EE16B
Designing Information Devices and Systems II

Lecture 13A
DFT and Spectral Analysis

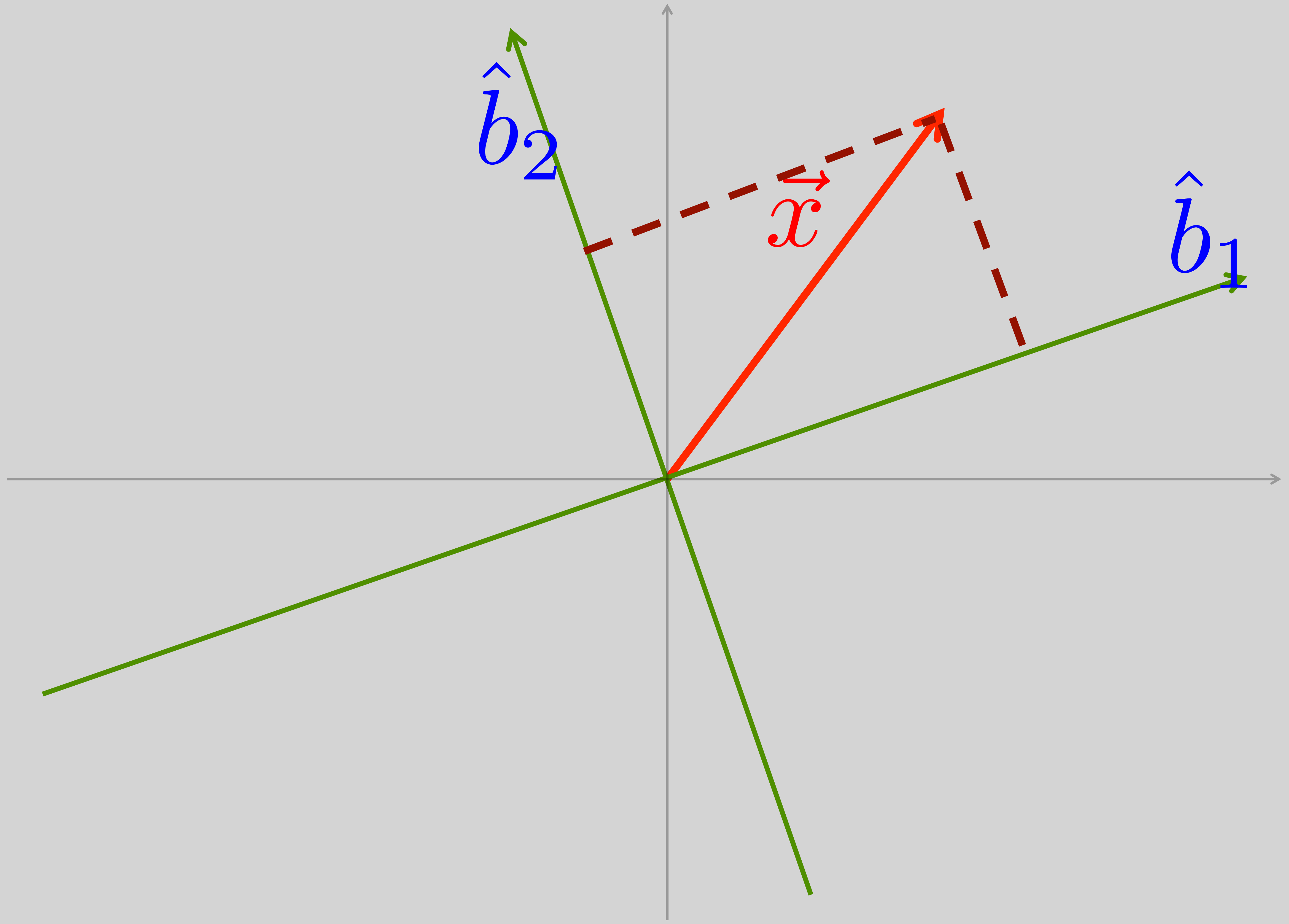
Change of Coordinates (Basis)

- We can compute new coordinates by projections onto orthonormal basis vectors

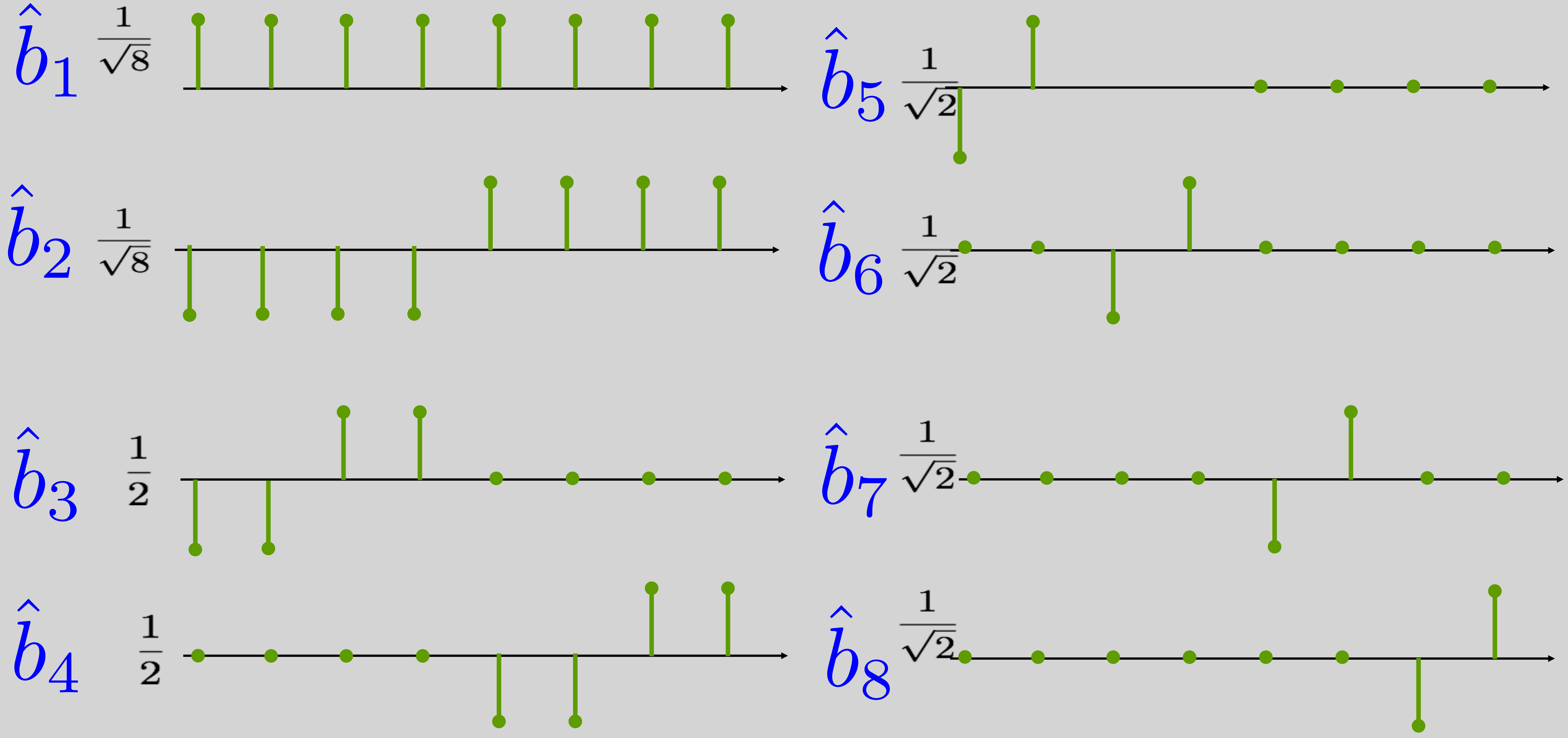
New coordinates:

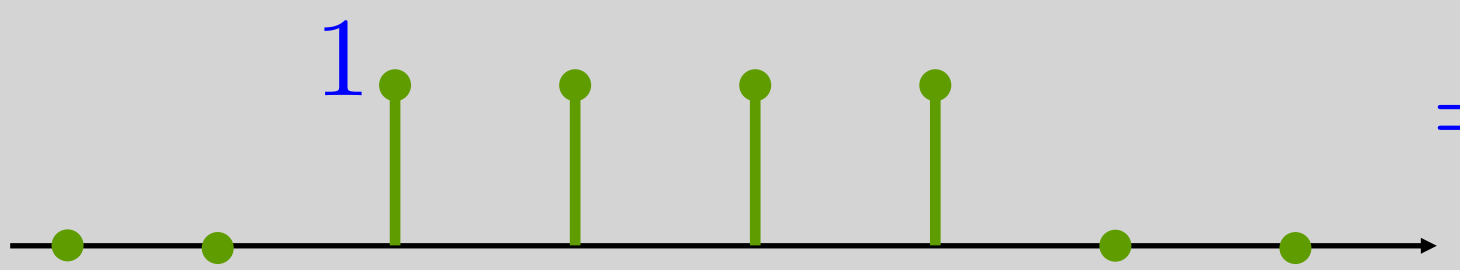
$$\begin{bmatrix} \hat{b}_1^* \vec{x} \\ \hat{b}_2^* \vec{x} \end{bmatrix} = \begin{bmatrix} \hat{b}_1 & \hat{b2} \end{bmatrix}^* \vec{x}$$

$$\Rightarrow \vec{x} = (\hat{b}_1^* \vec{x})\hat{b}_1 + (\hat{b}_2^* \vec{x})\hat{b}_2$$



Change of basis





$$= \quad \hat{b}_1 + \quad \hat{b}_2 + \quad \hat{b}_3 + \quad \hat{b}_4 + \quad \hat{b}_5 + \quad \hat{b}_6 + \quad \hat{b}_7 + \quad \hat{b}_8$$

Frequency Analysis Through Projections

- N-length normalized discrete frequency:

$$u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n} \qquad 0 \le n < N \qquad 0 \le \omega < 2\pi$$

$$\vec{u}_\omega = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\omega 0} \\ e^{j\omega 1} \\ \vdots \\ e^{j\omega (N-1)} \end{bmatrix} \qquad \Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$

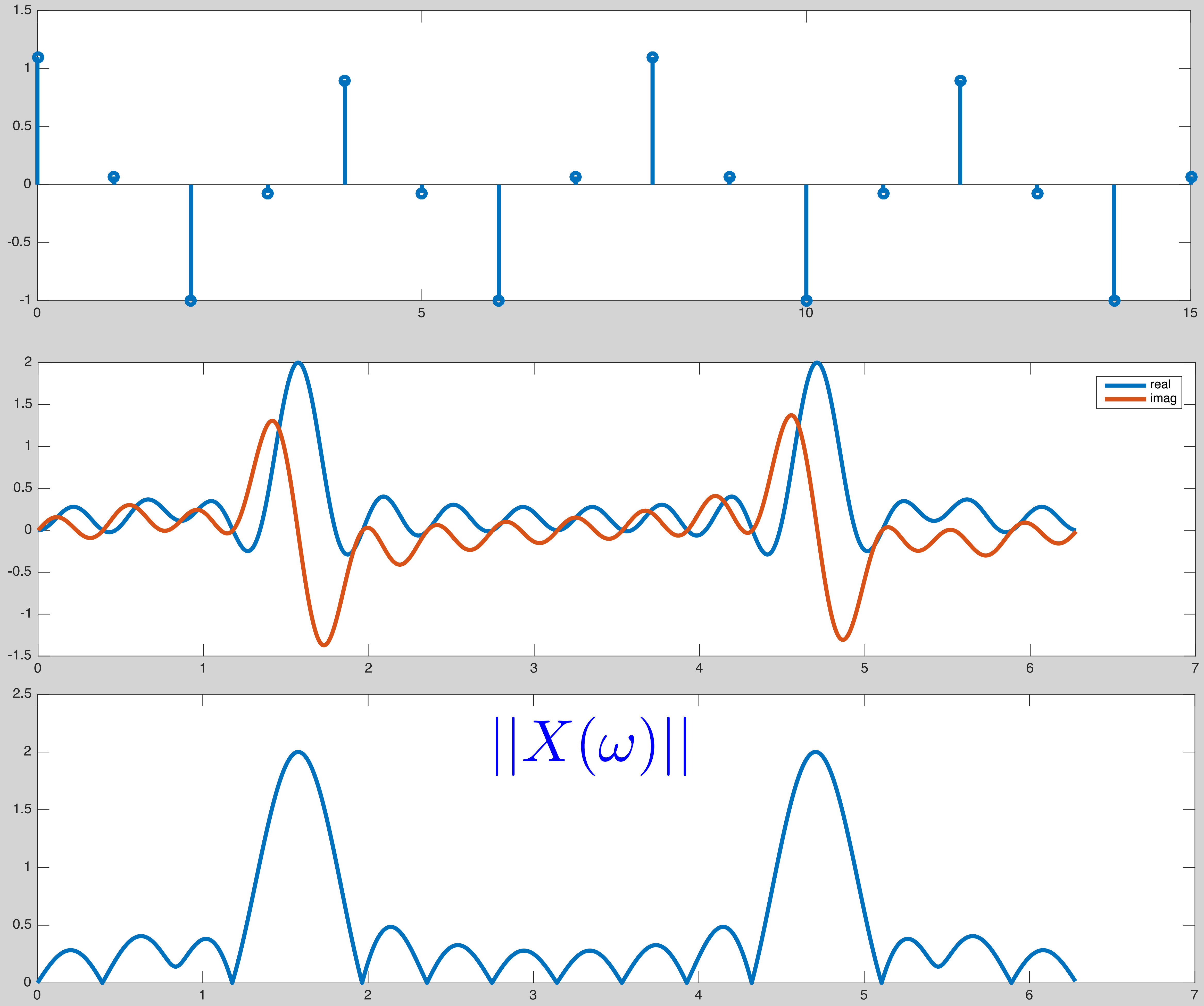
Also the DTFT of the finite sequence x

Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{2} n) + 0.1\cos(\frac{\pi}{4})$

N = 16

$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$

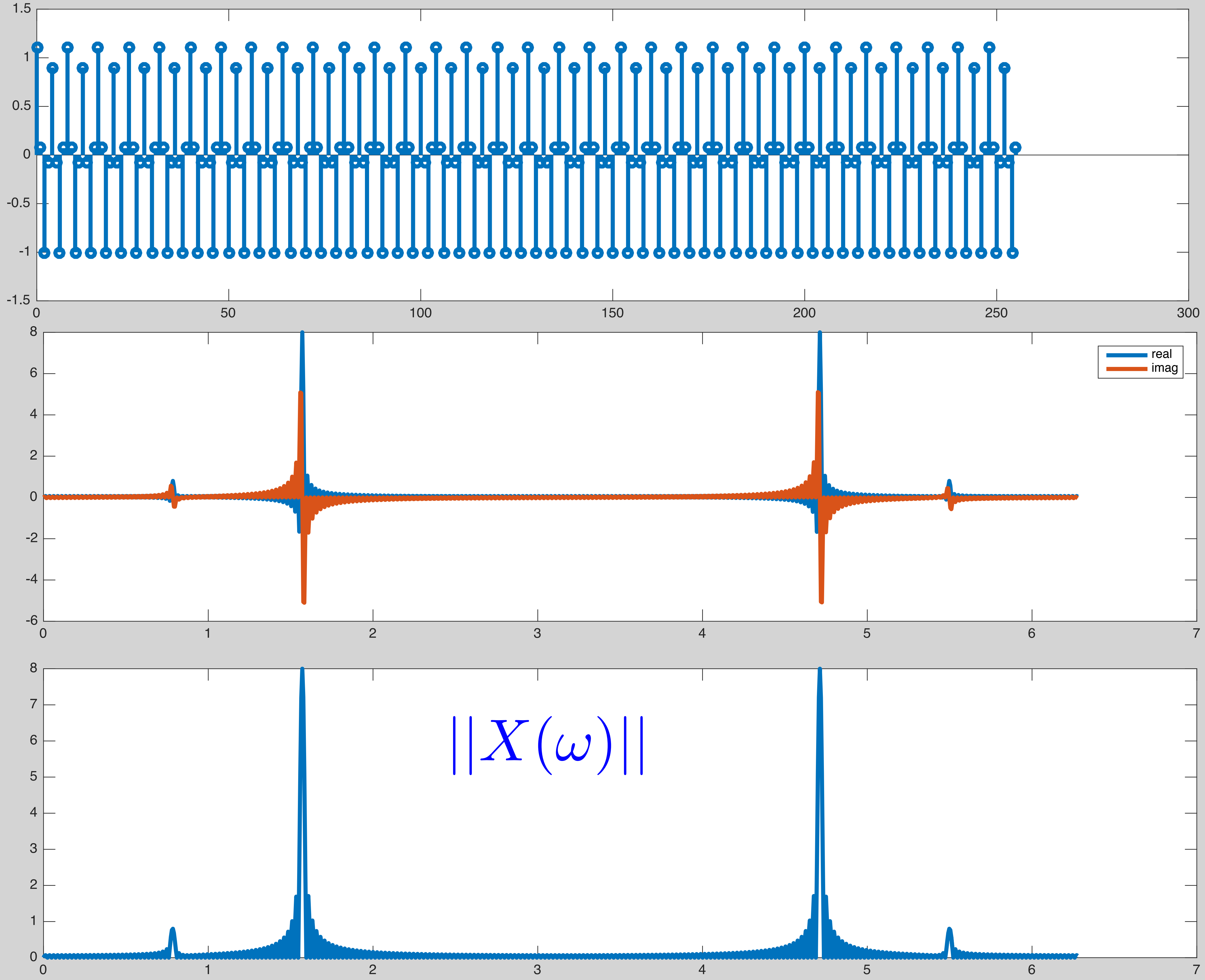


Frequency Analysis Through Projections

- Example: $x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4})$

N = 256

$$\Rightarrow X(\omega) = \vec{u}_w^* \vec{x}$$



Discrete Fourier Transform (DFT)

- For $u_\omega[n] = \frac{1}{\sqrt{N}} e^{j\omega n}$, pick a set of N frequencies, which will result in an orthogonal basis

- Choose: $\omega_k = \frac{2\pi k}{N} \Rightarrow \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N}n}$

$$k \in [0, N-1]$$

$$n \in [0, N-1]$$

$$W_N \triangleq e^{j2\pi/N} \quad \Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix} \quad k \in [0, N-1]$$

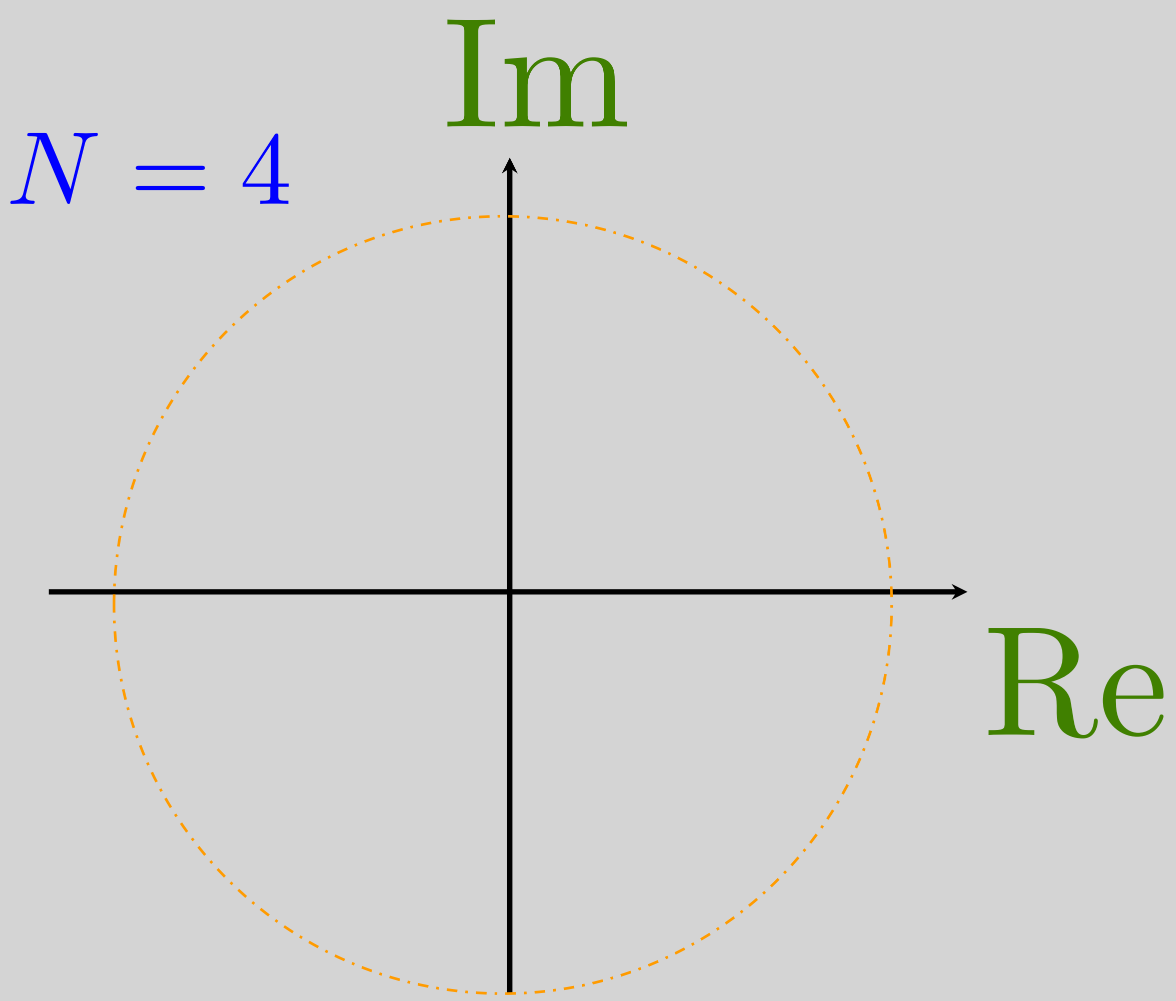
$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix} k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$k = 1, N = 4$



$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

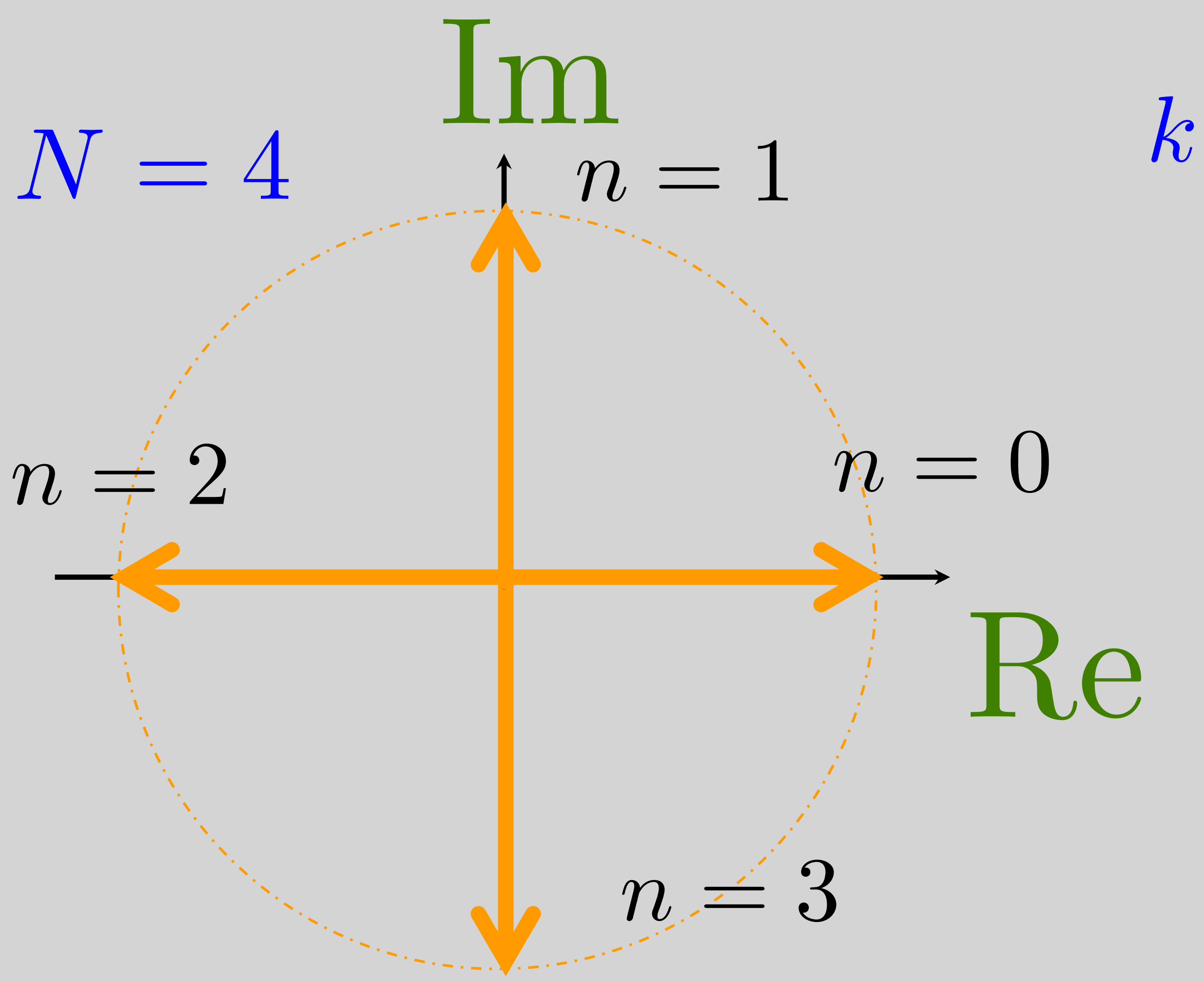
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix} k \in [0, N-1]$$

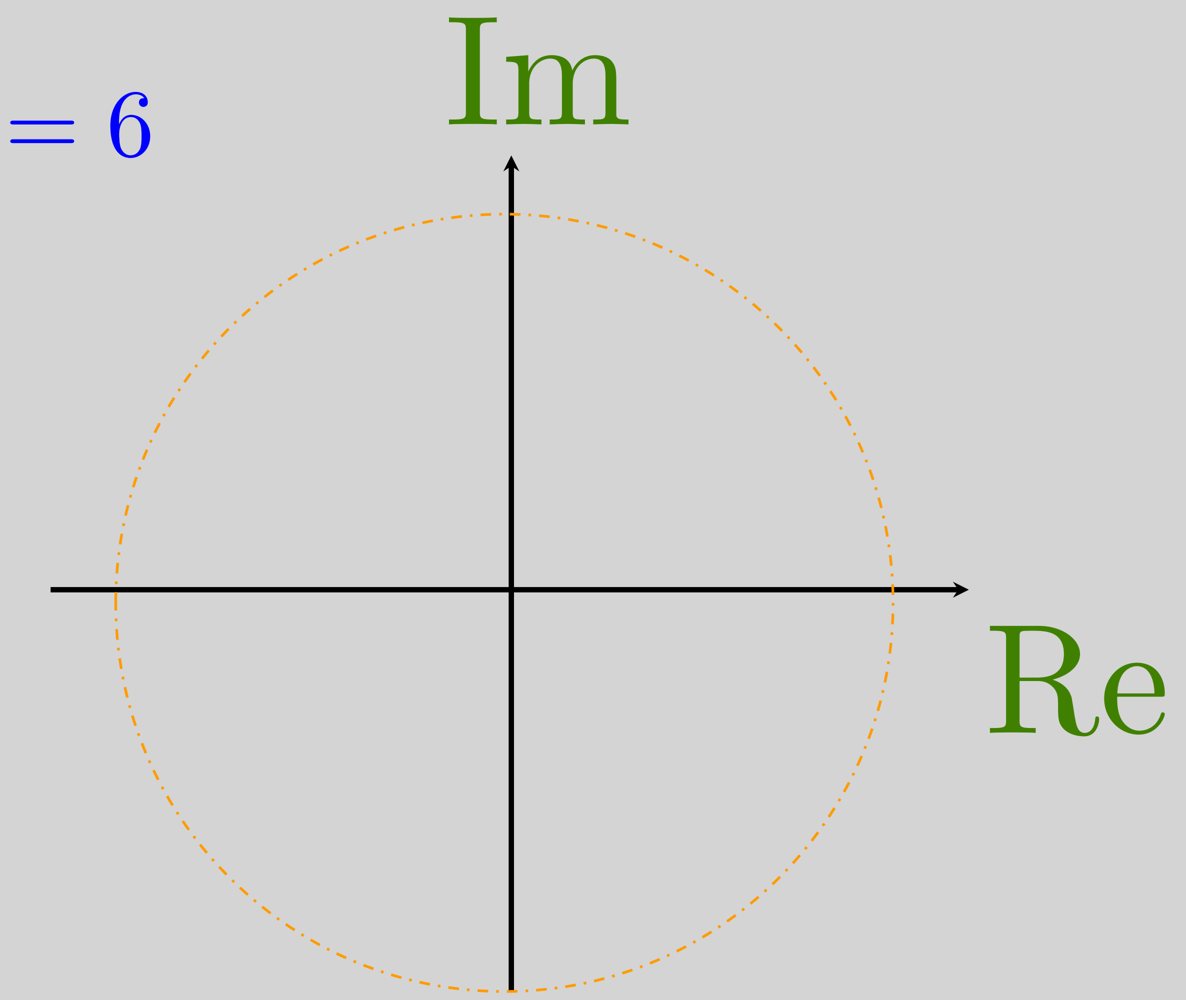
$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$

$k = 1, N = 4$



$k = 2, N = 6$

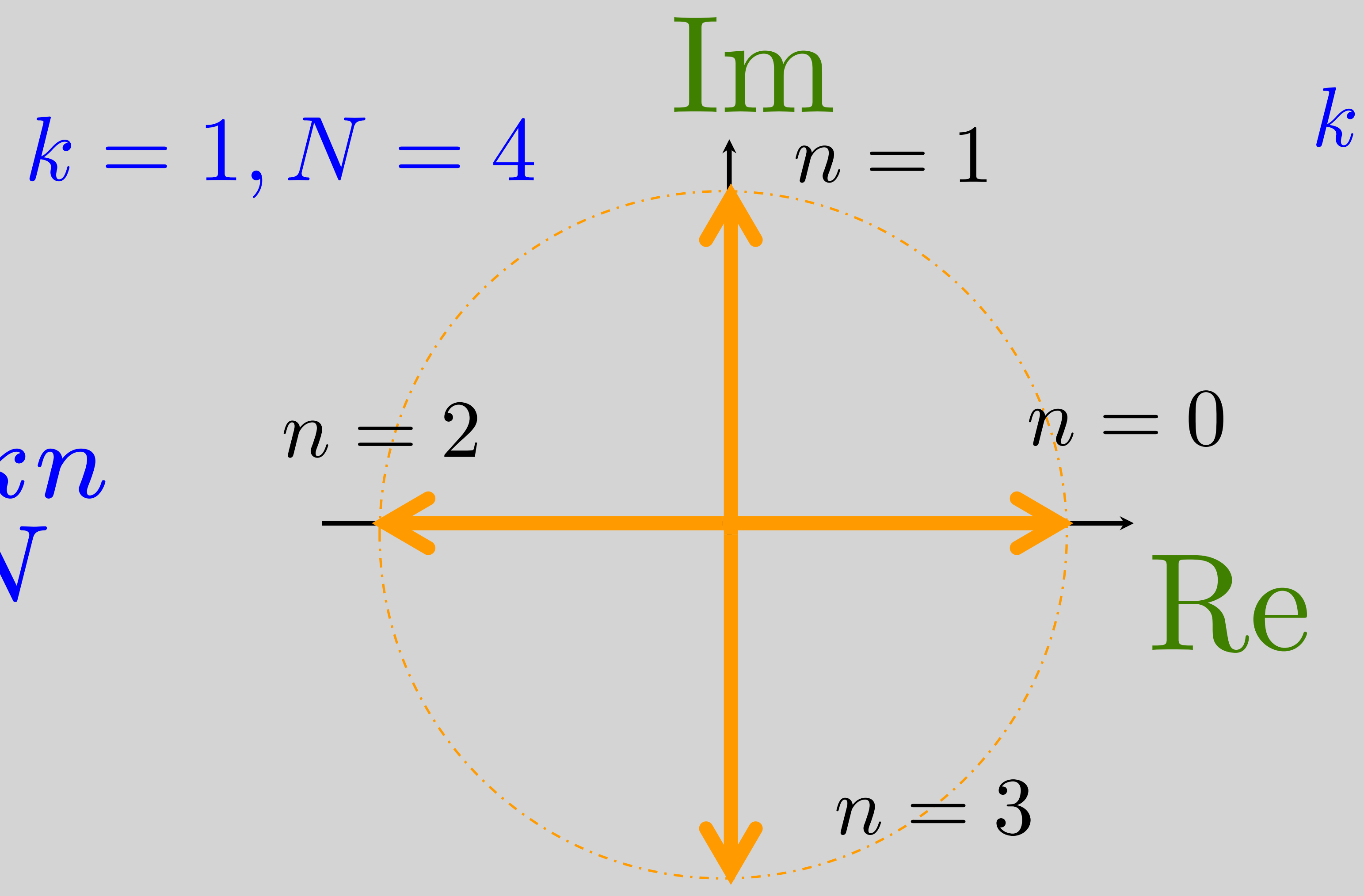


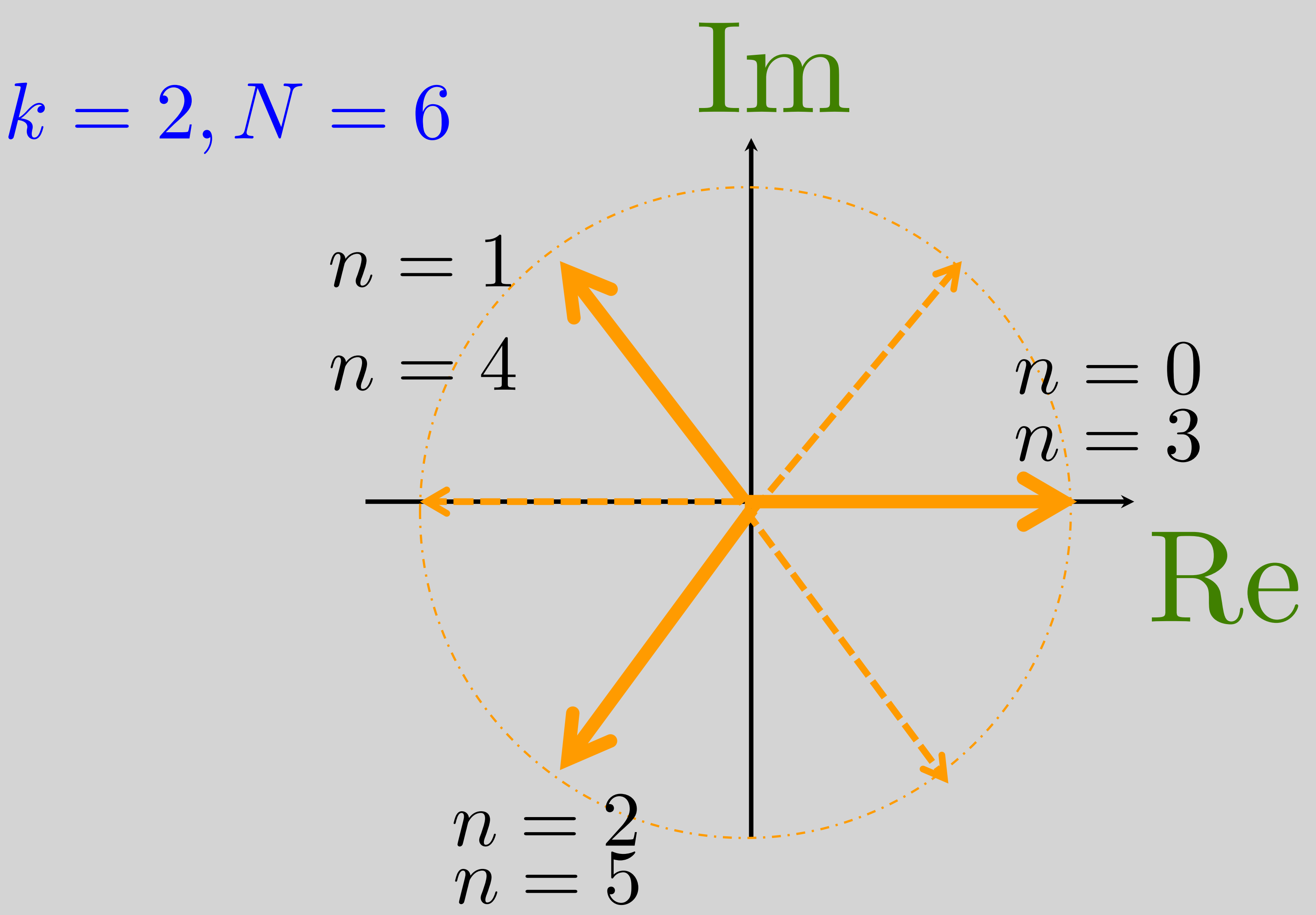
DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \quad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\Rightarrow \frac{1}{\sqrt{N}} W_N^{kn}$$





$$\sum_{n=0}^{N-1} W_N^{nk} = ? \; = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$$

Orthonormality of DFT Basis

- DFT basis vectors are orthonormal. Proof:

$$\vec{u}_k^* \vec{u}_m = \frac{1}{N}\sum_{n=0}^{N-1} W_N^{-nk} W_N^{nm} = \frac{1}{N}\sum_{n=0}^{N-1} W_N^{n(m-k)}$$

$$= \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$$

Example

$$N = 16$$

$$\vec{u}_k = \frac{1}{\sqrt{16}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{16}} \\ e^{j\frac{2\pi k \cdot 1}{16}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (15)}{16}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_{16}^{k\cdot 0} \\ W_{16}^{k\cdot 1} \\ \vdots \\ W_{16}^{k\cdot 15} \end{bmatrix}$$

$$x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4}) \quad = 0.5(e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}} + 0.1e^{j\frac{\pi n}{4}} + 0.1e^{-j\frac{\pi n}{2}})$$

$$= 0.5(e^{j\frac{2\pi 4n}{16}} + e^{-j\frac{2\pi 4n}{16}} + 0.1e^{j\frac{2\pi 2n}{16}} + 0.1e^{-j\frac{2\pi 2n}{16}})$$

$$= 0.5(e^{j\frac{2\pi 4n}{16}} + e^{j\frac{2\pi 12n}{16}} + 0.1e^{j\frac{2\pi 2n}{16}} + 0.1e^{j\frac{2\pi 14n}{16}})$$

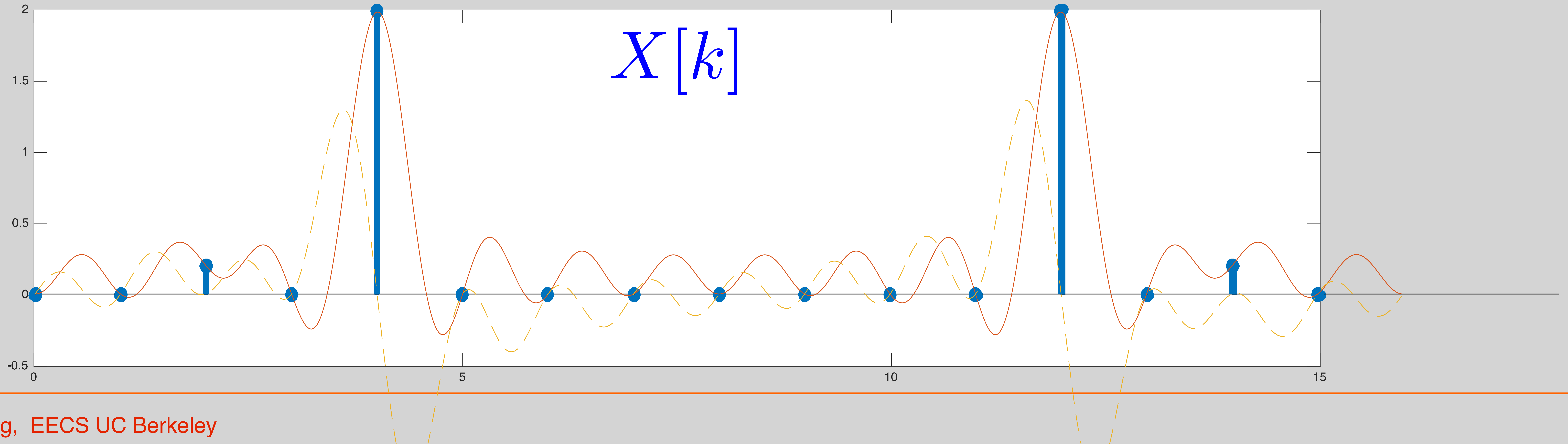
$$= \frac{2}{\sqrt{16}}W_{16}^{4n} + \frac{2}{\sqrt{16}}W_{16}^{12n} + \frac{0.2}{\sqrt{16}}W_{16}^{2n} + \frac{0.2}{\sqrt{16}}W_{16}^{14n}$$

Example

$$x[n] = \cos(\frac{\pi}{2}n) + 0.1\cos(\frac{\pi}{4})$$

$$= \frac{2}{\sqrt{16}}W_{16}^{4n} + \frac{2}{\sqrt{16}}W_{16}^{12n} + \frac{0.2}{\sqrt{16}}W_{16}^{2n} + \frac{0.2}{\sqrt{16}}W_{16}^{14n}$$

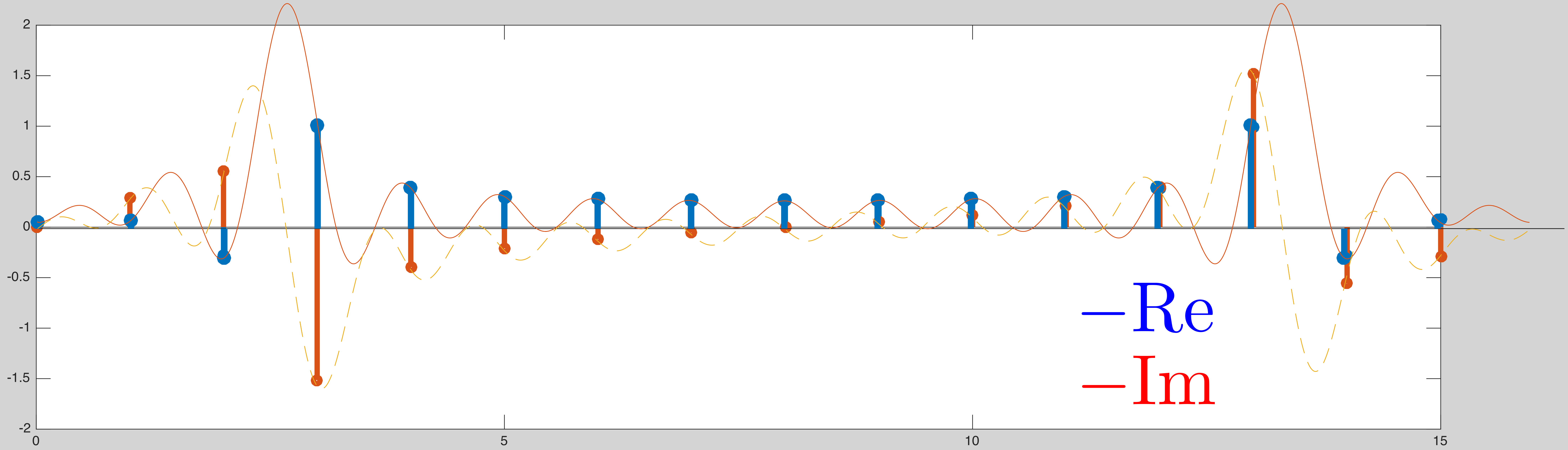
$$= 0.2\vec{u}_2 + 2\vec{u}_4 + 2\vec{u}_{12} + 0.2\vec{u}_{14}$$



Example 2

What if there is no integer k to fit the frequency $\omega_k = \frac{2\pi k}{N}$

$$x[n] = \cos(\frac{\pi}{3}n) + 0.1\cos(\frac{\pi}{6})$$



DFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \qquad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\vec{X} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}^*}_{\triangleq F^*} \vec{x}$$

DFT

- DFT Analysis

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} - & \vec{u}_0^* & - \\ - & \vec{u}_1^* & - \\ & \vdots & \\ - & \vec{u}_{N-1}^* & - \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\vec{X} = F^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

DFT

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

- DFT Synthesis

$$\begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\vec{x} = F\vec{X} \qquad = F(F^* \vec{x})$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] W_N^{+nk}$$

Quiz

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix}$$

Compute a 2 point DFT of: $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{u}_1 =$

$\vec{u}_2 =$

$\vec{u}_1^* \vec{x} =$

$\vec{u}_2^* \vec{x} =$

$\vec{X} =$

Example cont

- DFT$_2$ matrix:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\vec{X} = F^* \vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$