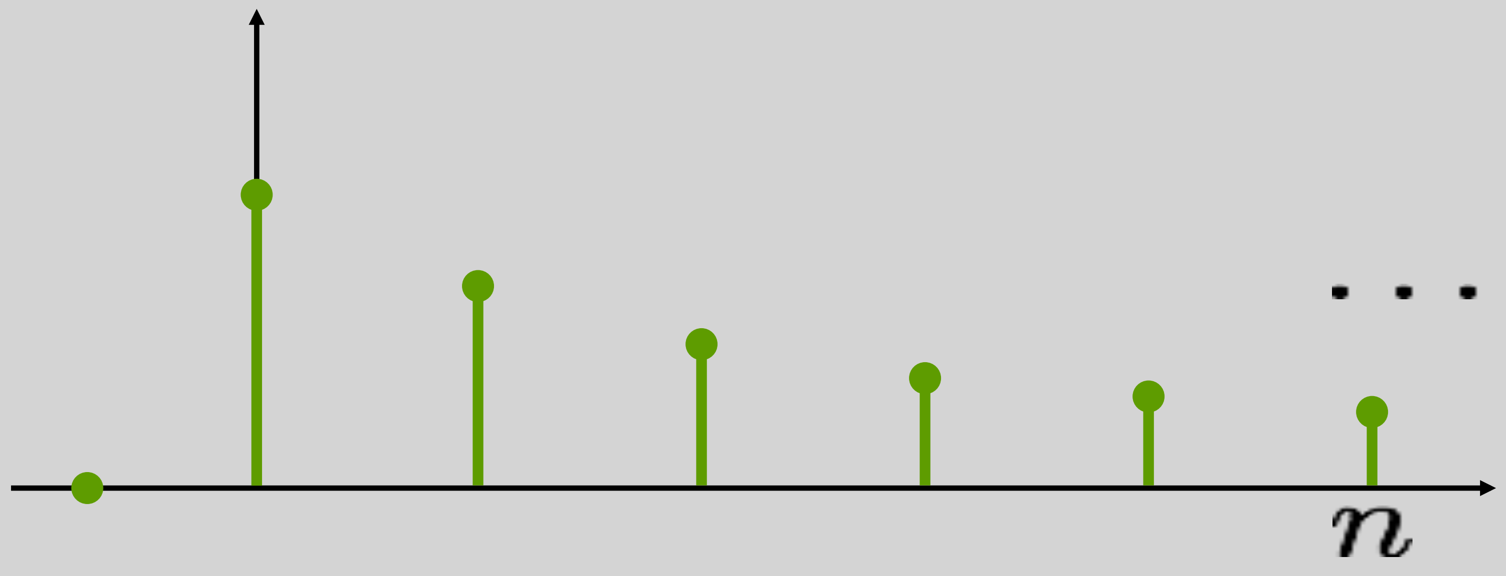
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] * h[n]$$

Example:

$$y[n] = ay[n-1] + x[n]$$

$$h[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

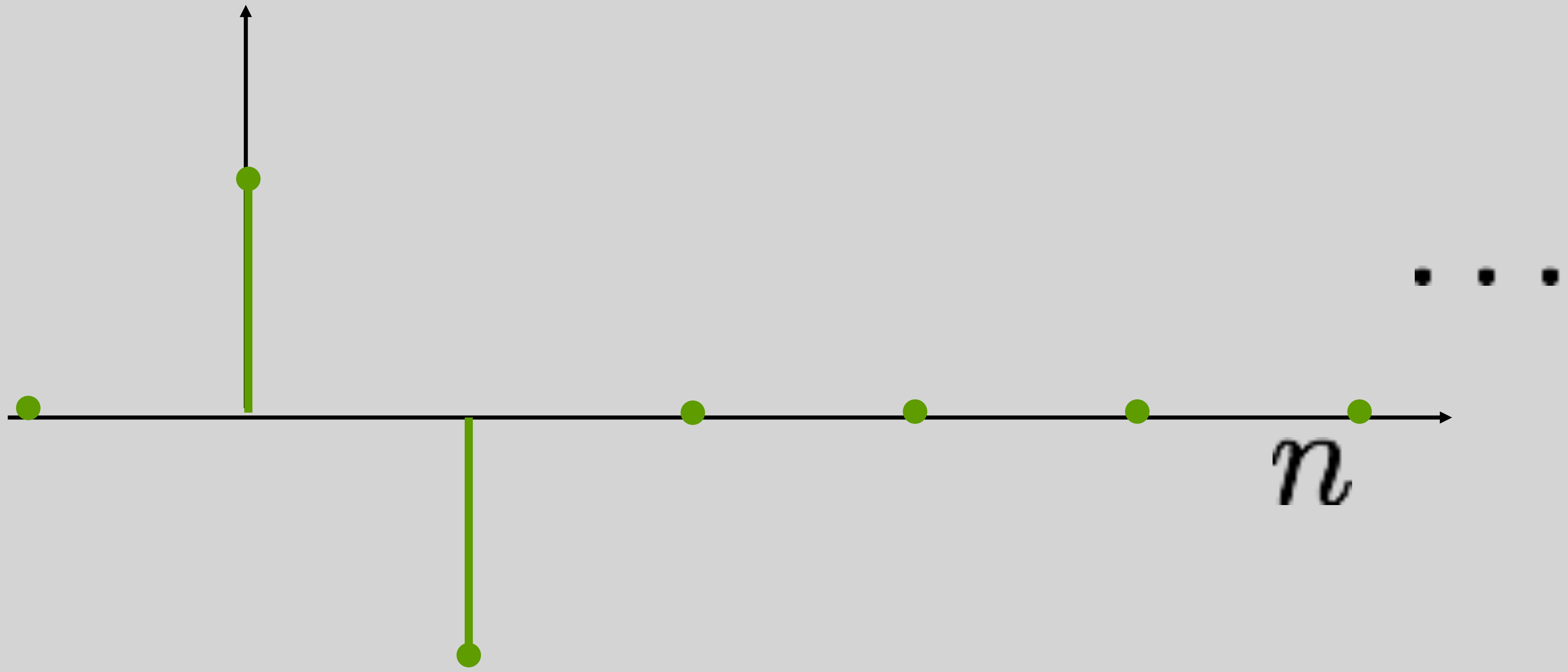
Infinite impulse response (IIR)



$$y[n] = x[n] - x[n-1]$$

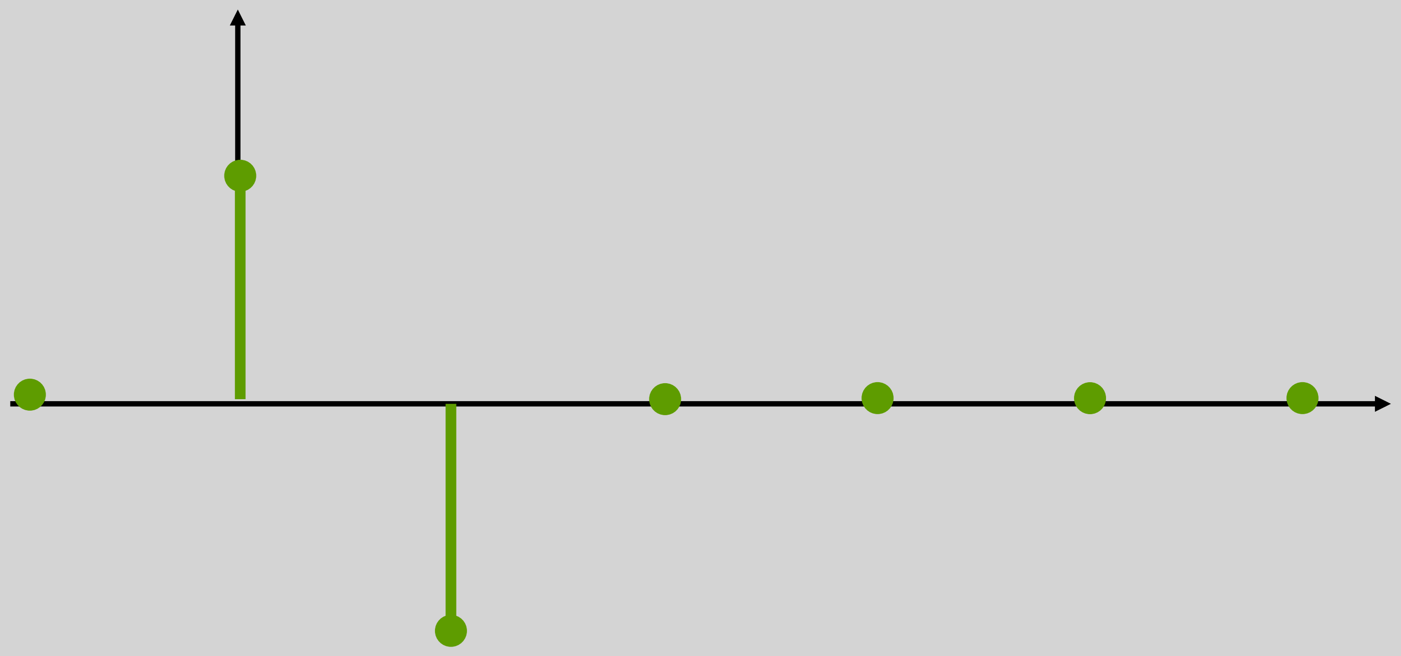
$$h[n] = \delta[n] - \delta[n-1]$$

finite impulse response (FIR)



Convolution Sum

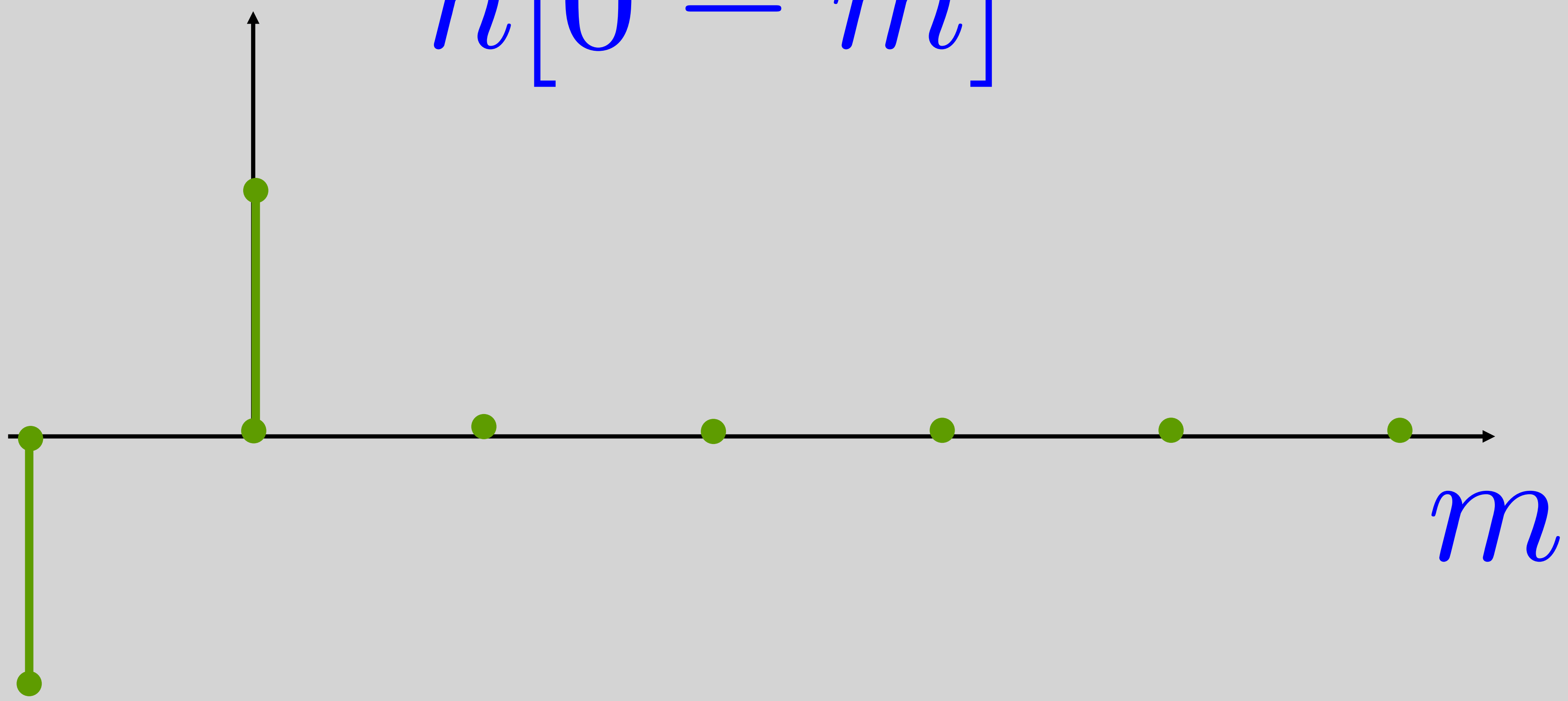
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$



- What is h[n-m] for different n's?

$$h[n] = \delta[n] - \delta[n-1]$$

$$h[0-m]$$



Convolution Sum

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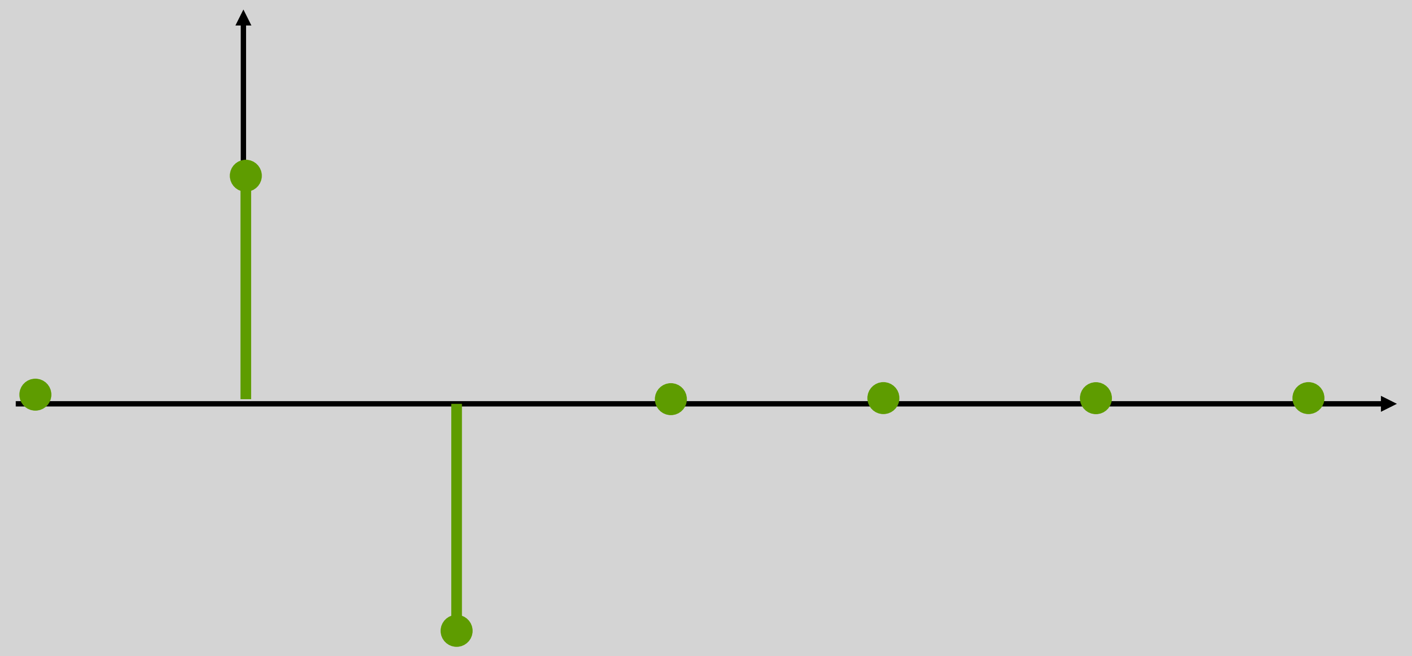
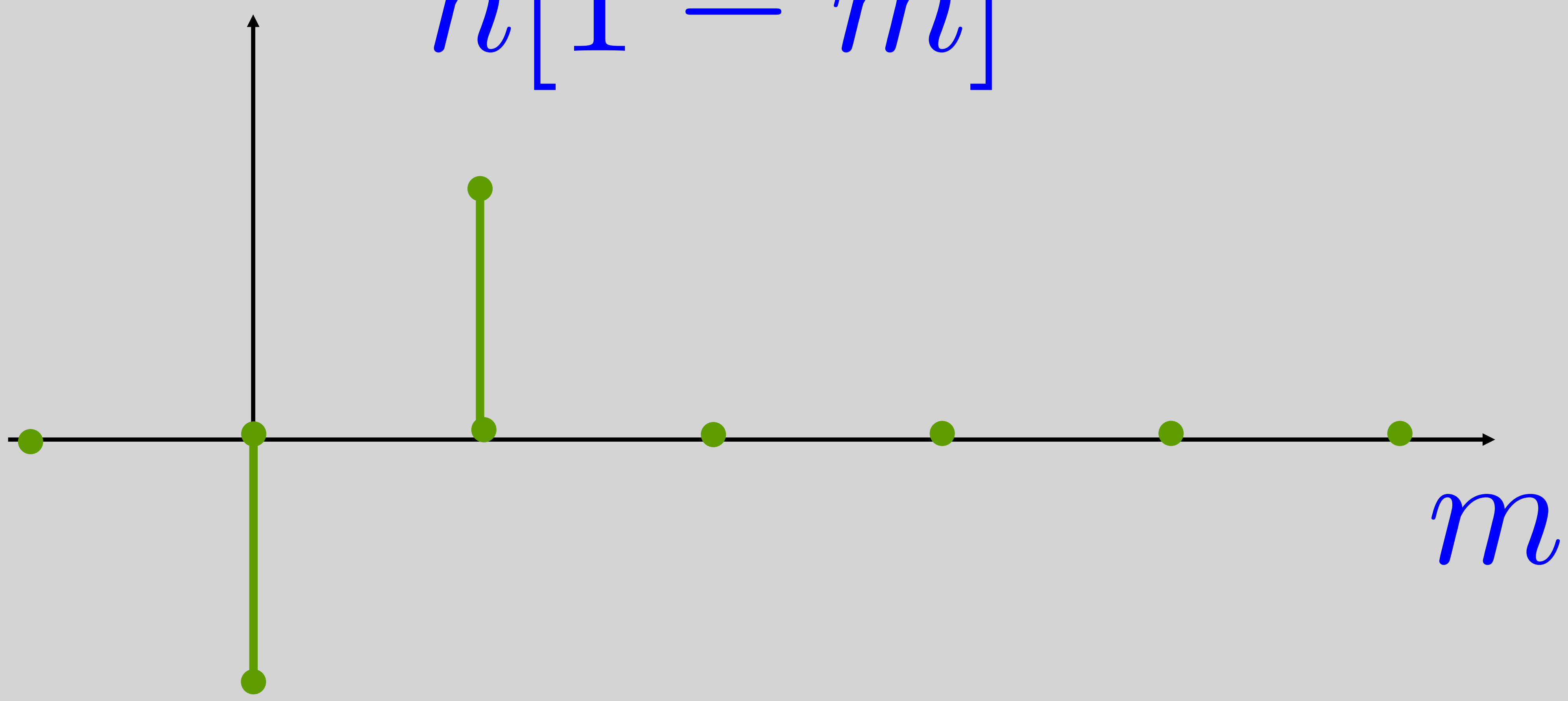


$$h[n] = \delta[n] - \delta[n-1]$$

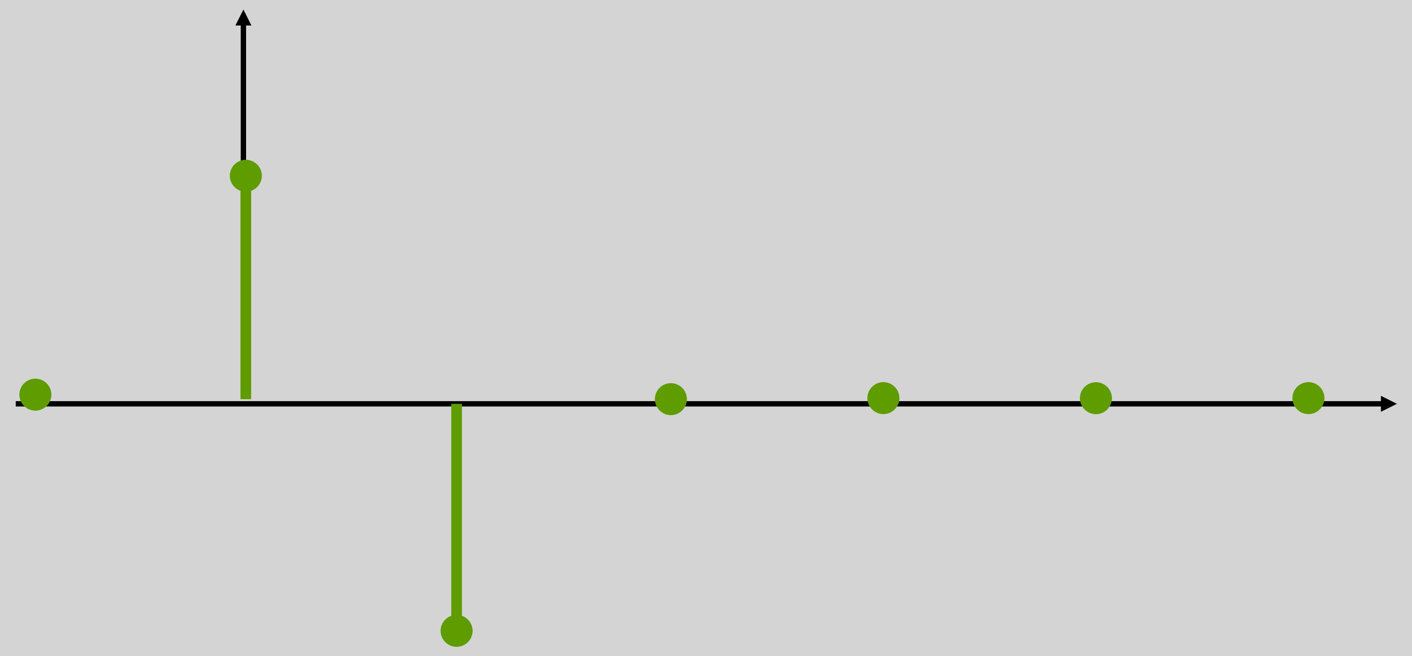
- What is h[n-m] for different n's?

$$h[1-m]$$



Convolution Sum

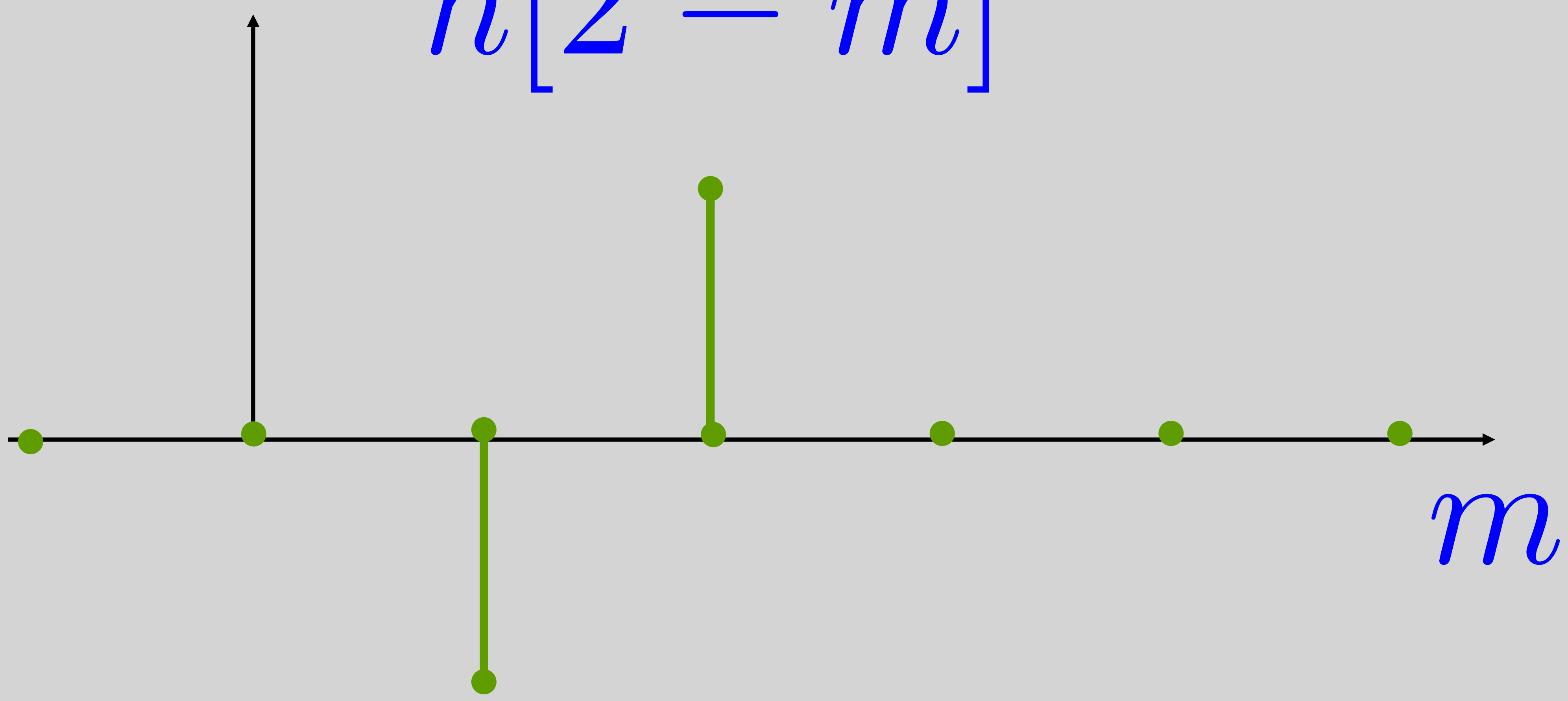
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$



- What is h[n-m] for different n's?

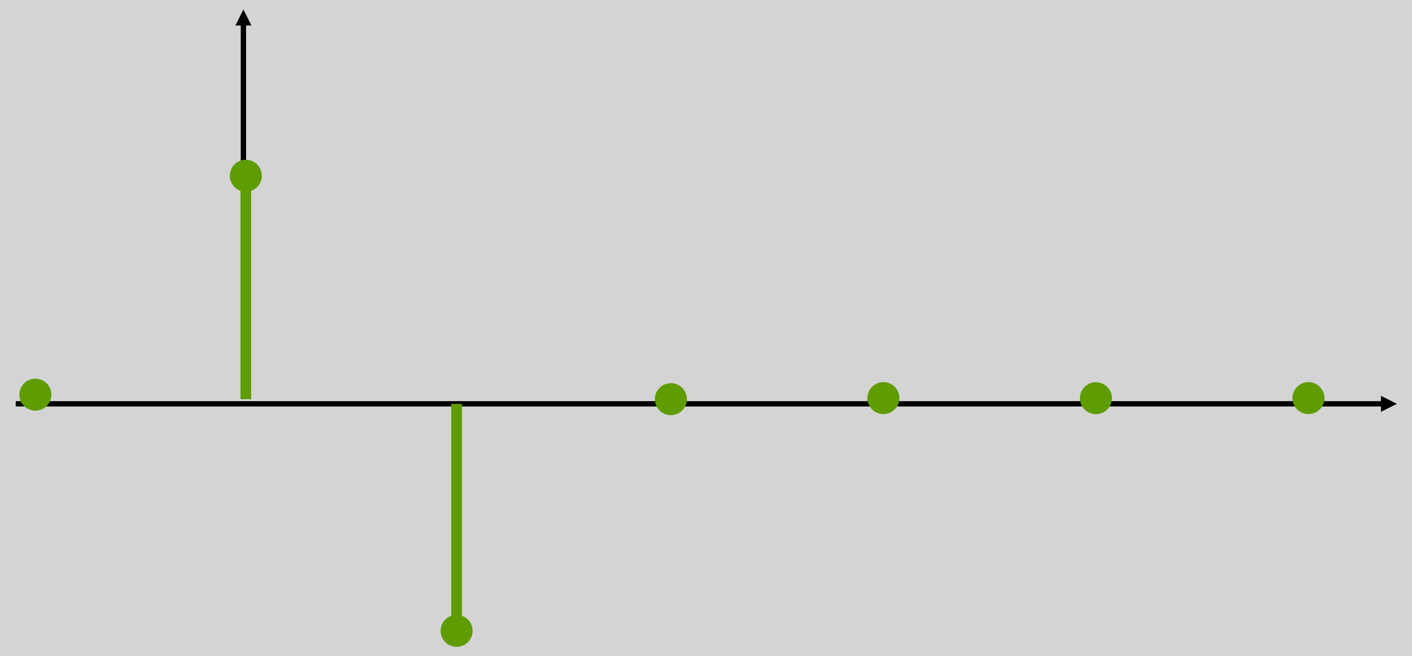
$$h[n] = \delta[n] - \delta[n-1]$$

$$h[2-m]$$



Convolution Sum

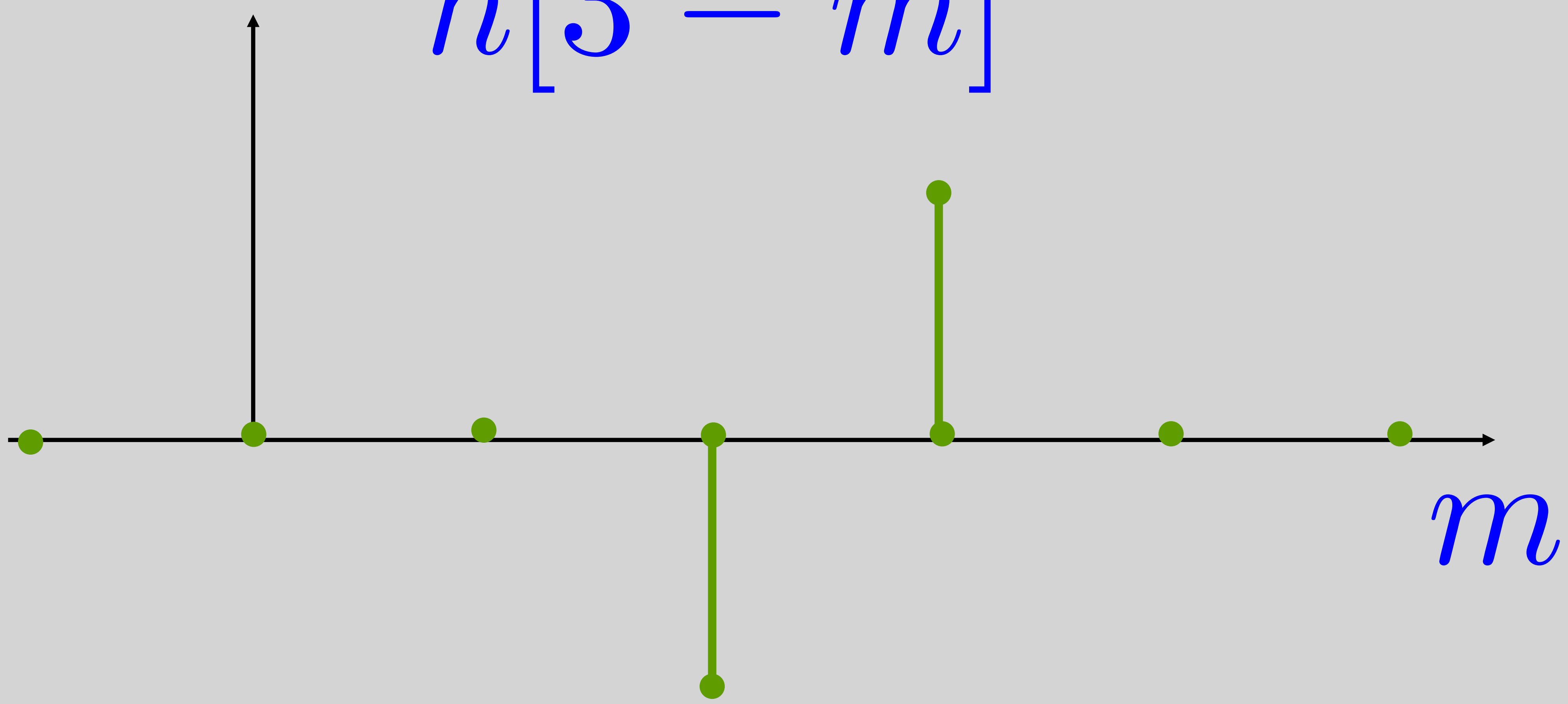
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$



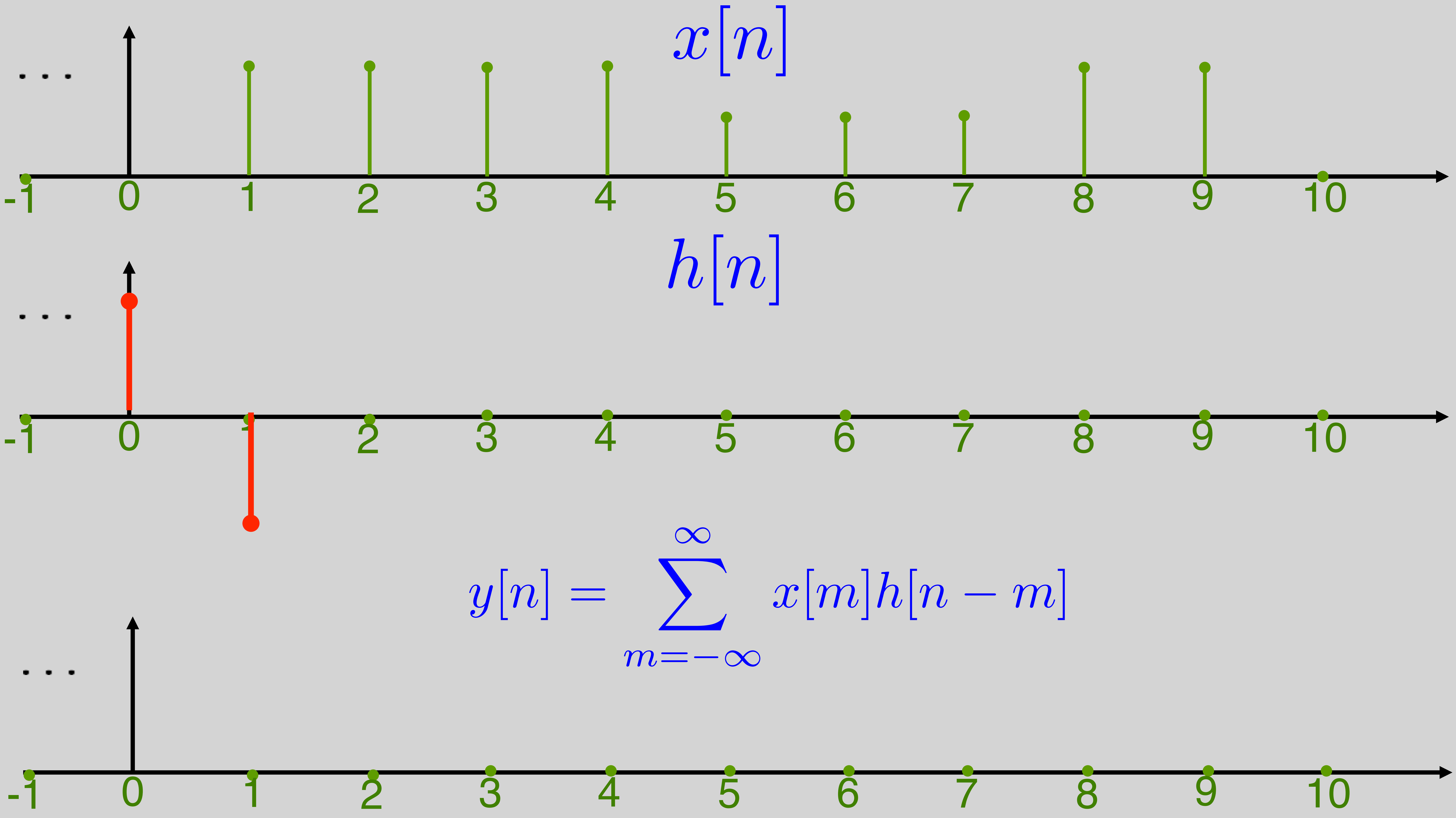
$$h[n] = \delta[n] - \delta[n-1]$$

- What is h[n-m] for different n's?

$$h[3-m]$$



Graphical Example of Convolution



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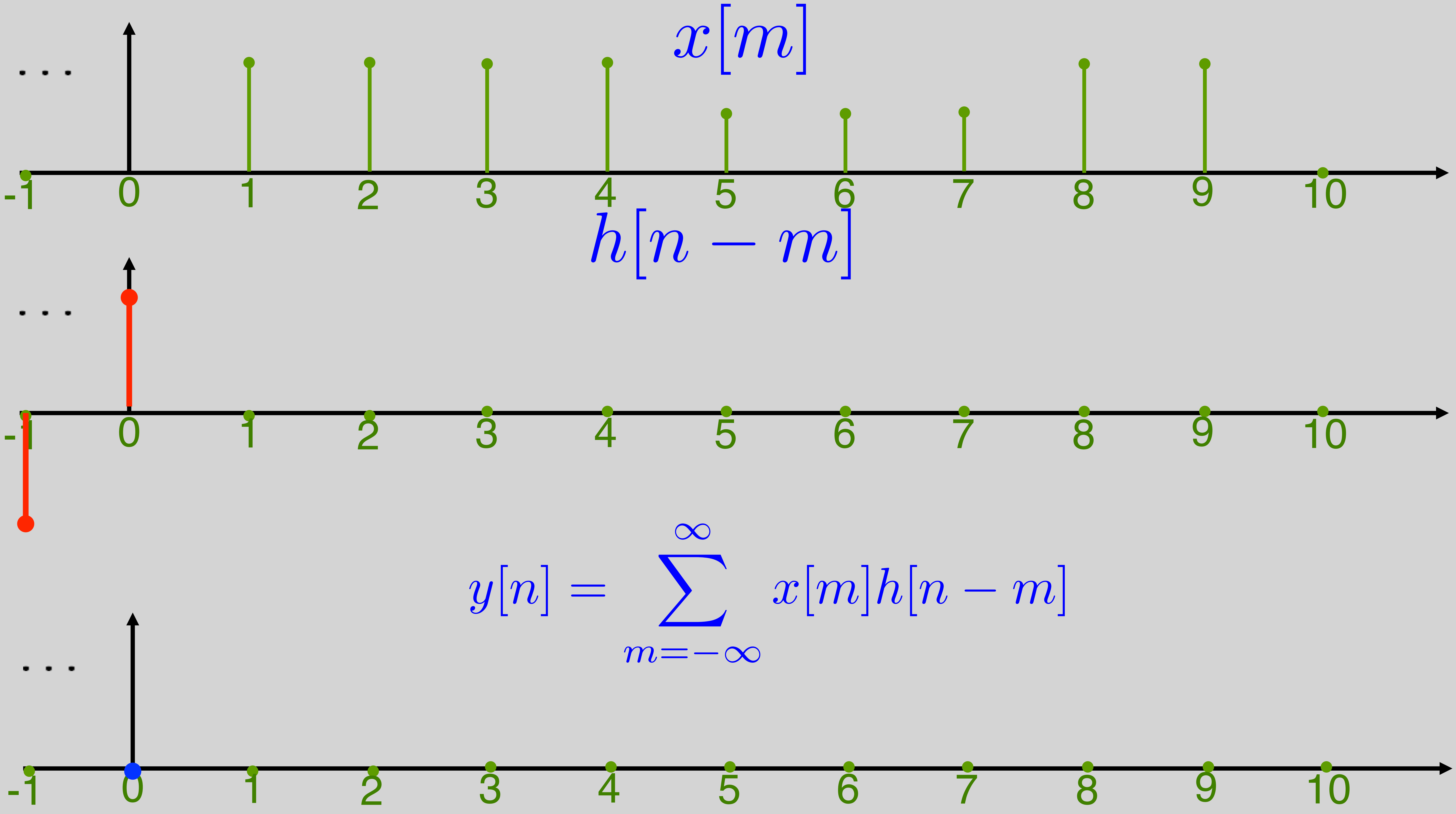
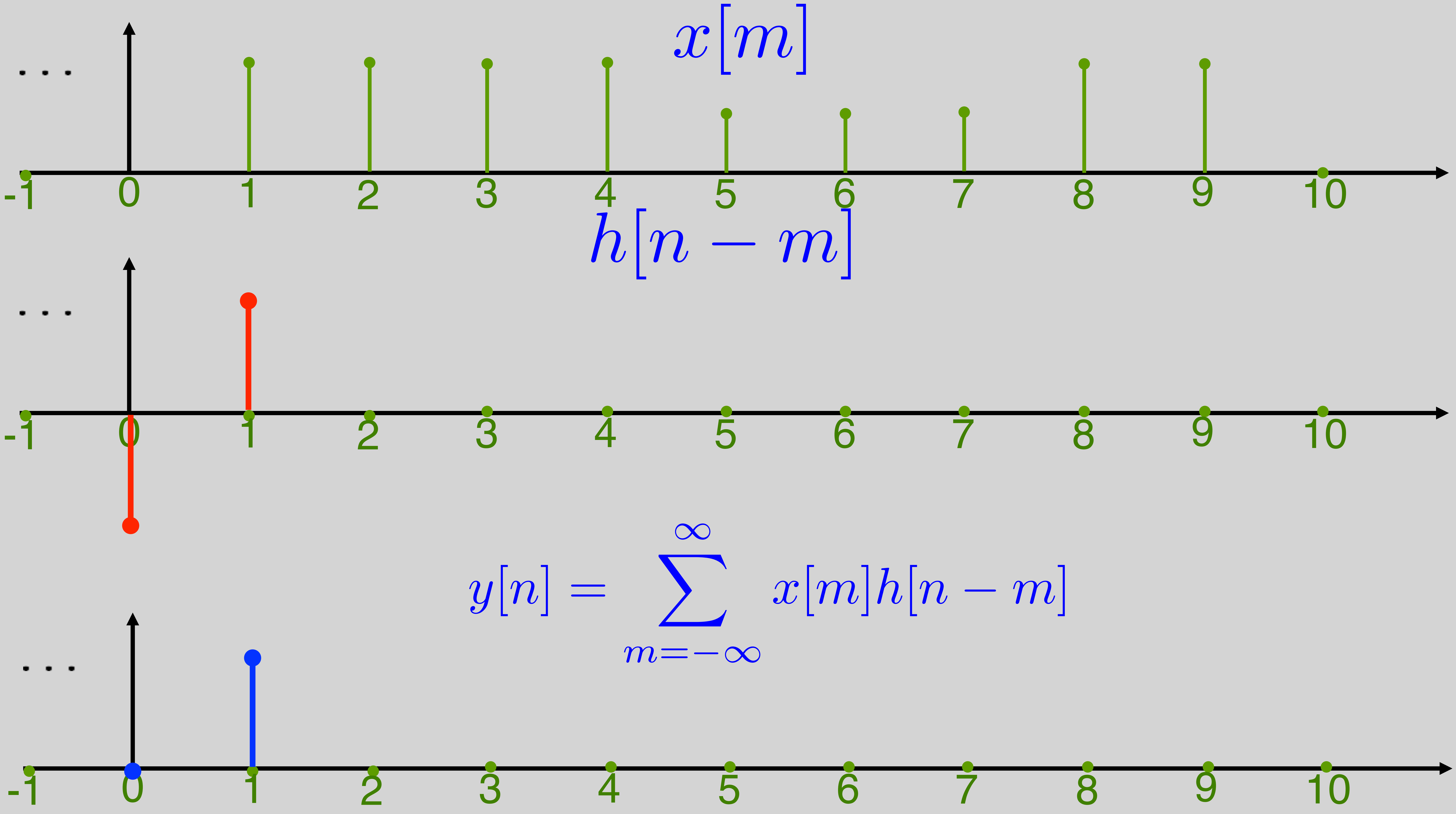


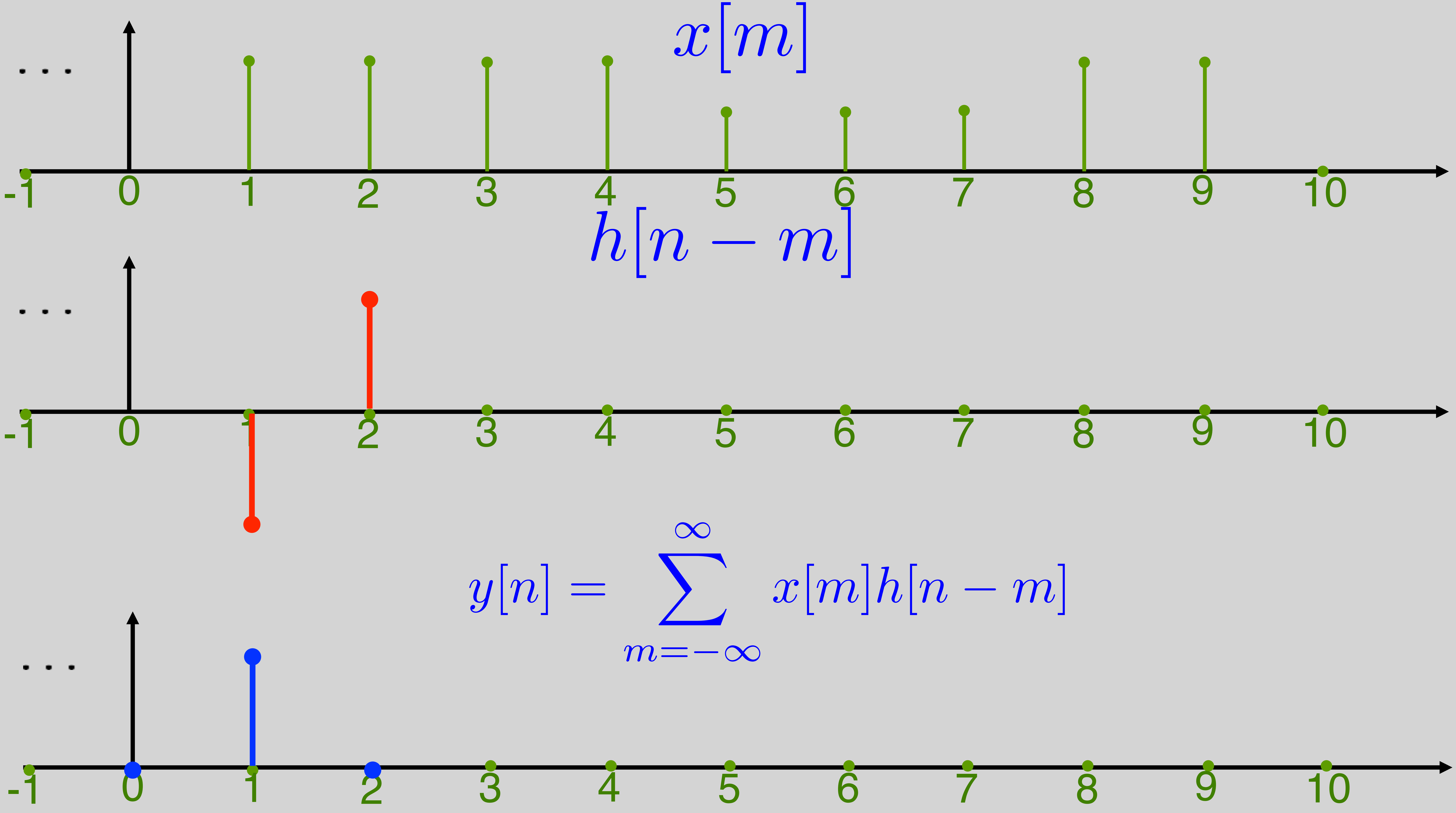
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



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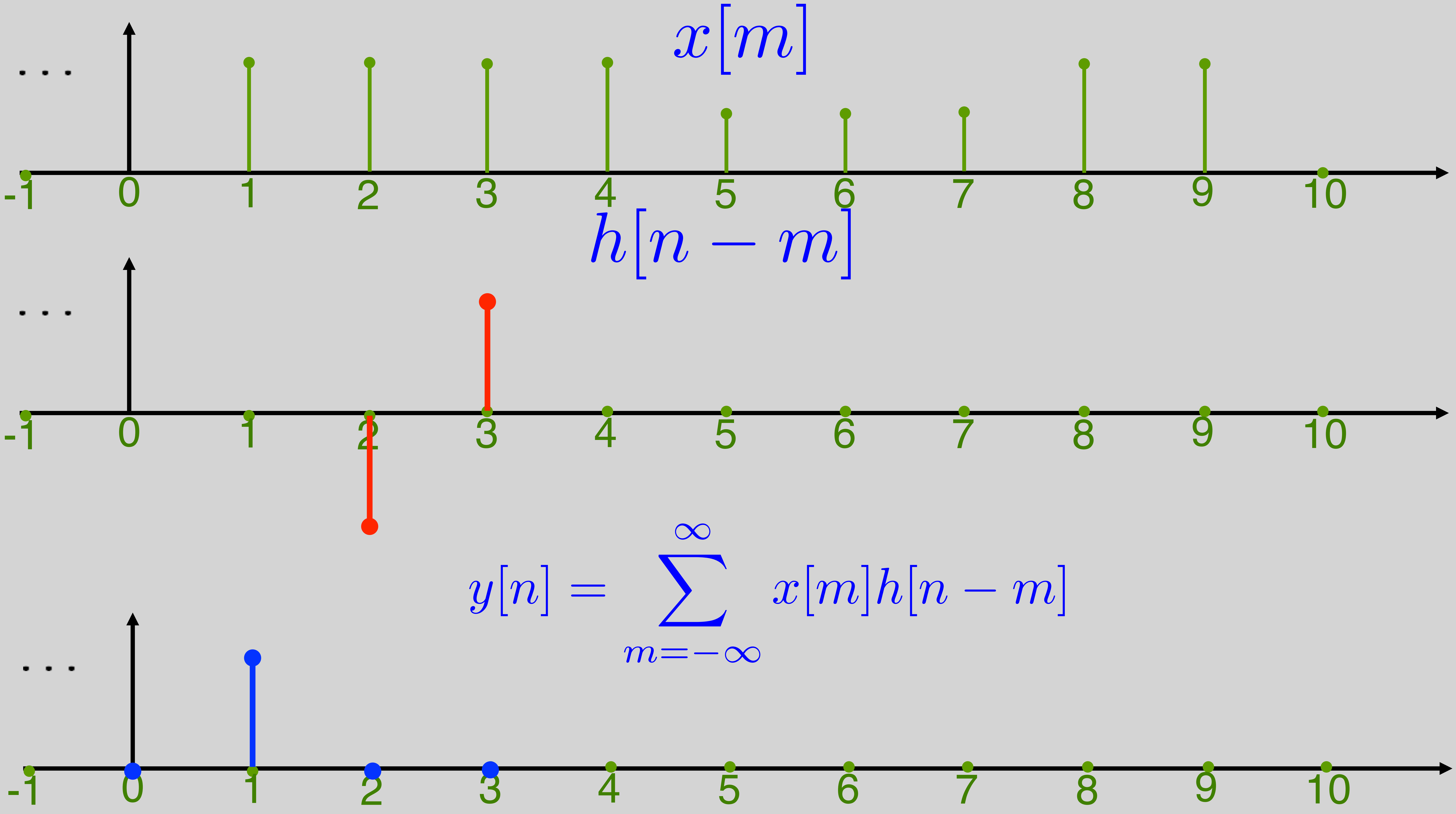

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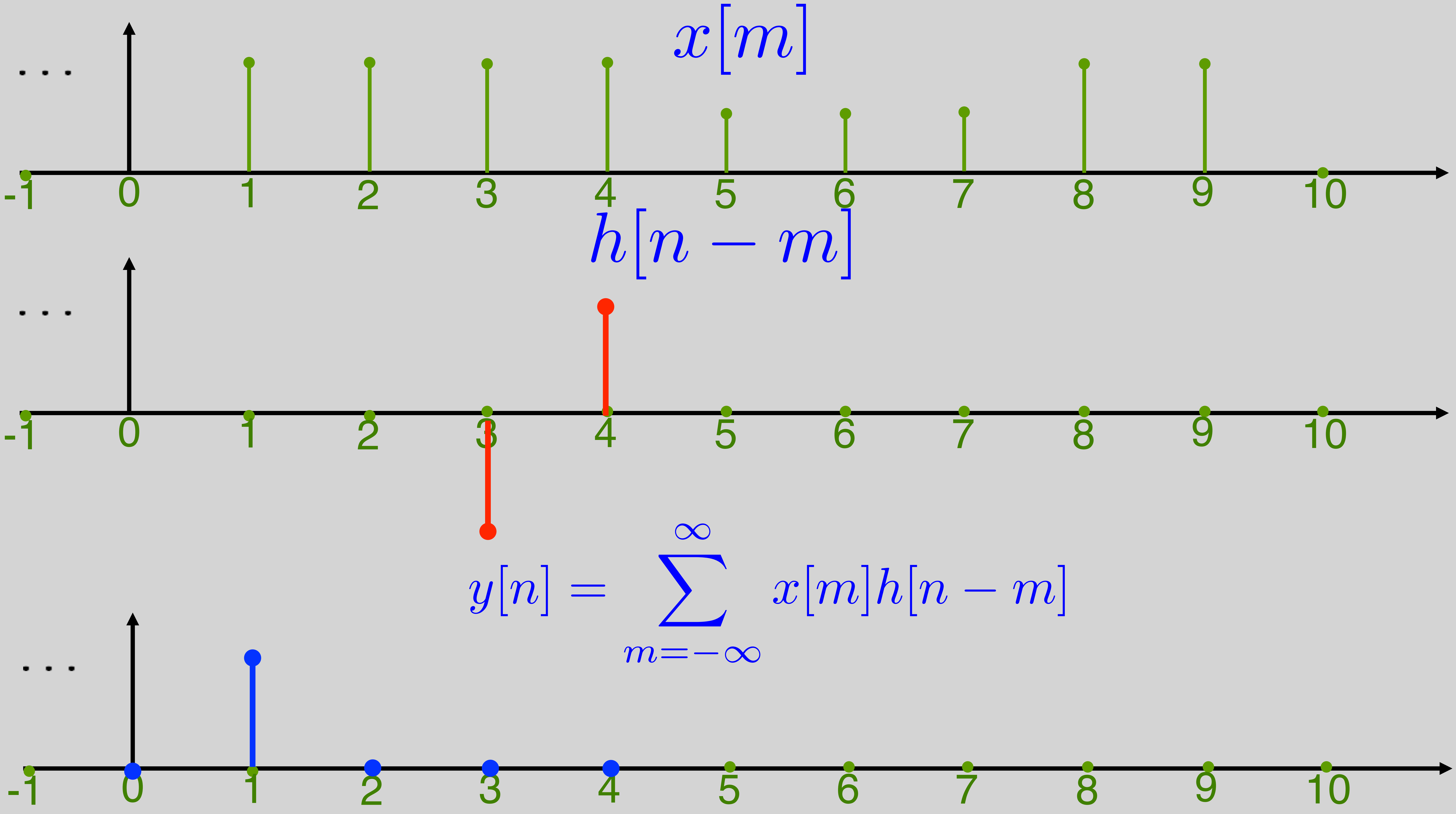
Graphical Example of Convolution



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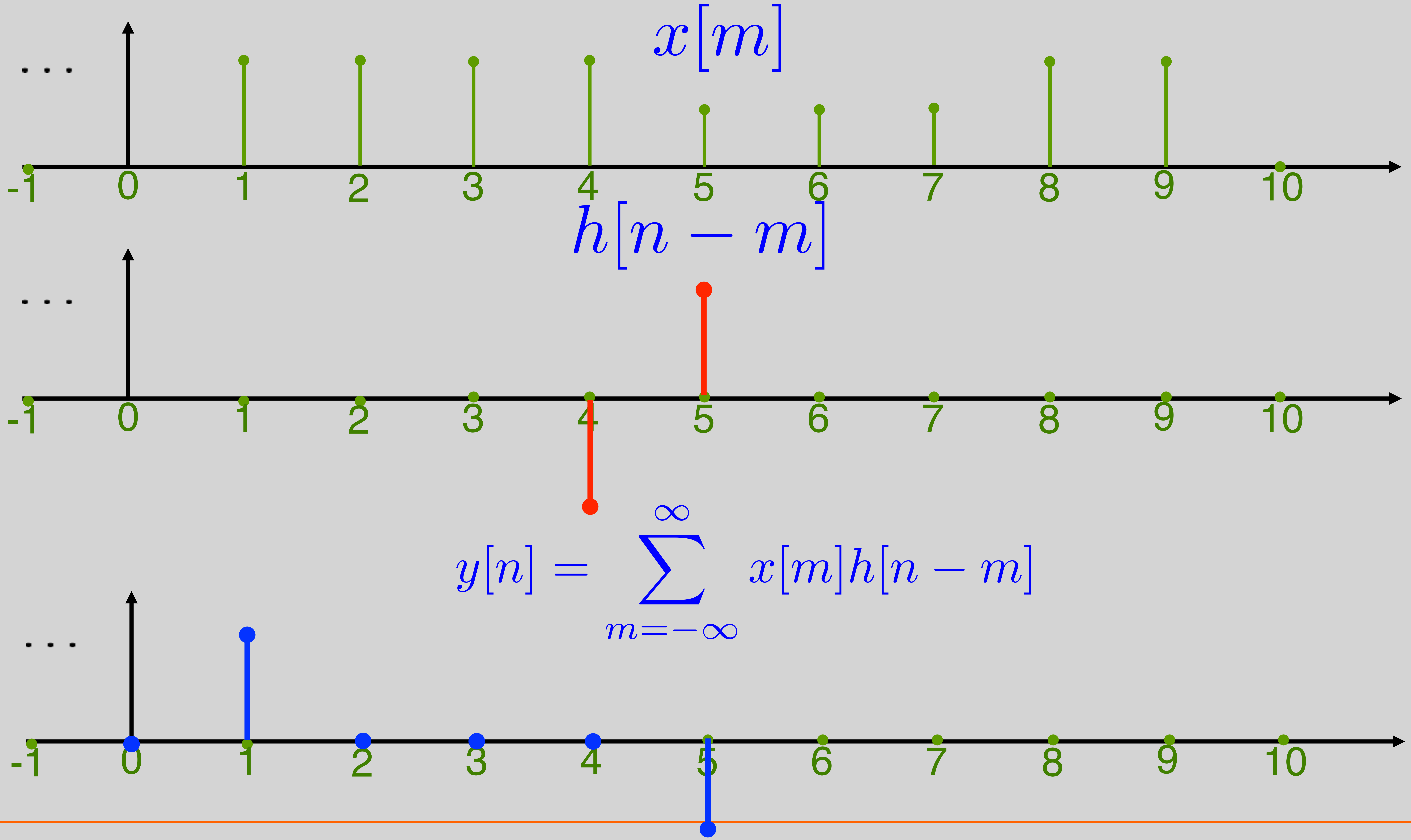
Graphical Example of Convolution



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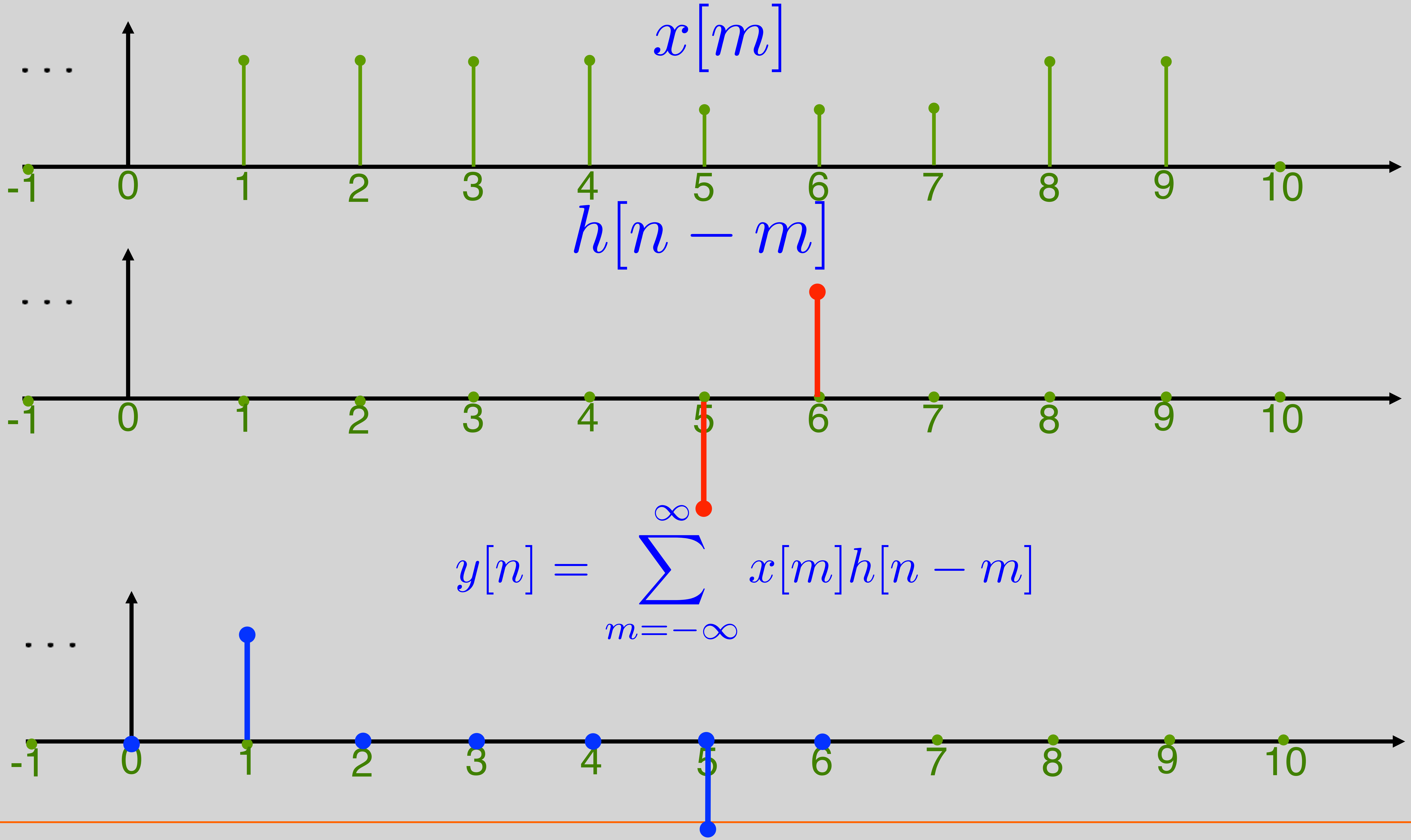
Graphical Example of Convolution



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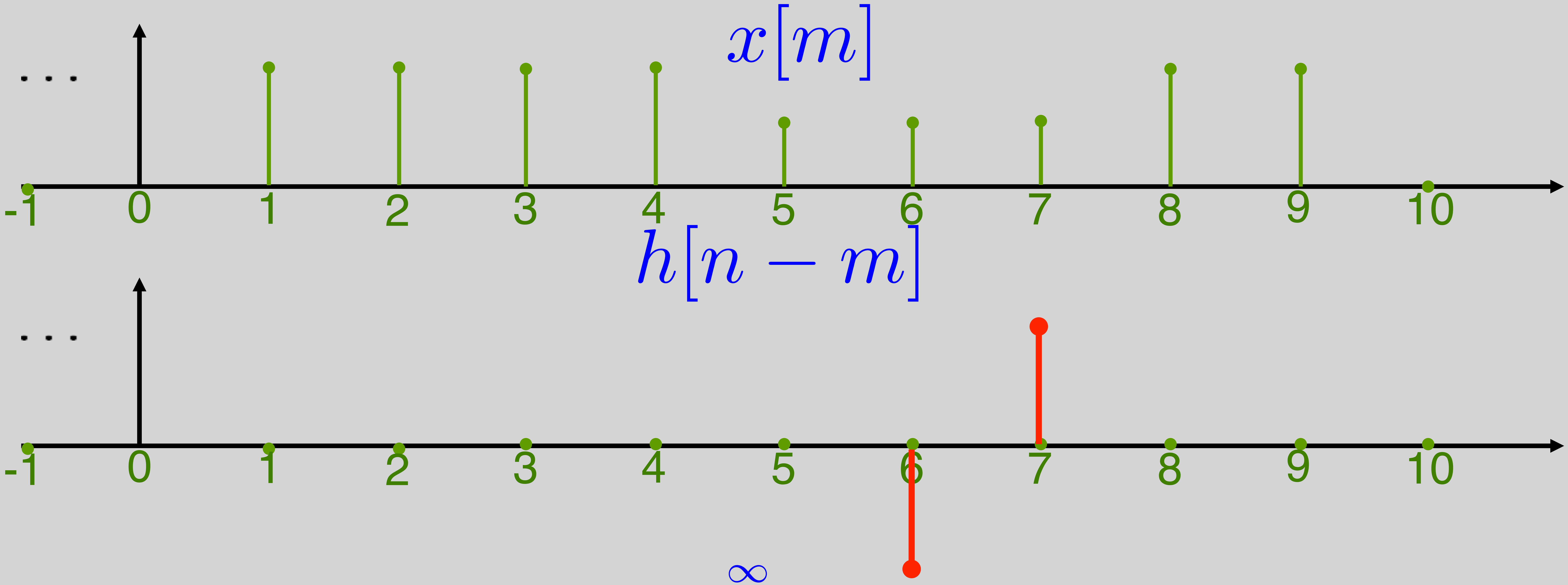
Graphical Example of Convolution



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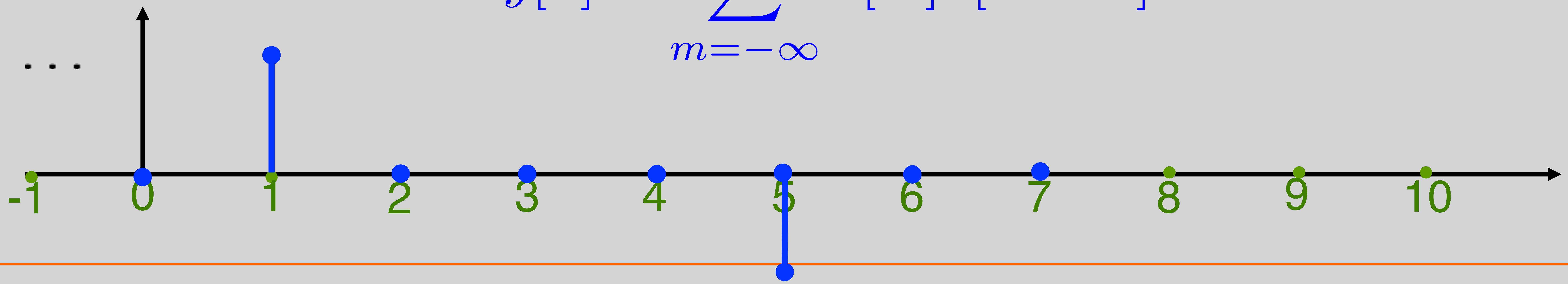
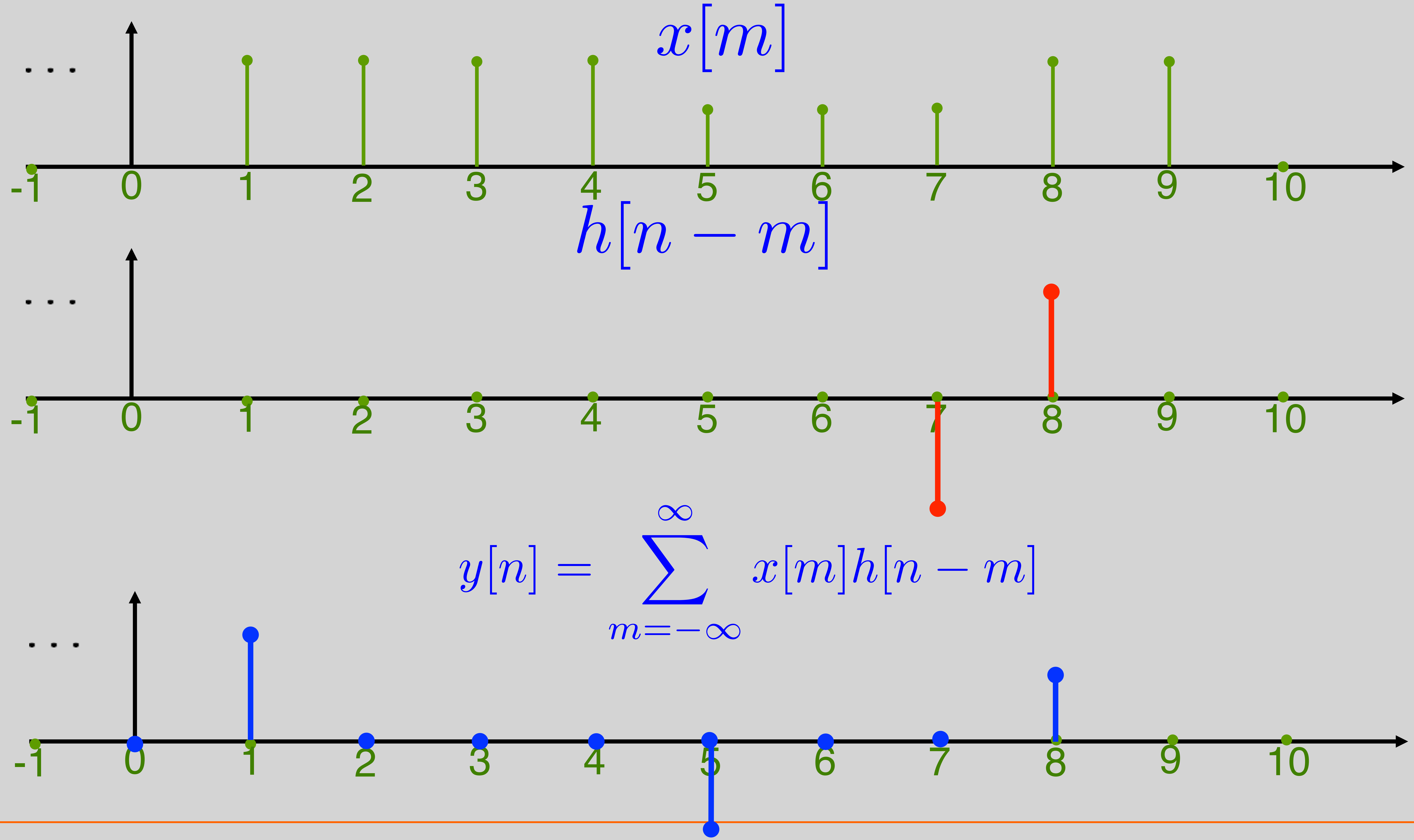


Graphical Example of Convolution



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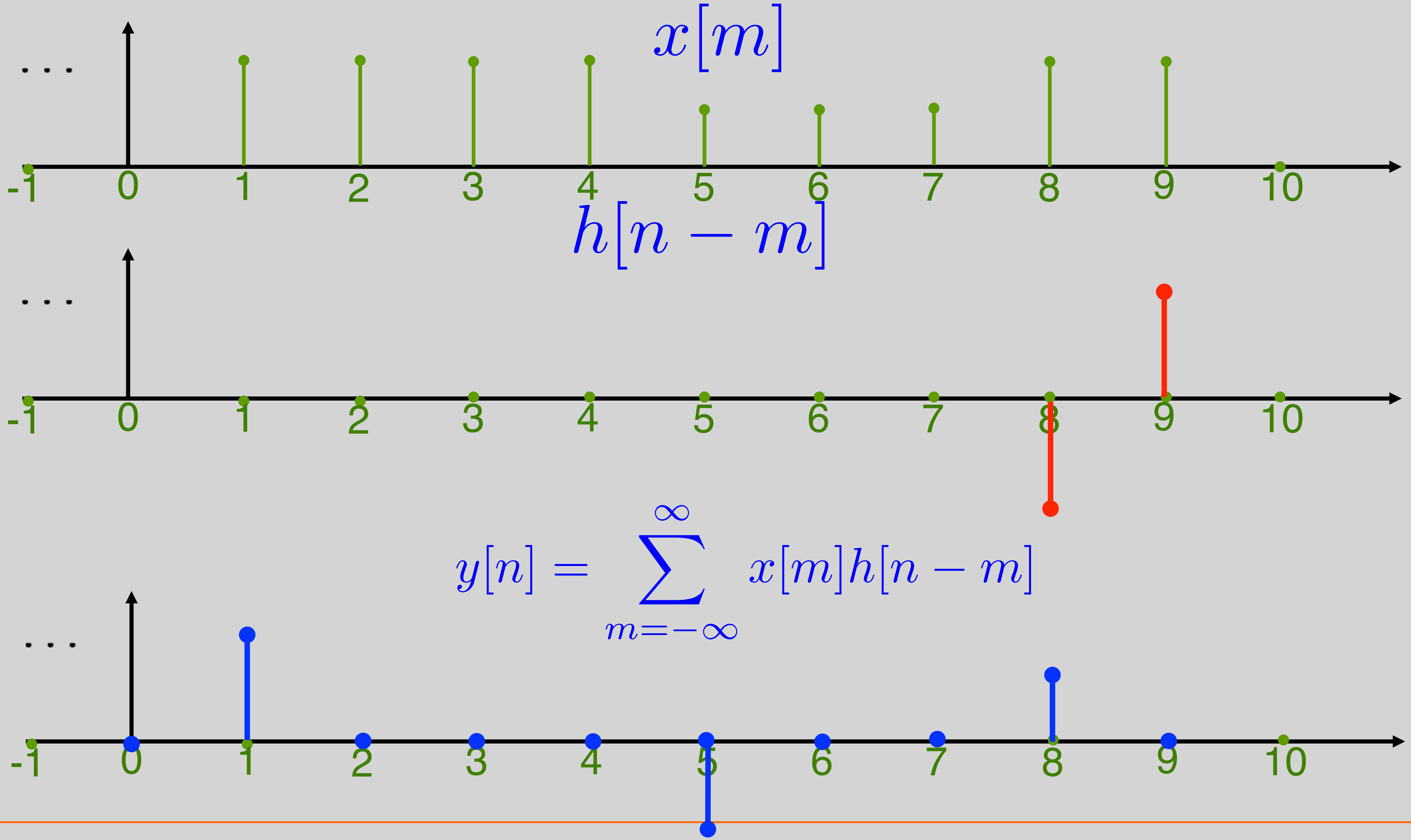
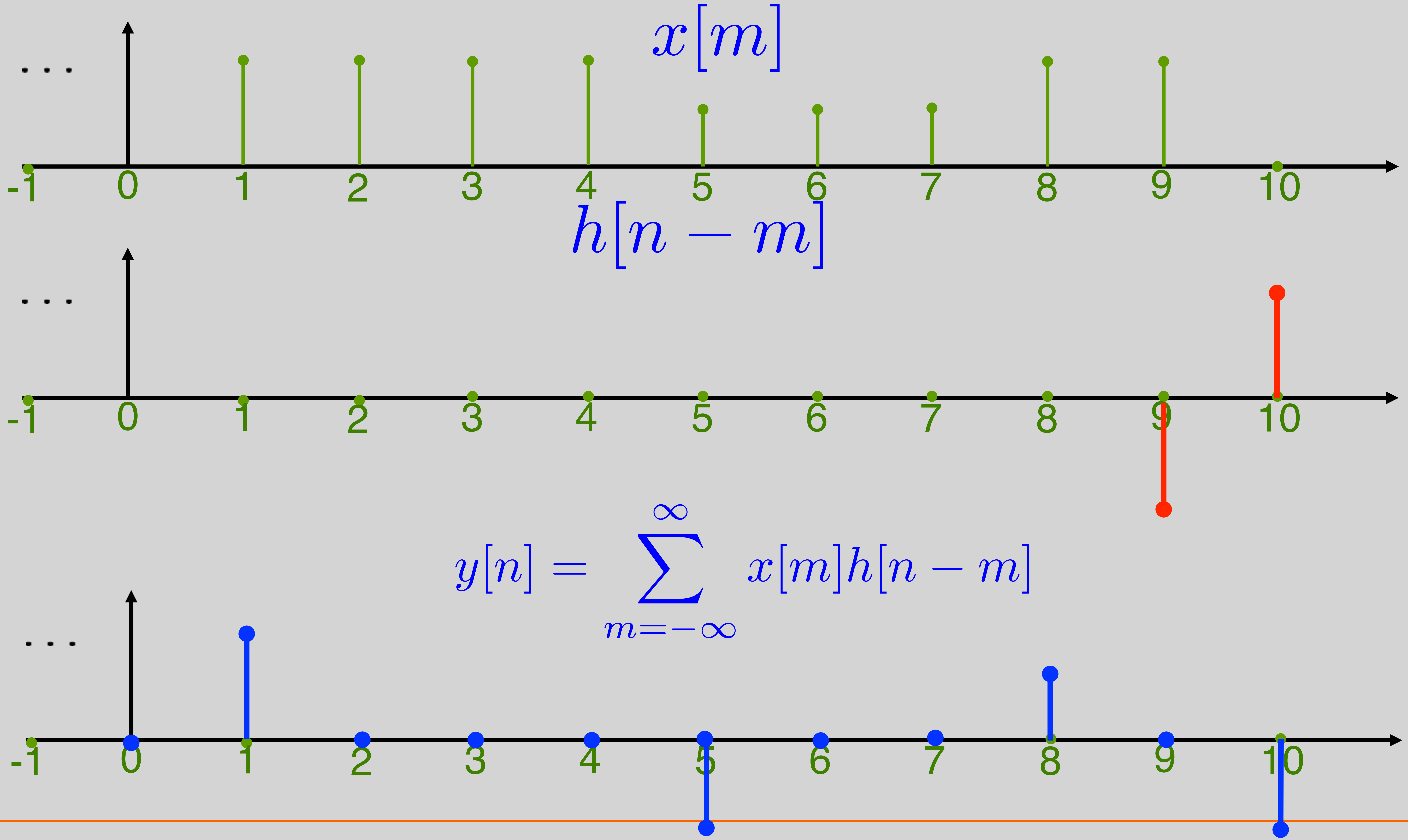
Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution

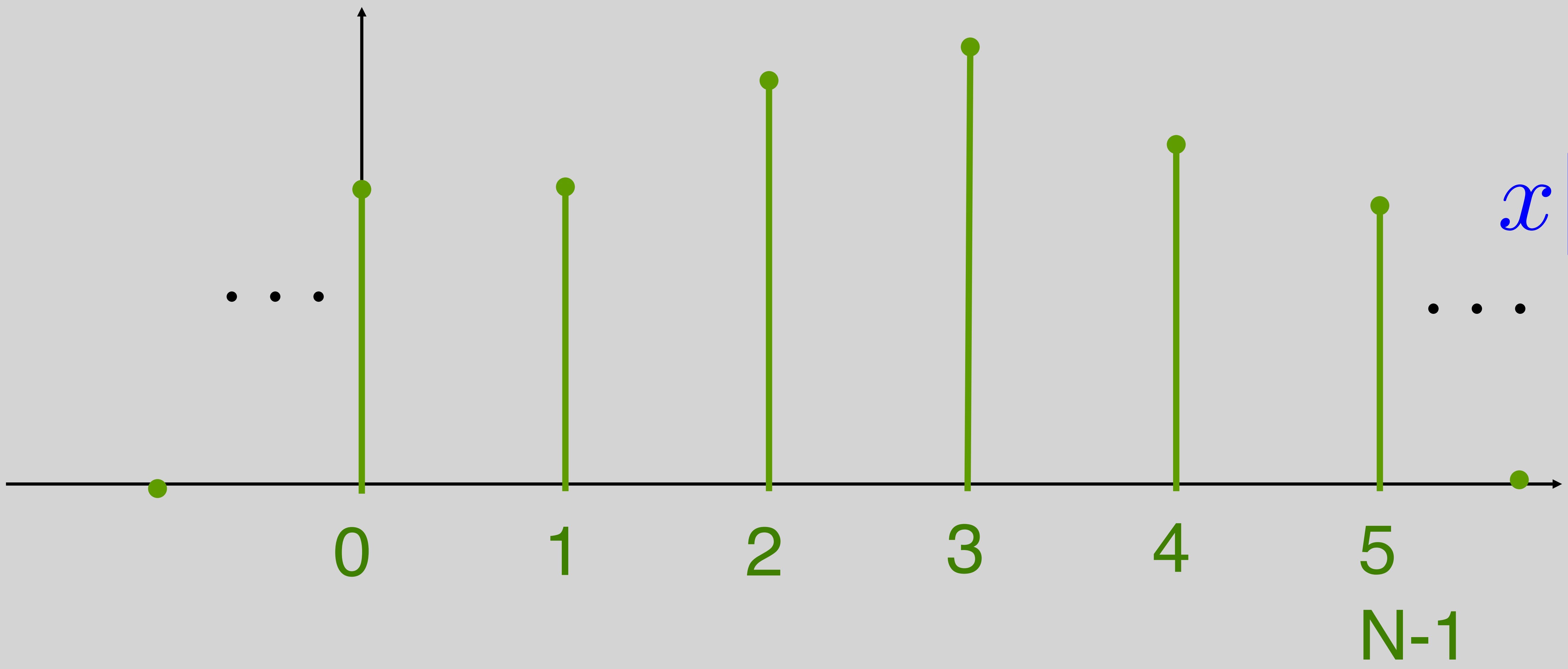


$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Example

Finite Sequences

- Consider a finite sequence of length N



$$x[n] = \begin{cases} \text{something} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

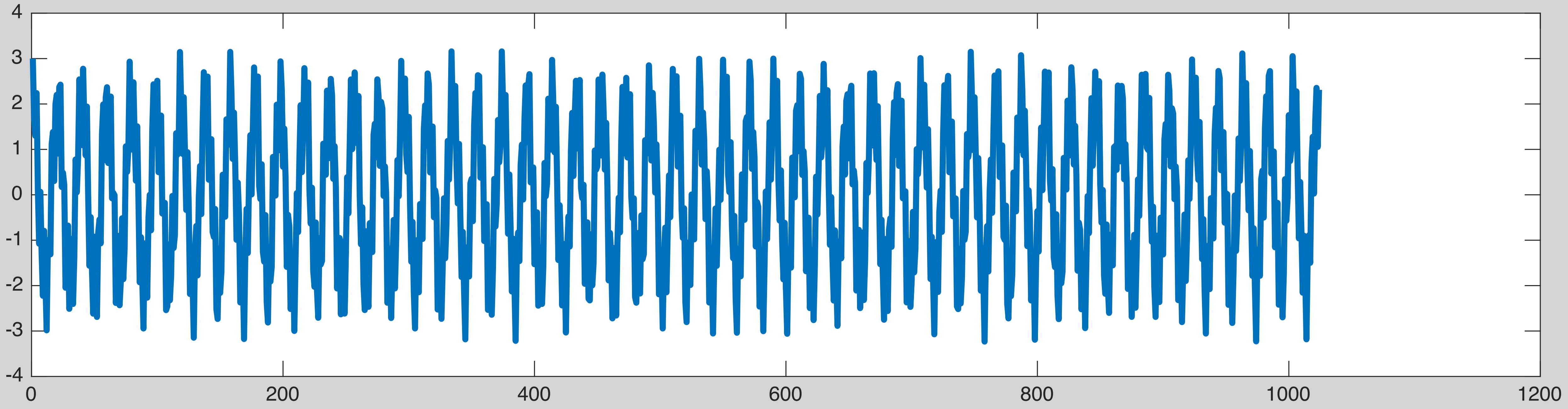
- Can also be written as a vector

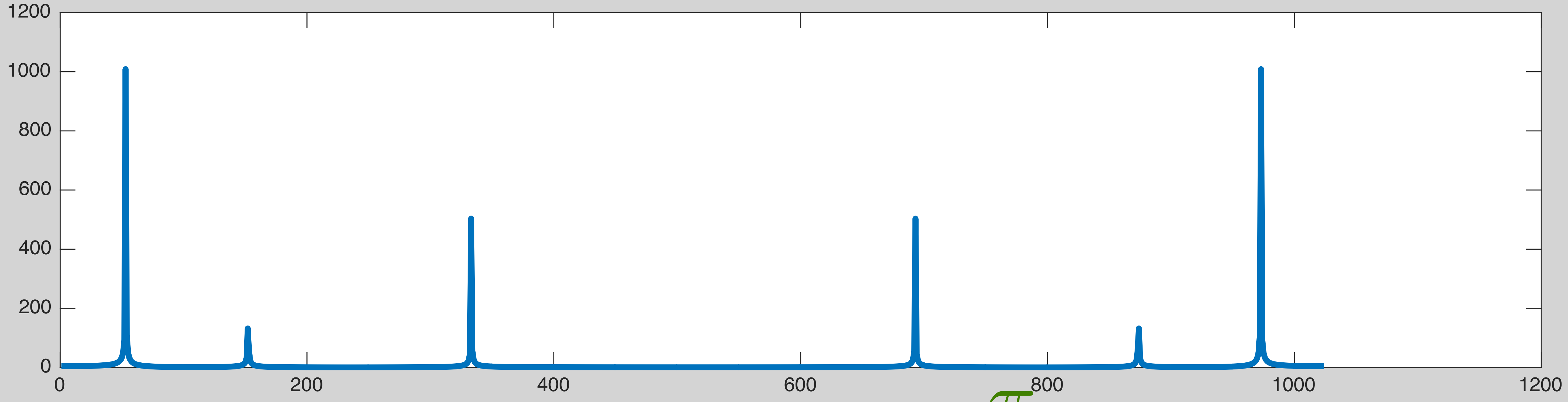
$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Why?

- To compute this:

$x[n] \Rightarrow X[k]$





$\frac{\pi}{2}$ π $-\frac{\pi}{2}$

Finite Sequences as Vectors

- Define an inner-product (for R^N):

$$< \vec{x}, \vec{y} > = \vec{x} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]y[n] =$$

$$= \vec{x}^T \vec{y}$$

So,

$$< \vec{x}, \vec{x} > = \sum_{n=0}^{N-1} x[n]x[n] = \sum_{n=0}^{N-1} x^2[n] = ||\vec{x}||^2$$

$$\Rightarrow \vec{x}^T \vec{x} = ||\vec{x}||^2$$

Finite Sequences as Vectors

- What about complex?

$$x \cdot x = x^2 = (x_r + jx_i)(x_r + jx_i) = x_r^2 - x_i^2 + 2jx_r x_i \neq ||x||^2$$

but,

$$x^* \cdot x = (x_r - jx_i)(x_r + jx_i) = x_r^2 + x_i^2 = ||x||^2$$

- Transpose vs Transpost conjugate

$$\vec{x} = \begin{bmatrix} 1 \\ j \\ 1+j \end{bmatrix}$$

$$\vec{x}^T = \begin{bmatrix} 1 & j & 1+j \end{bmatrix}$$

$$\vec{x}^* = \begin{bmatrix} 1 & -j & 1-j \end{bmatrix}$$

Finite Sequences as Vectors

- Define Complex inner product

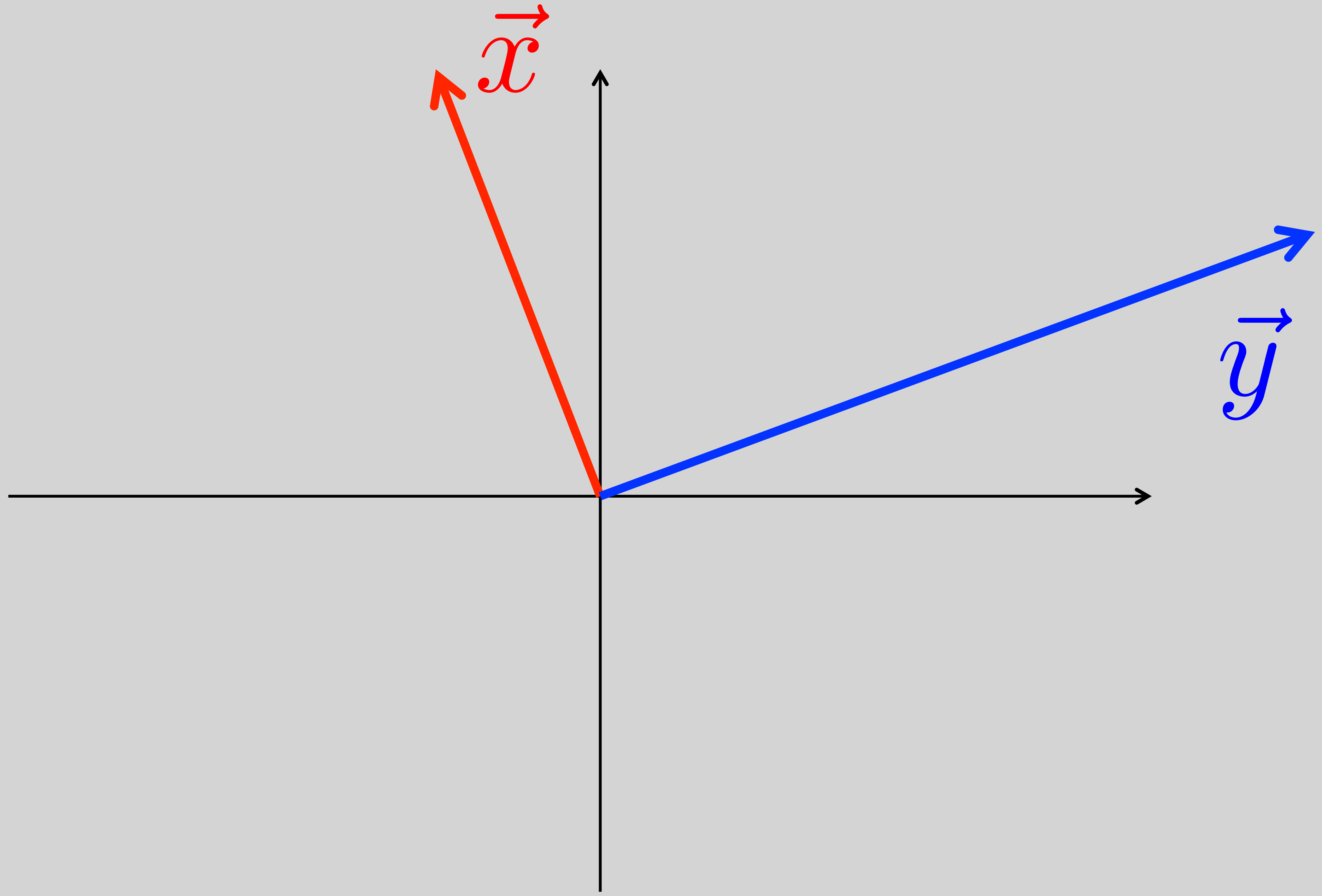
$$< \vec{x}, \vec{y} > = \overline{\vec{x}} \cdot \vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = \vec{x}^* \vec{y} = \vec{x}^H \vec{y}$$

$$\vec{x} = \begin{bmatrix} 1 \\ j \end{bmatrix} \Rightarrow \vec{x}^* x = \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} = 2$$

Projections

- Orthogonality:

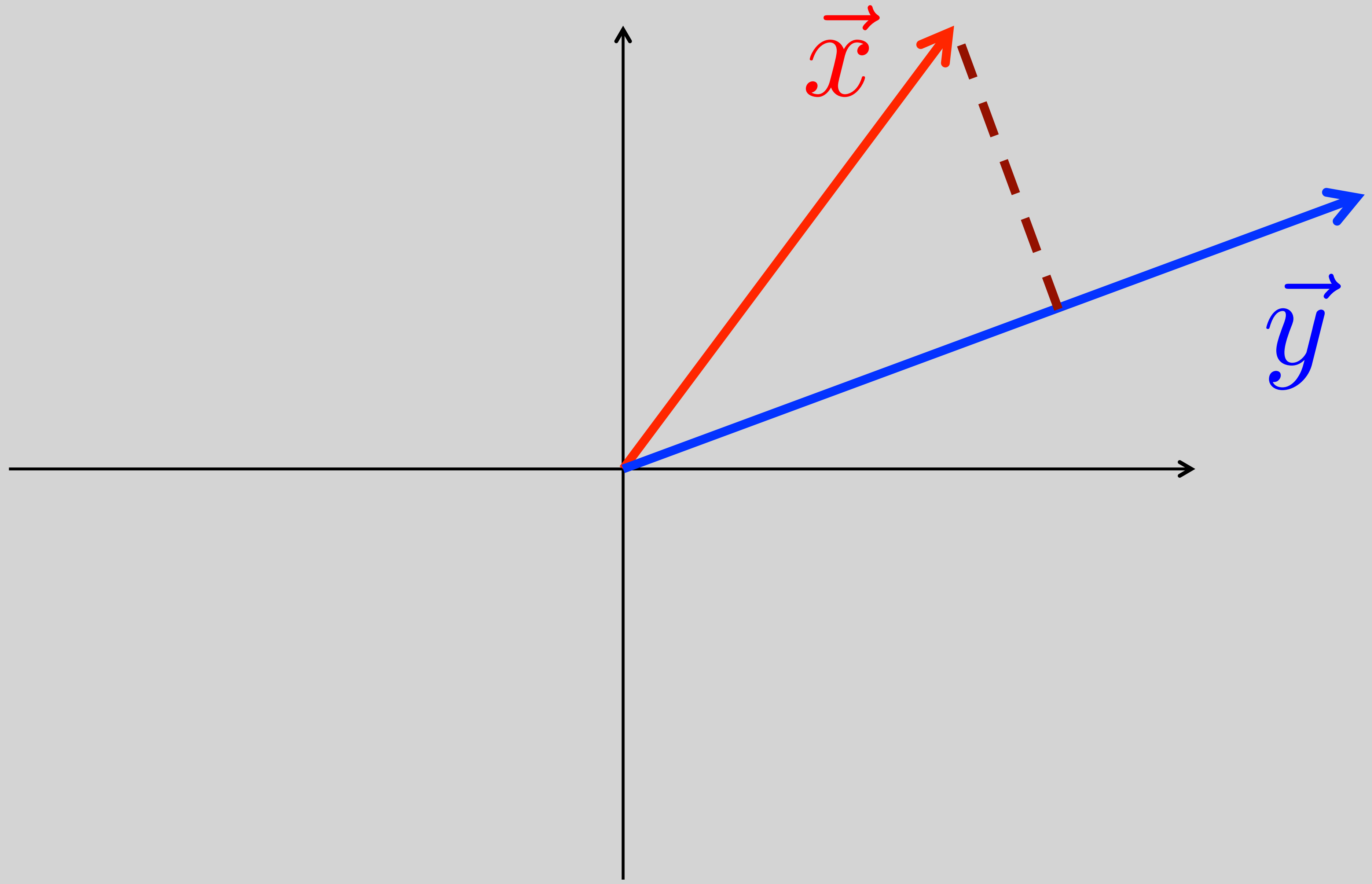
$$\vec{x}^*\vec{y} = \sum_{n=0}^{N-1} x[n]^* y[n] = 0$$



- Unit vector: $||\hat{x}|| = 1$

$$\hat{x} = \frac{\vec{x}}{||\vec{x}||}$$

- Define projection as: $\dfrac{\vec{y}^* x}{||\vec{y}||}$

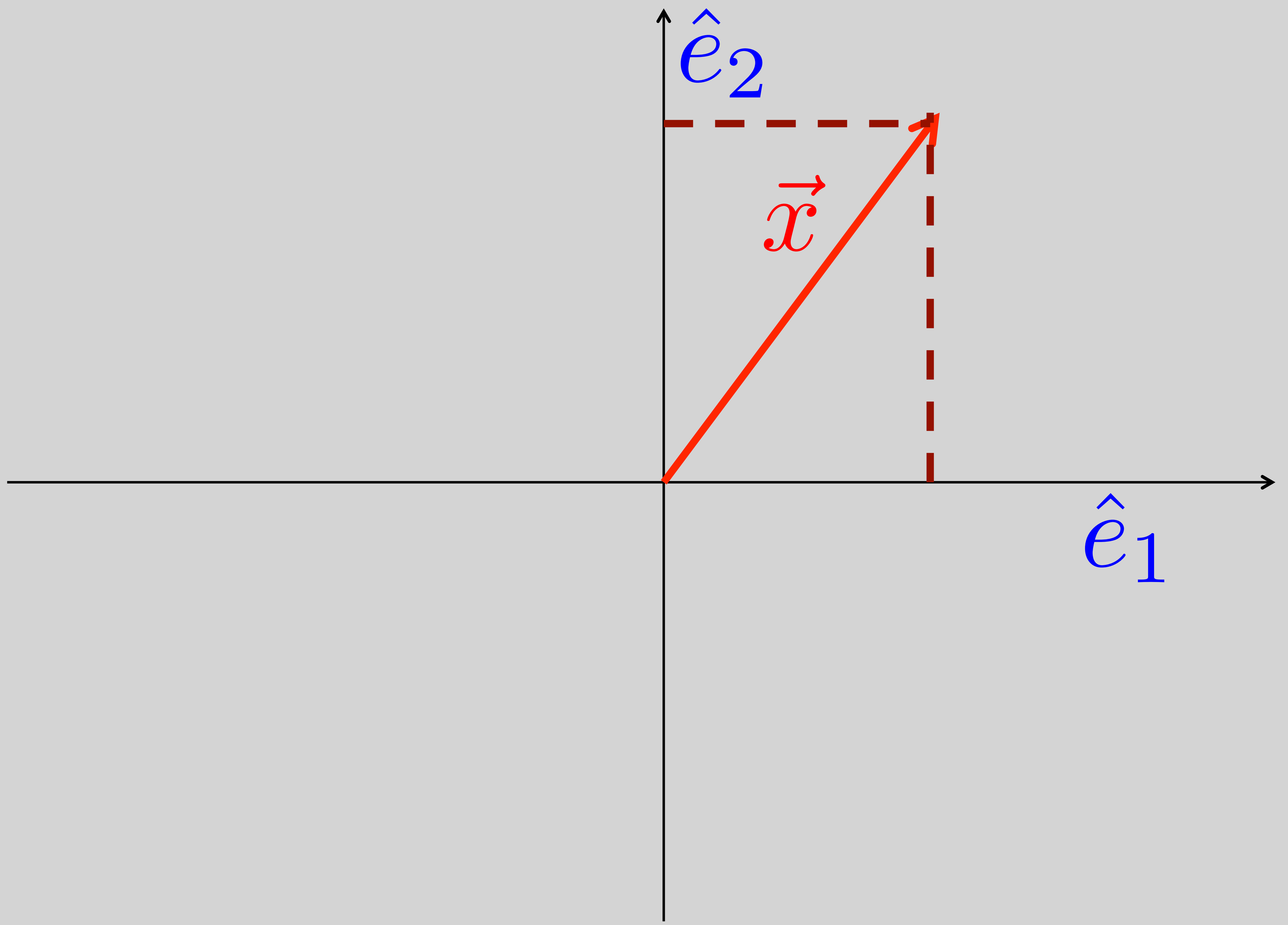


Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

$$\hat{e}_1^* \vec{x} = [1 \quad 0]\vec{x} = x_1$$

$$\hat{e}_2^* \vec{x} = [0 \quad 1]\vec{x} = x_2$$



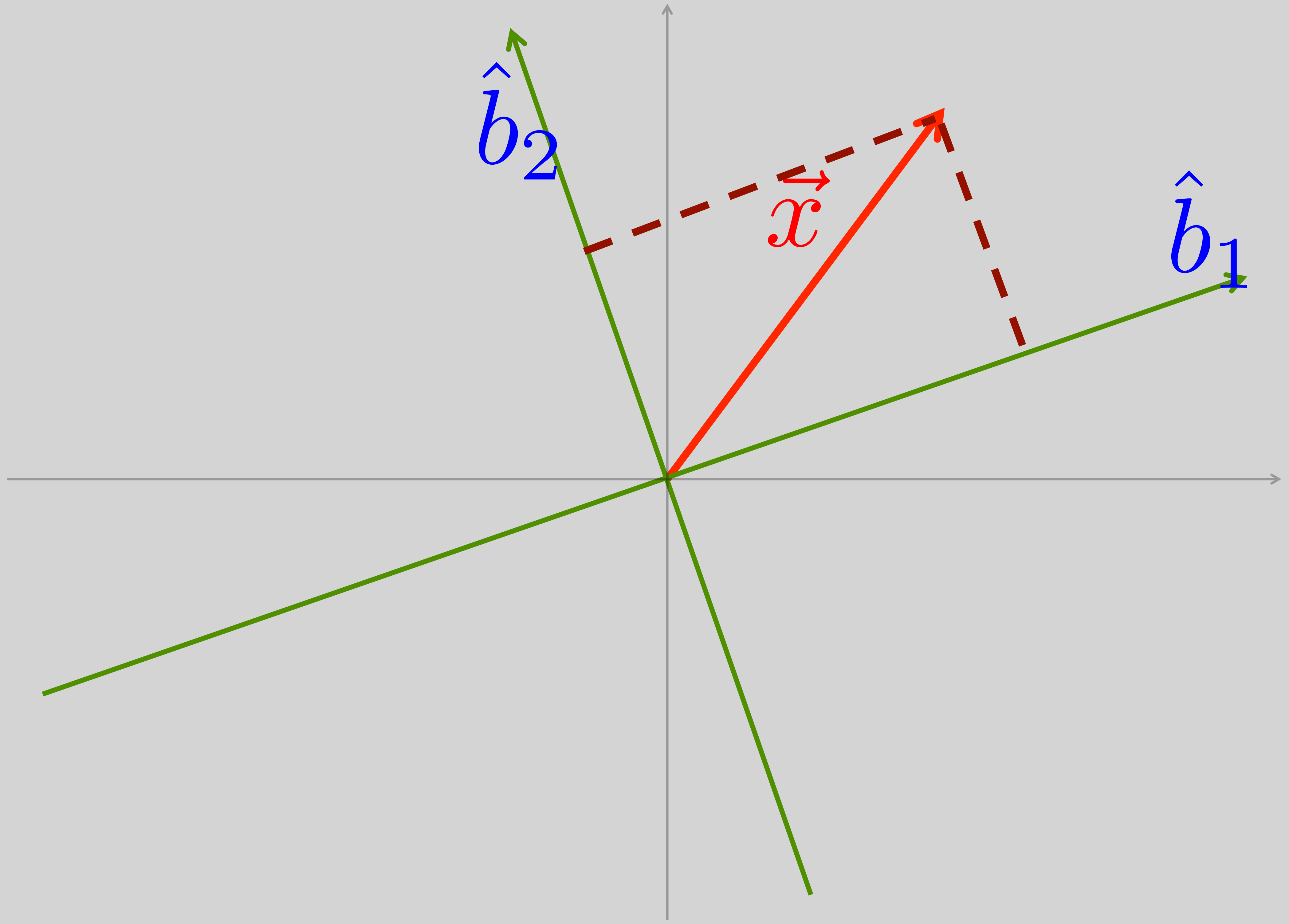
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

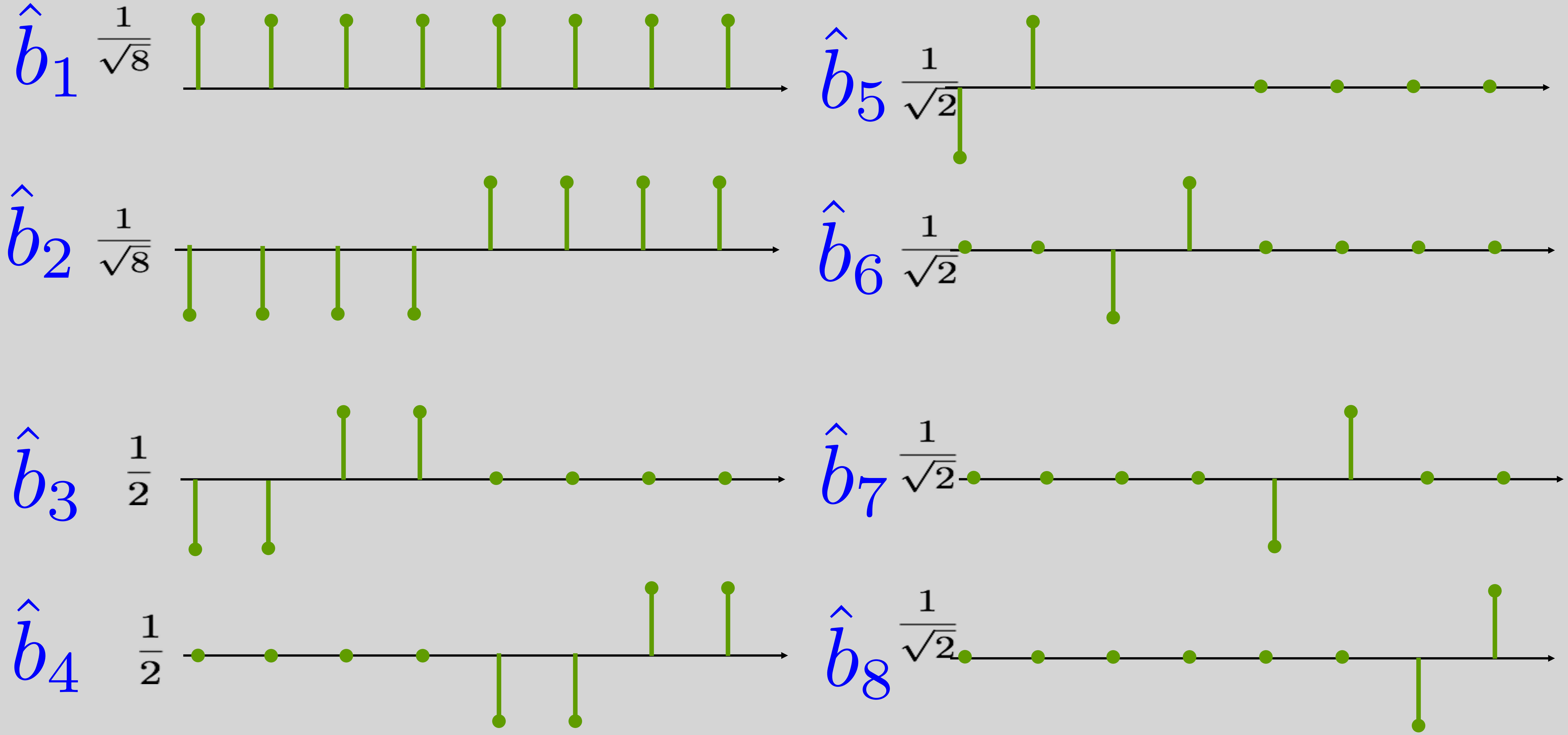
New coordinates:

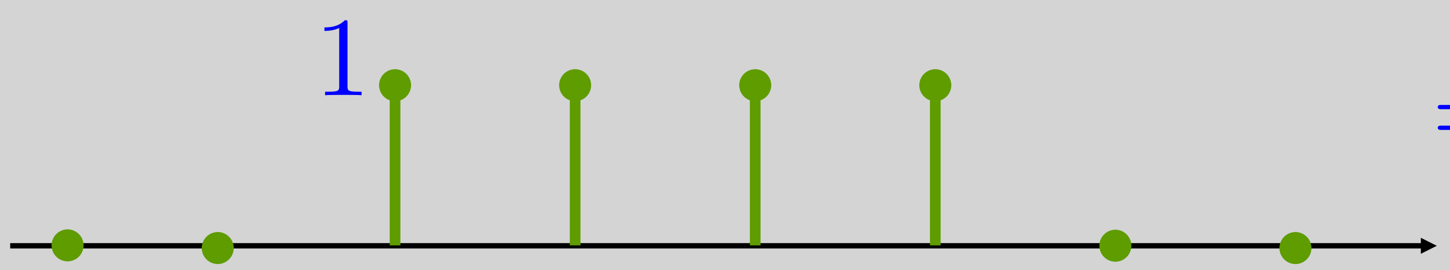
$$\begin{bmatrix} \hat{b}_1^* \vec{x} \\ \hat{b}_2^* \vec{x} \end{bmatrix} = \begin{bmatrix} \hat{b}_1 & \hat{b2} \end{bmatrix}^* \vec{x}$$

$$\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$



Change of basis





$$= \quad \hat{b}_1 + \quad \hat{b}_2 + \quad \hat{b}_3 + \quad \hat{b}_4 + \quad \hat{b}_5 + \quad \hat{b}_6 + \quad \hat{b}_7 + \quad \hat{b}_8$$

Frequency Analysis Through Projections

- N-length normalized discrete frequency:

$$b_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n} \qquad 0 \le n < N \qquad 0 \le \omega < 2\pi$$

$$\vec{b}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix} \qquad \Rightarrow X(\omega) = \vec{b}_w^* \vec{x}$$

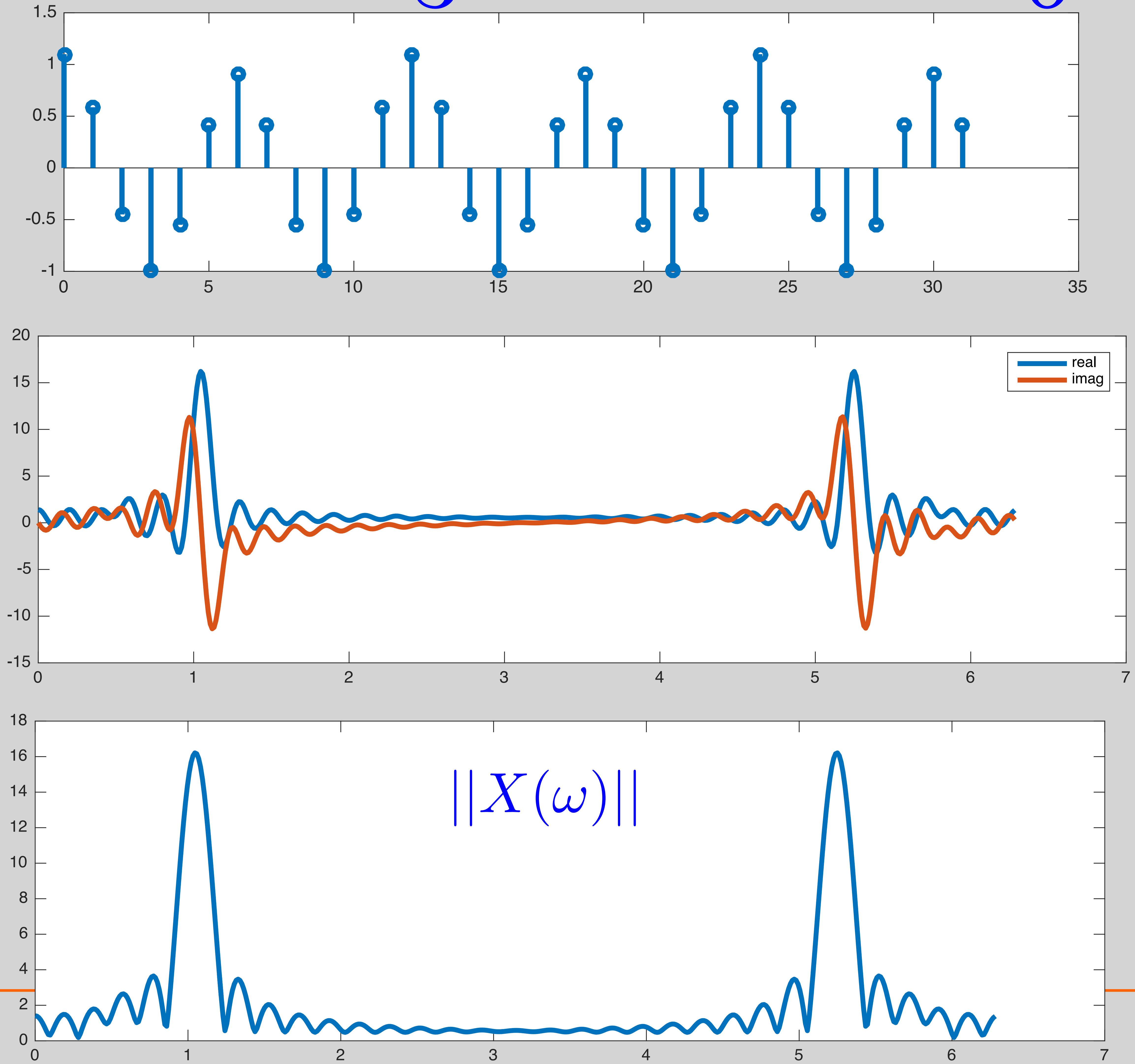
Also the DTFT of the finite sequence x

Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{3}n) + 0.1\cos(\frac{\pi}{6})$

N = 32

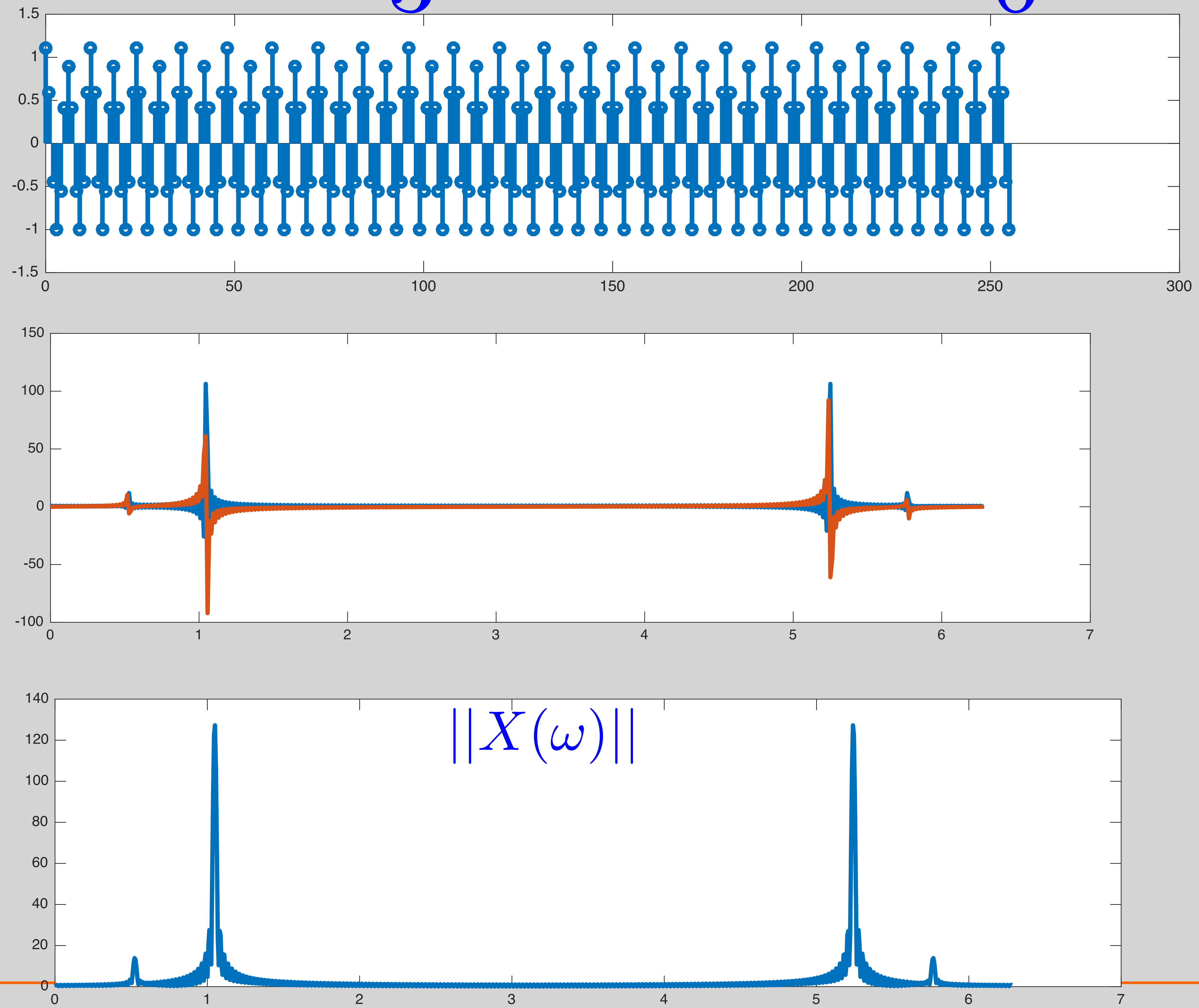
$$\Rightarrow X(\omega) = \vec{b}_w^* \vec{x}$$



Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{3}n) + 0.1\cos(\frac{\pi}{6})$

N = 256



$$\Rightarrow X(\omega) = \vec{b}_w^* \vec{x}$$

Discrete Fourier Transform (DFT)

- For $b_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n}$, pick a set of N frequencies, which will result in an orthogonal basis

- Choose: $$\omega_k = \frac{2\pi k}{N} \Rightarrow \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N}n}$$

$$W_n \triangleq e^{j2\pi/N} \quad \Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

DFT Matrix

$$W_n \triangleq e^{j2\pi/N} \Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

- N = 2