EE16B
Designing Information Devices and Systems II

Lecture 14A
Properties of the DFT

DFT Basis

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k \cdot 0} \\ W_N^{k \cdot 1} \\ \vdots \\ W_N^{k \cdot (N-1)} \end{bmatrix} \quad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

DFT

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix} \qquad k \in [0, N-1]$$

$$\Rightarrow X[k] = \vec{u}_k^* \vec{x}$$

$$\vec{X} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}^*}_{\triangleq F^*} \vec{x}$$

DFT

- DFT Analysis

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} - & \vec{u}_0^* & - \\ - & \vec{u}_1^* & - \\ & \vdots & \\ - & \vec{u}_{N-1}^* & - \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\vec{X} = F^* \vec{x}$$

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

DFT

- DFT Synthesis

$$F = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} | & | & & | \\ \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1} \\ | & | & & | \end{bmatrix} \begin{bmatrix} X[0] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$\vec{x} = F\vec{X} \qquad = F(F^*\vec{x})$$

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] W_N^{+nk}$$

Example cont

- DFT$_2$ matrix:

$$F = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

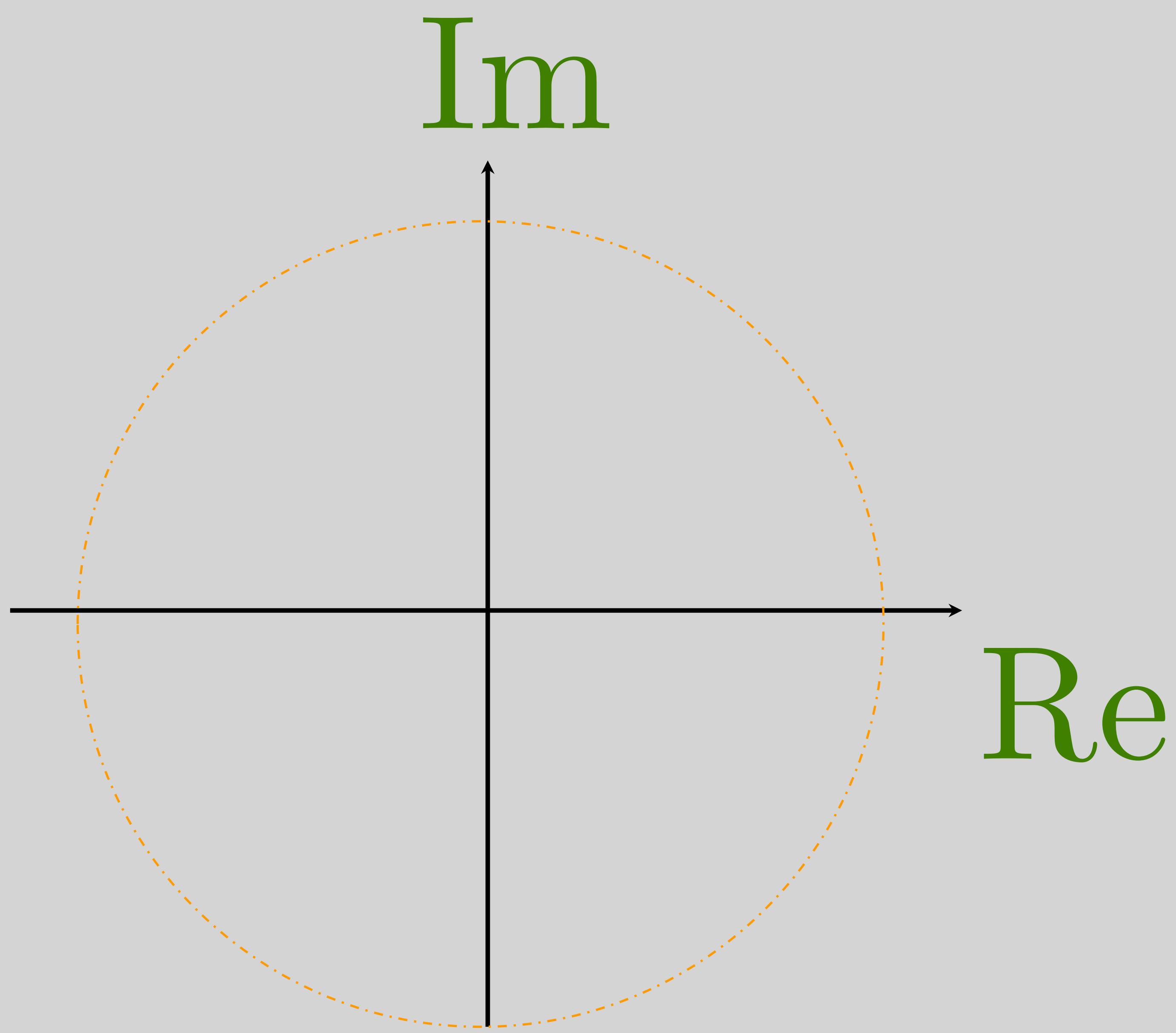
$$\vec{X} = F^* \vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

Example

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k\cdot 0}{N}} \\ e^{j\frac{2\pi k\cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k\cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix}$$

- Compute the inverse DFT_4 of: $\vec{X} = [1\ 1\ 1\ 1]^*$

$$\vec{x} = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{\triangleq F} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



Example

$$\vec{u}_k = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{j\frac{2\pi k \cdot 0}{N}} \\ e^{j\frac{2\pi k \cdot 1}{N}} \\ \vdots \\ e^{j\frac{2\pi k \cdot (N-1)}{N}} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{k\cdot 0} \\ W_N^{k\cdot 1} \\ \vdots \\ W_N^{k\cdot (N-1)} \end{bmatrix}$$

- Compute the inverse DFT_4 of: $\vec{X} = [1\ 1\ 1\ 1]^*$

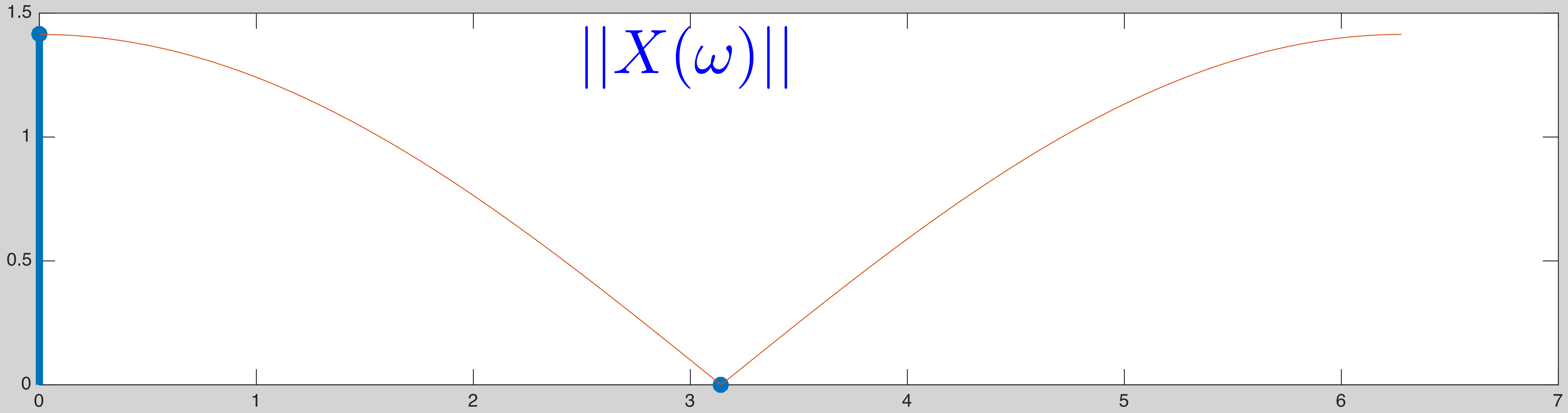
$$\vec{x} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Spectral Analysis with DFT

- Recall:

$$\vec{X} = F^*\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

$$\Rightarrow X(\omega) = \vec{u}_w^*\vec{x}$$



$$k \in [0, N-1] \qquad \omega_k = \frac{2\pi k}{N} \Rightarrow \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N}n}$$

Zero-Padding For Frequency Analysis

- What does it mean to compute a DFT_4 of an N=2 sequence?

- Assume sequence is zero elsewhere

Example: Compute DFT_4 of: $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

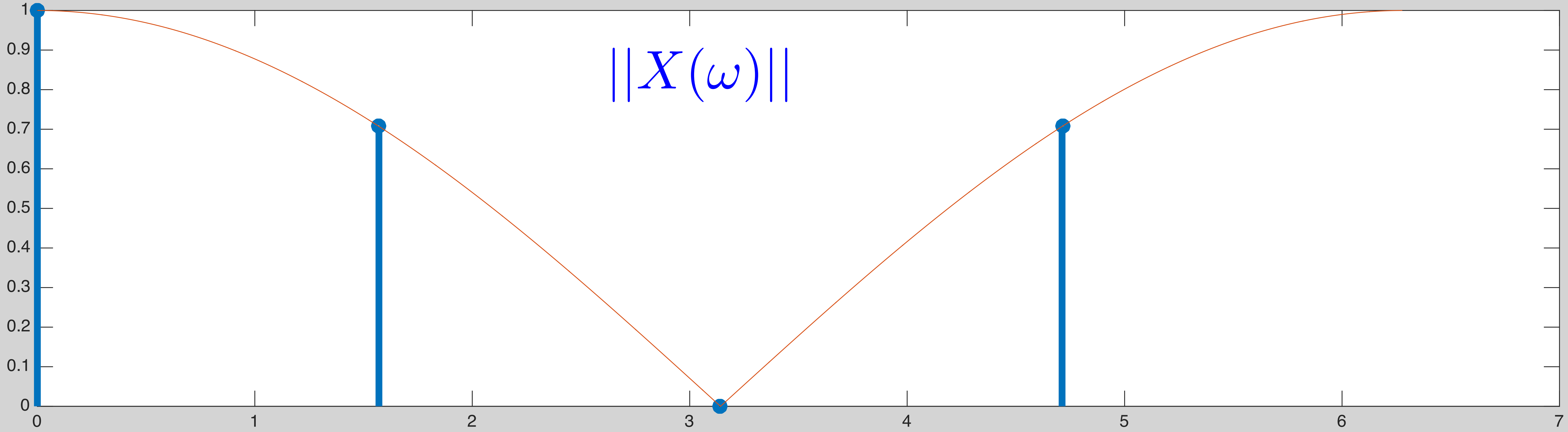
Zeropad:

$$\vec{X} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Zero-Padding For Frequency Analysis

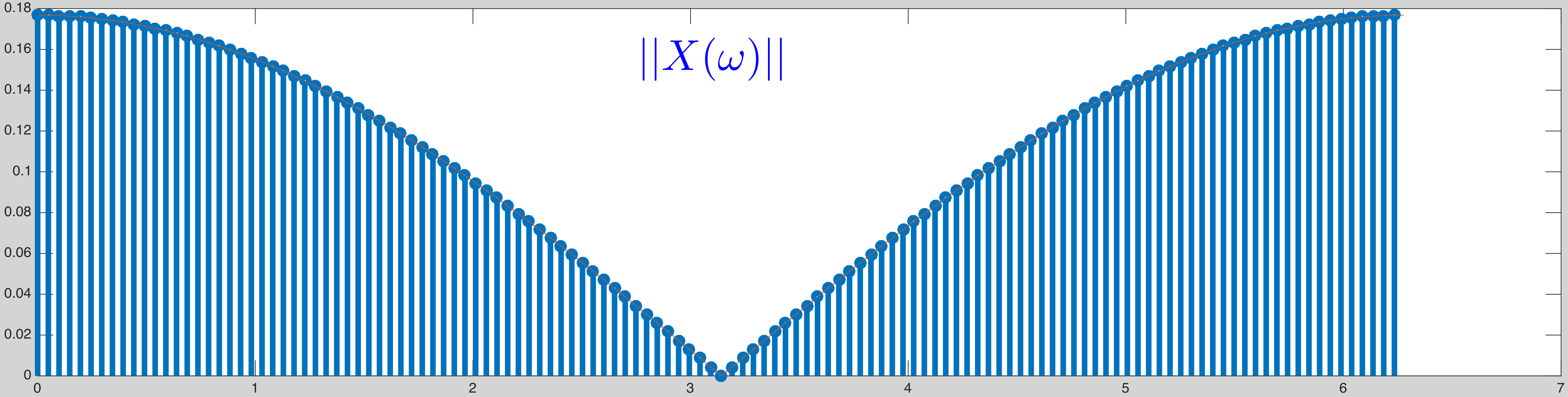
$$\vec{X} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cdots & W_4^{-n\cdot 0} & \cdots \\ \cdots & W_4^{-n\cdot 1} & \cdots \\ \cdots & W_4^{-n\cdot 2} & \cdots \\ \cdots & W_4^{-n\cdot 3} & \cdots \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -j \\ 1 & -1 \\ 1 & +j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & W_2^{-1\cdot 0} \\ 1 & W_2^{-1\cdot 0.5} \\ 1 & W_2^{-1\cdot 1} \\ 1 & W_2^{-1\cdot 1.5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Zeropadding

- Zero-pad to 128



- Note that result should be scaled by $\frac{\sqrt{N_{\text{zp}}}}{\sqrt{N}}$

Properties of the DFT

- Scaling and superposition: $\vec{X} = F^*\vec{x} \qquad \vec{Y} = F^*\vec{y}$

$$F^*(a\vec{x}) = aF^*\vec{x} = a\vec{X}$$

$$F^*(\vec{x}+\vec{y}) = F^*\vec{x} + F^*\vec{y} = \vec{X} + \vec{Y}$$

Properties of the DFT

- Parseval's relation (Energy conservation)

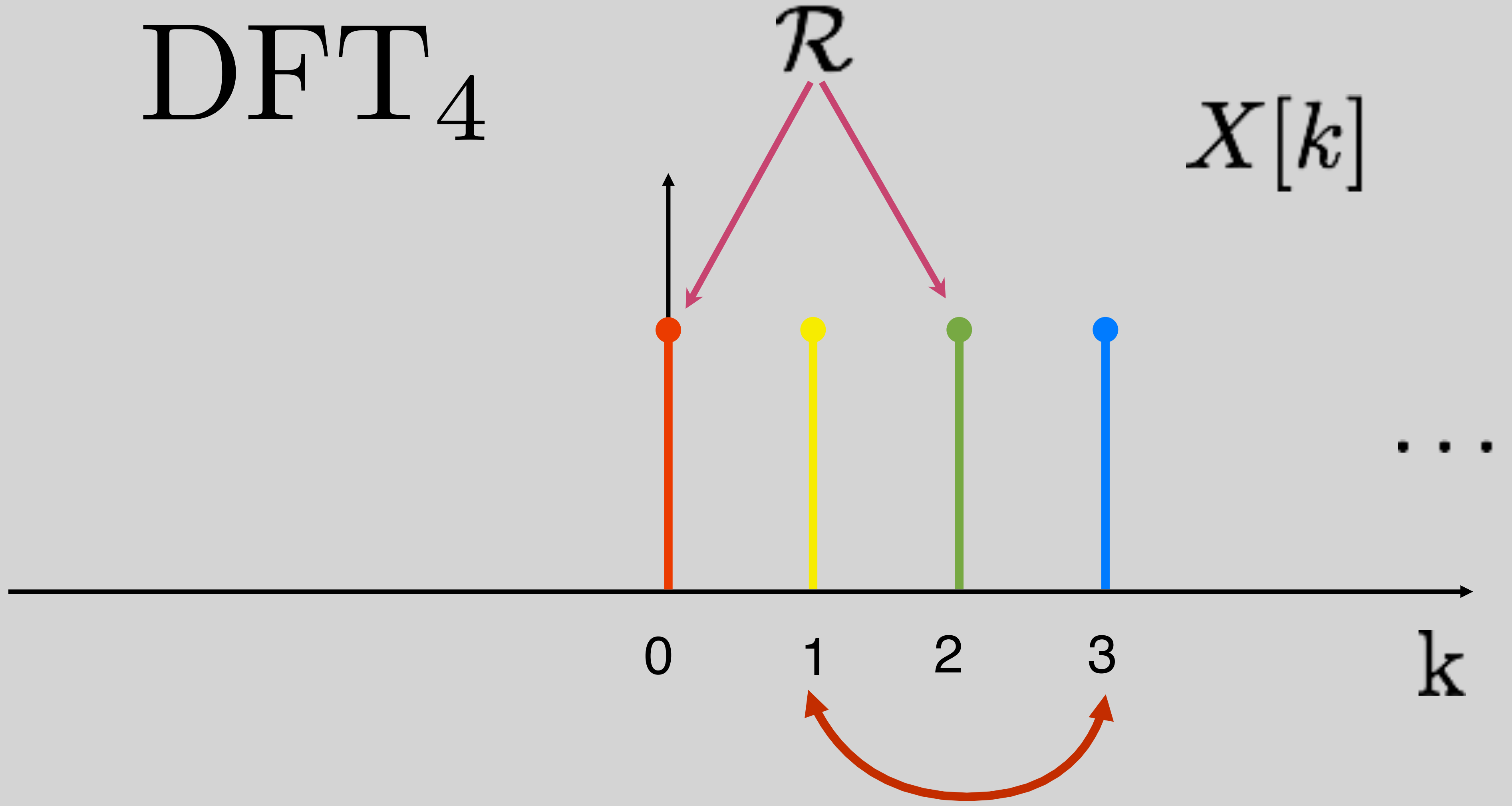
$$\vec{X} = F^* \vec{x} \;\; \Rightarrow \;\; \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} |X[n]|^2$$

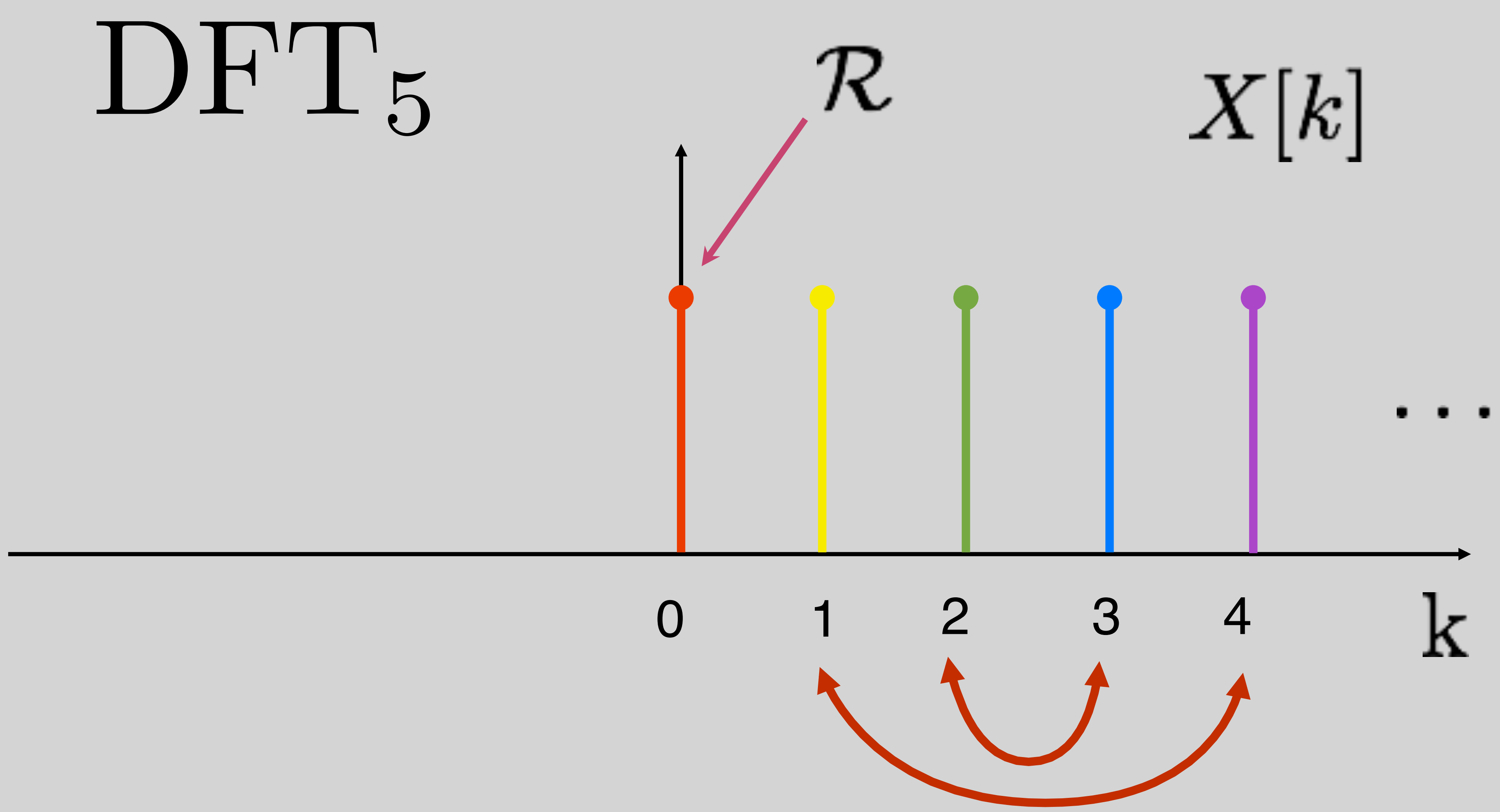
$$\Rightarrow \vec{x}^* \vec{x} = (F\vec{X})^*(F\vec{X}) = \vec{X}^* F^* F \vec{X} = \vec{X}^* \vec{X}$$

Conjugate Symmetry

- When $\vec{x} \in \mathrm{R}^N$
  the DFT coefficients satisfy:

$$X[N-k] = X^*[k] \qquad k = 1, 2, \cdots, N-1$$





Proof: Use properties of: $W_N^{(N-k)} = (W_N^k)^*$, DFT and the realness of x

Modulation and Circular shift

Modulation – Circular shift

$$x[n]e^{j\frac{2\pi n}{N}k_0} = x[n]W_N^{nk_0} \Rightarrow X[\text{mod}_N(k-k_0)]$$

$$\Rightarrow \text{DFT}_N\{x[n]W_N^{nk_0}\} = \sum_{n=0}^{N-1} x[n]W_N^{nk_0}W_N^{-nk}$$

$$= \sum_{n=0}^{N-1} x[n]W_N^{-n(k-k_0)} = X[\text{mod}_N(k-k_0)]$$

Similarly, circular shift - modulation

$$x[\text{mod}_N(n-n_0)] \Rightarrow X[k]W_N^{-kn_0}$$

DFT Matrix and Circulant Matrices

- DFT diagonalizes Circulant matrices:

$$C = \begin{bmatrix} c[0] & c[N-1] & \cdots & c[2] & c[1] \\ c[1] & c[0] & c[N-1] & & c[2] \\ \vdots & c[1] & c[0] & \ddots & \vdots \\ c[N-2] & \vdots & \ddots & \ddots & c[N-1] \\ c[N-1] & c[N-2] & \cdots & c[1] & c[0] \end{bmatrix}$$

$$F^*CF = \sqrt{N} \begin{bmatrix} C[0] & & & \\ & C[1] & & \\ & & \ddots & \\ & & & C[N-1] \end{bmatrix}$$

where, $\vec{C} = F^*\vec{c}$

DFT Matrix and Circulant Matrices

$$F^*\begin{bmatrix} c[0] & c[N-1] & \cdots & c[2] & c[1] \\ c[1] & c[0] & c[N-1] & & c[2] \\ \vdots & c[1] & c[0] & \ddots & \vdots \\ c[N-2] & \vdots & \ddots & \ddots & c[N-1] \\ c[N-1] & c[N-2] & \cdots & c[1] & c[0] \end{bmatrix} F = \begin{bmatrix} C[0] & C[0] & & C[0] \\ C[1] & W_N^{-1\cdot 1}C[1] & & W_N^{-(N-1)1}C[1] \\ C[k] & W_N^{-1\cdot k}C[k] & \cdots\vdots\cdots & W_N^{-(N-1)k}C[k] \\ C[N-1] & W_N^{-1\cdot(N-1)}C[N-1] & & W_N^{-(N-1)(N-1)}C[N-1] \end{bmatrix} F$$

$$= \begin{bmatrix} C[0] & & & 0 \\ & C[1] & & \\ & & \ddots & \\ 0 & & & C[N-1] \end{bmatrix} \begin{bmatrix} 1 & 1 & & 1 \\ 1 & W_N^{-1\cdot 1} & & W_N^{-(N-1)1} \\ 1 & W_N^{-1\cdot k} & \cdots\vdots\cdots & W_N^{-(N-1)k} \\ 1 & W_N^{-1\cdot(N-1)} & & W_N^{-(N-1)(N-1)} \end{bmatrix} F$$

$$= \sqrt{N}\begin{bmatrix} C[0] & & & 0 \\ & C[1] & & \\ & & \ddots & \\ 0 & & & C[N-1] \end{bmatrix} F^*F = \sqrt{N}\begin{bmatrix} C[0] & & & 0 \\ & C[1] & & \\ & & \ddots & \\ 0 & & & C[N-1] \end{bmatrix}$$

Fast Circulant Matrix Vector Multiplication

- Given : $\vec{X} = F^*\vec{x} \qquad \vec{C} = F^*\vec{c} \qquad \vec{Y} = F^*\vec{y}$

$$C \leftarrow \text{circulant}$$

- If, $\vec{y} = C\vec{x}$ then, $\vec{Y} = \sqrt{N}(\vec{C} \cdot \vec{X})$

$$F^*\vec{y} = F^*C\vec{x}$$

$$F^*\vec{y} = F^*CFF^*\vec{x}$$

$$\vec{Y} = \sqrt{N}\begin{bmatrix} C[0] & & & 0 \\ & C[1] & & \\ & & \ddots & \\ 0 & & & C[N-1] \end{bmatrix}\vec{X}$$

Fast Circulant Matrix Vector Multiplication

- Why bother?
- Option I, compute: $\vec{y} = C\vec{x} \qquad \Rightarrow O(N^2)$
- Option II, compute: $\vec{y} = F\big((F^*\vec{c}) \cdot (F^*\vec{x})\big) \quad \Rightarrow O(N^2)$

Using the fast Fourier Transform (FFT)
calculation of the DFT (and inverse) is O(N log N)

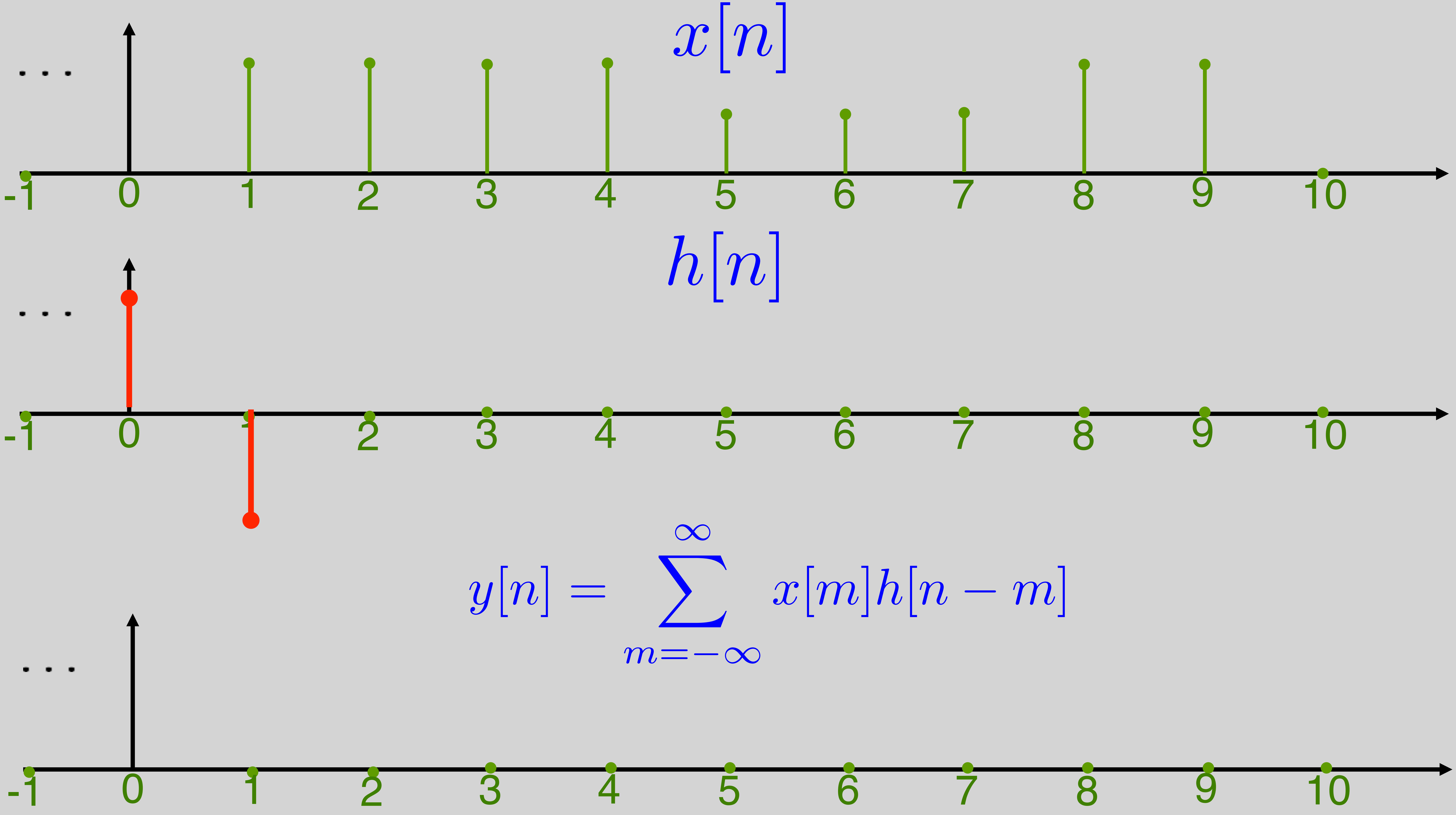
For N = 1000: N^2 = 1,048,576 whereas, N log N = 10240

Fast Convolution Sum using the DFT

- We can write linear operators on finite sequences as matrix vector multiplication

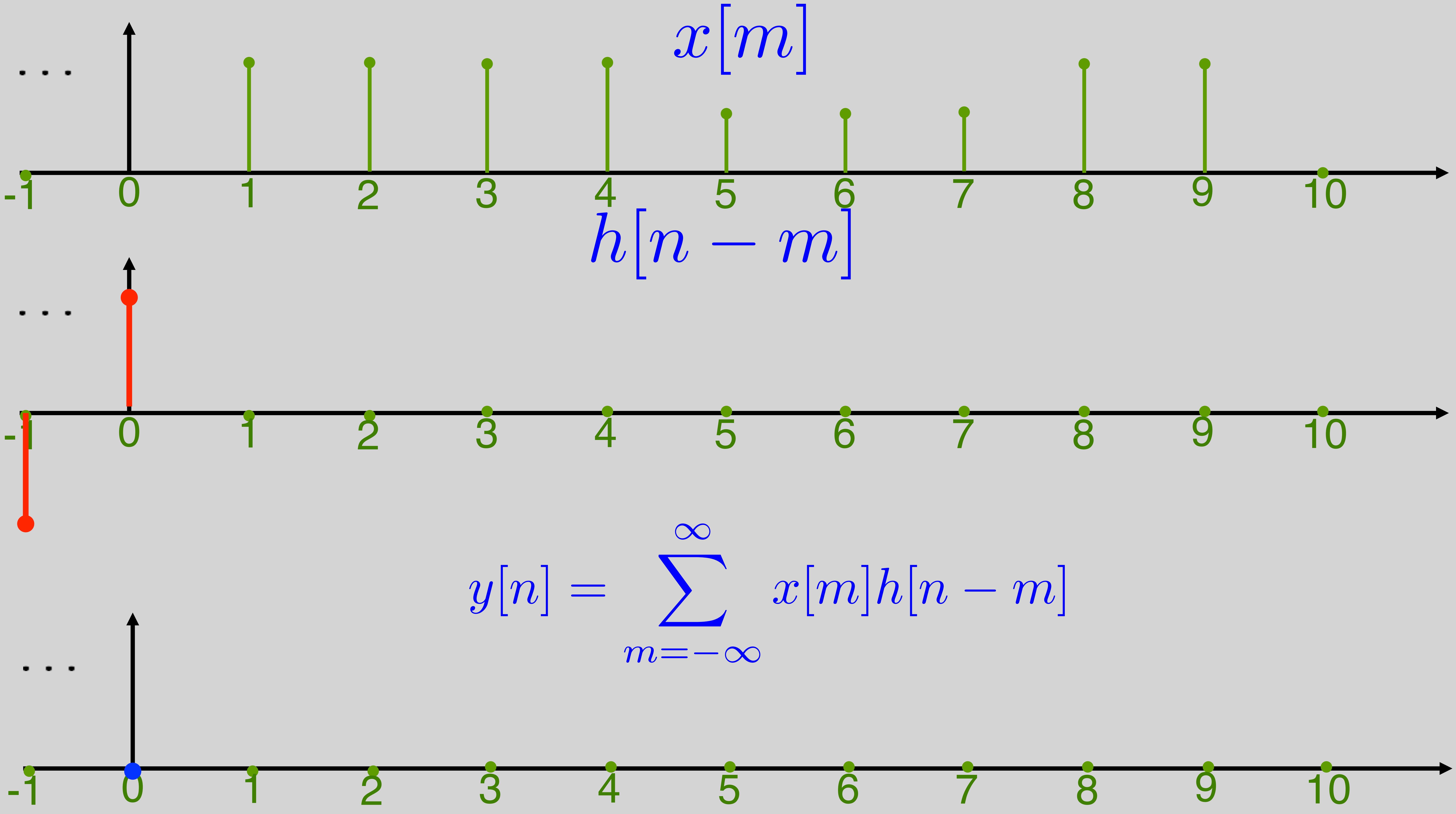
- Recall… convolution sum.....

Graphical Example of Convolution



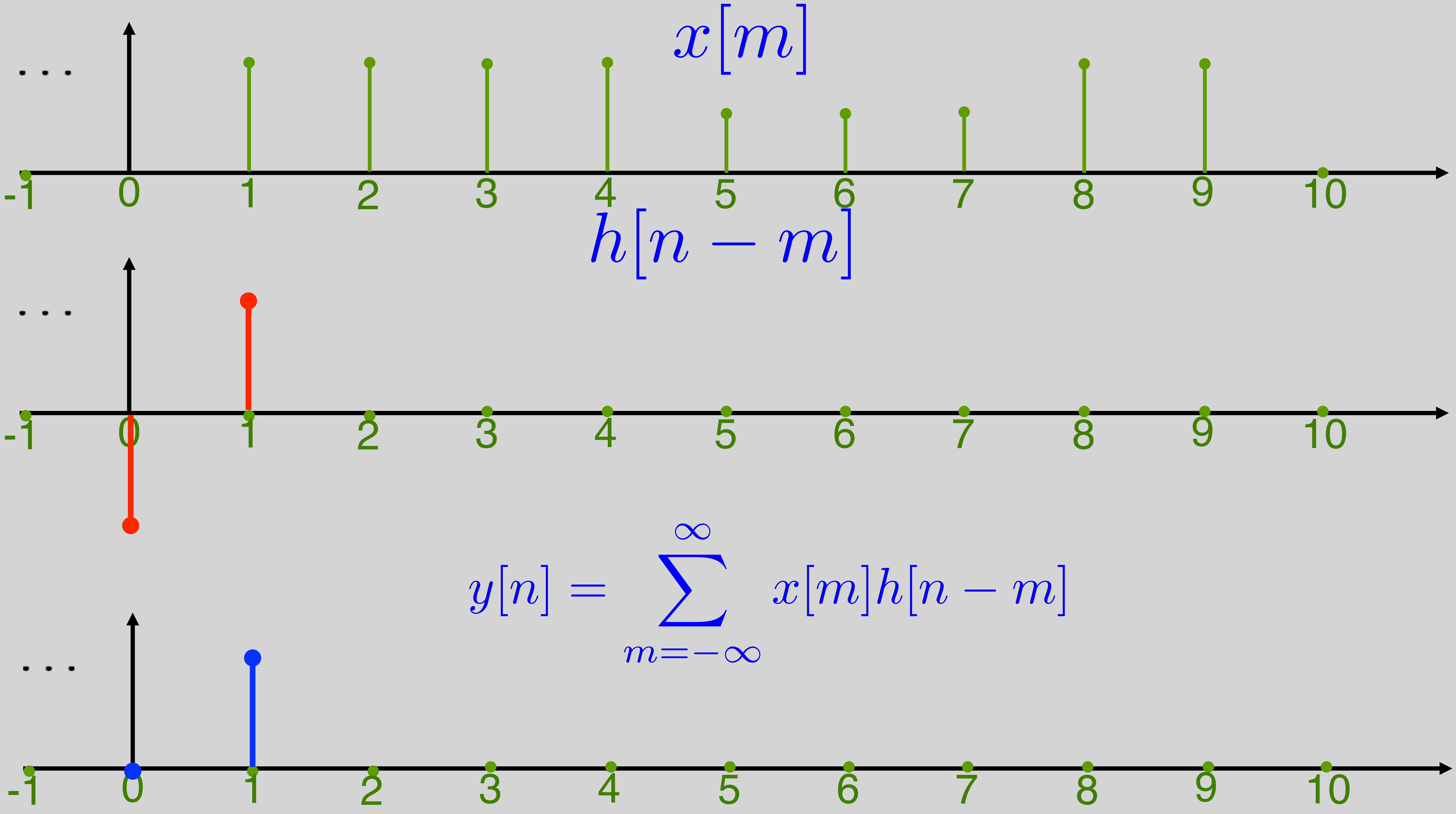
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



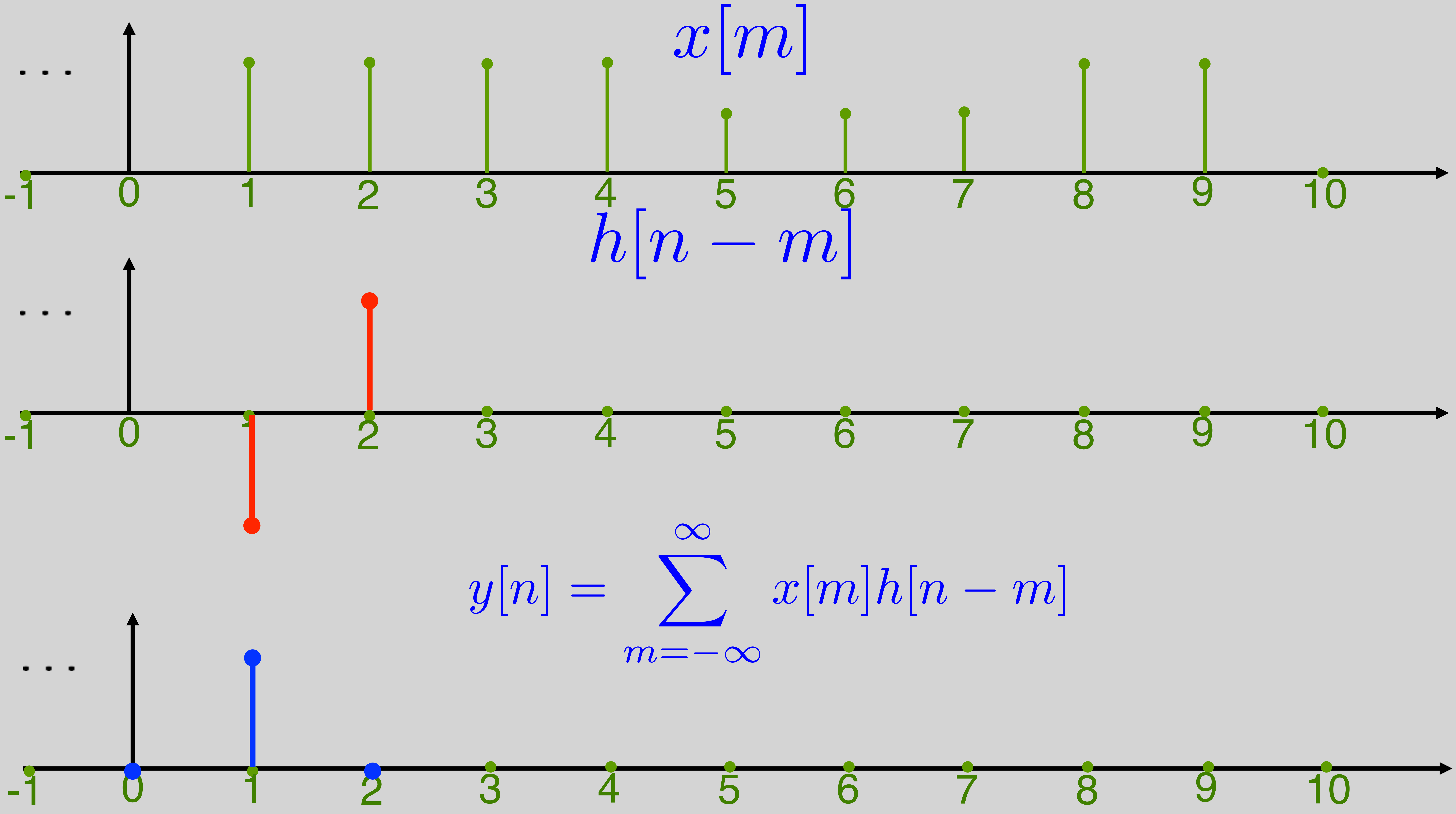
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



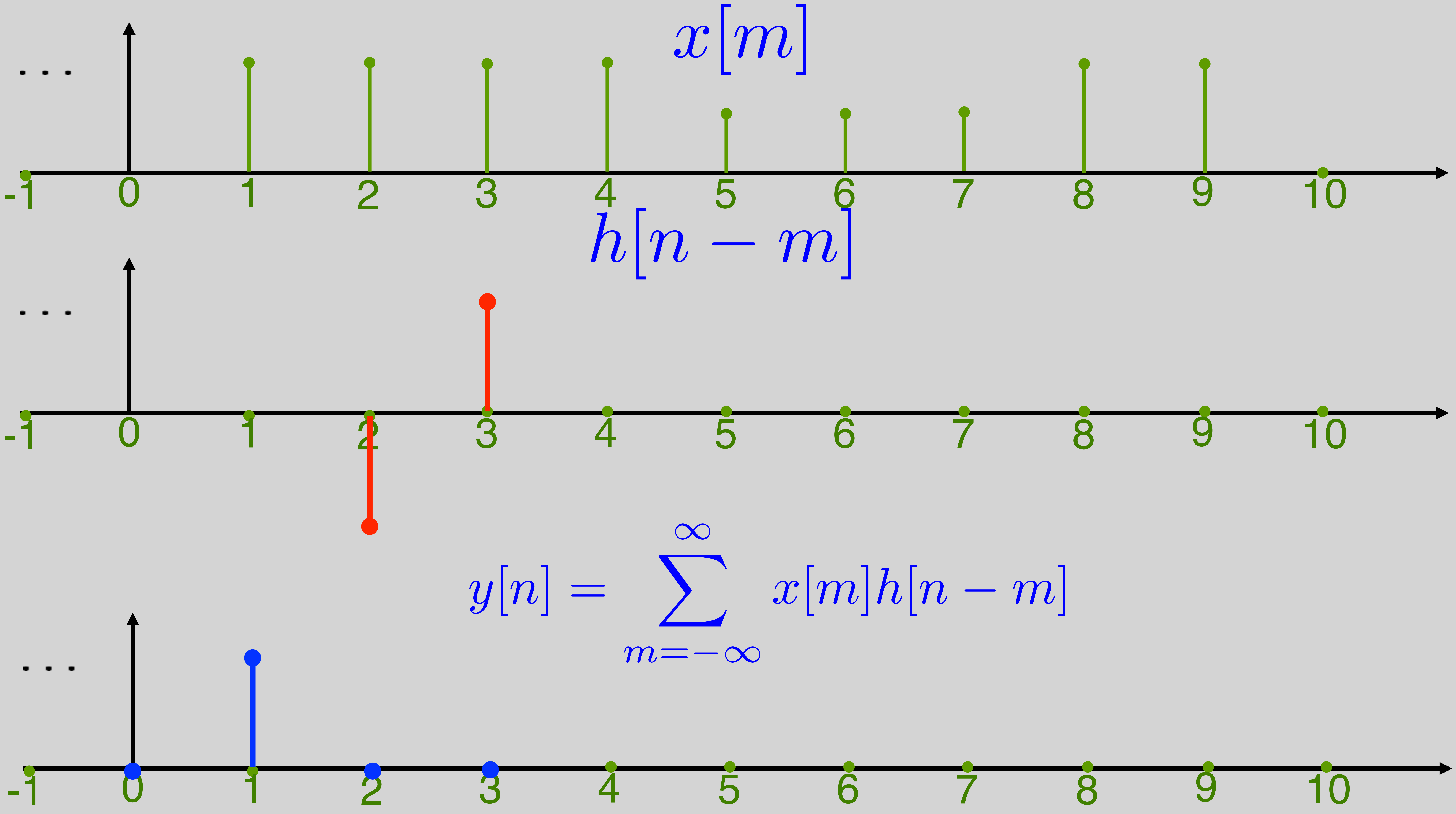
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

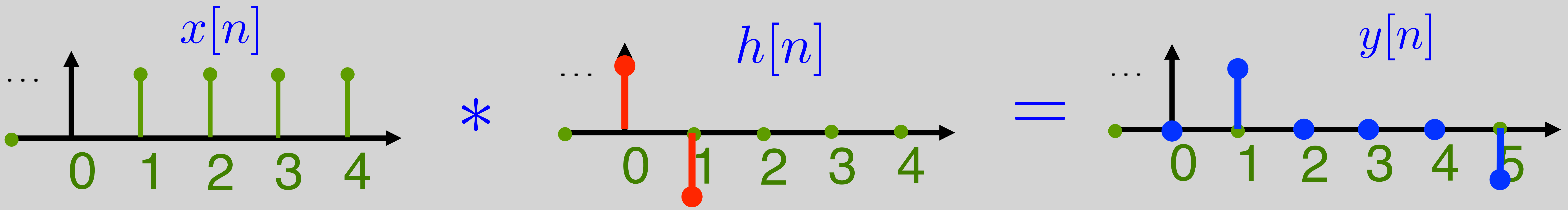
Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

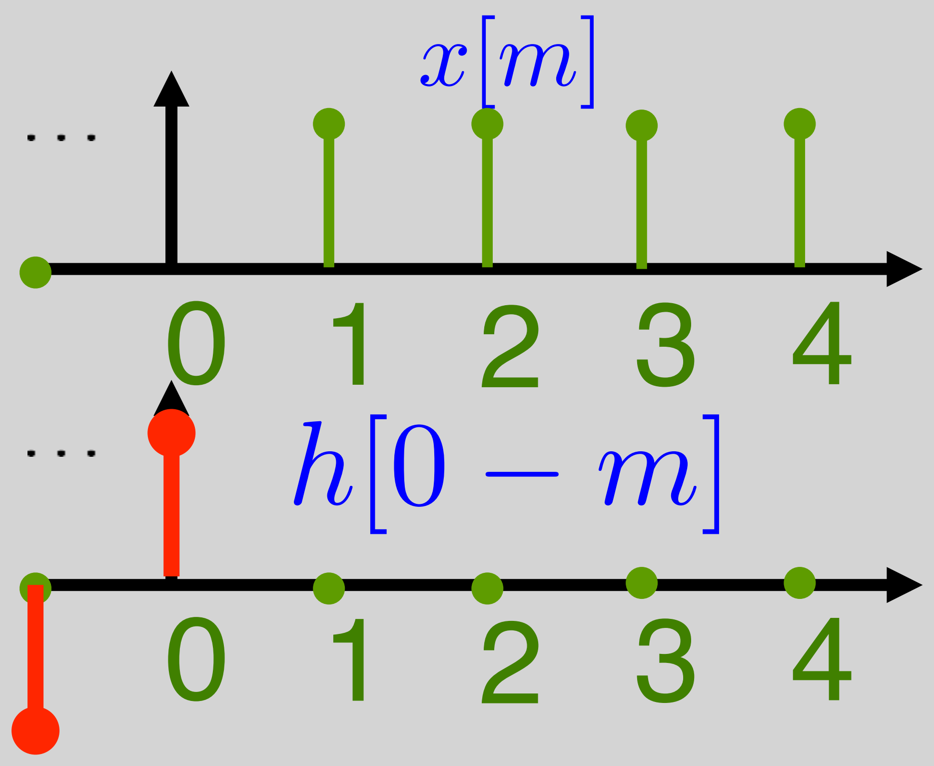
Example:

If h[n] is length 2 and x[n] is length 5, what is the length of their convolution sum?



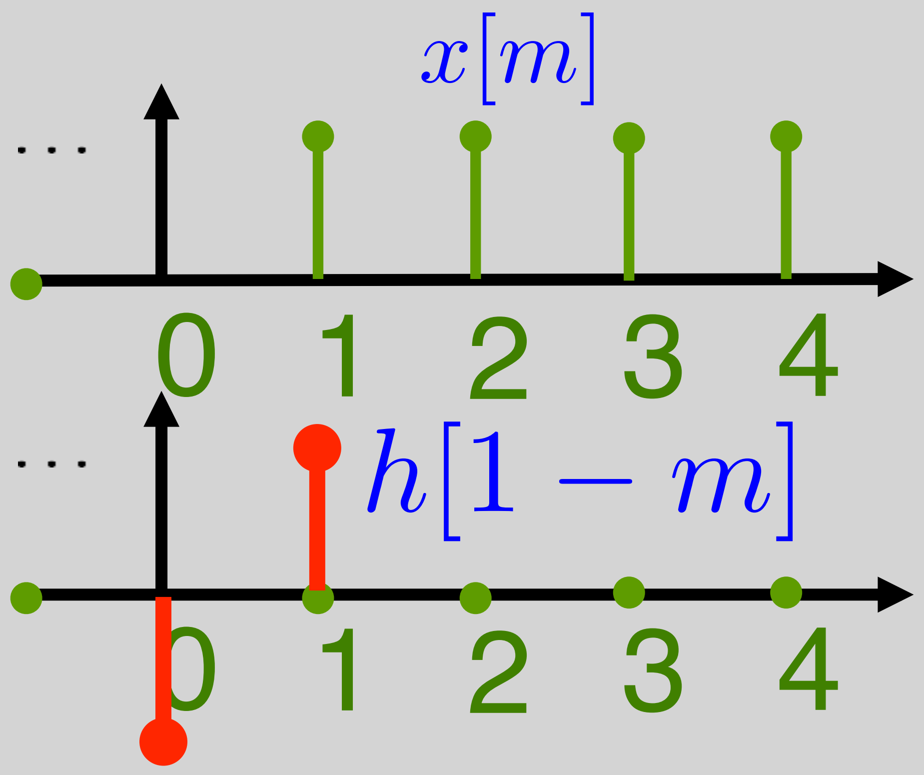
Example

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$



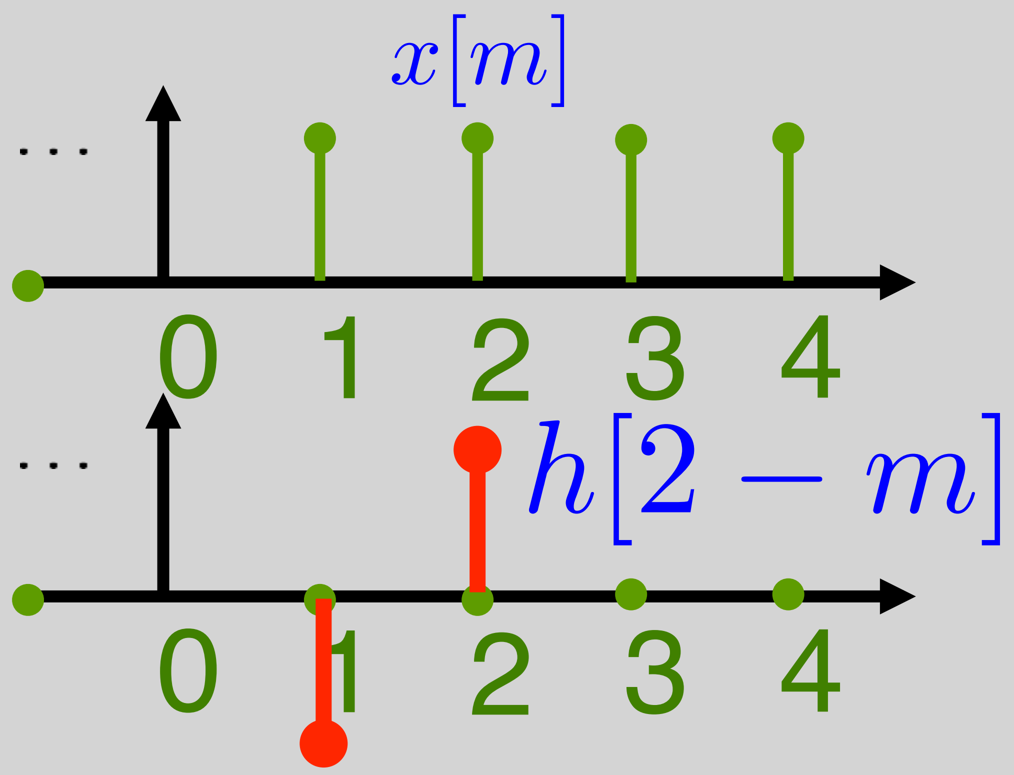
Example

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$



Example

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$



Example

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$

- This matrix is called a Toeplitz matrix
  - But.. Not square… not circulant....

Example

- Convert system to be square circulant by zero-padding

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ 0 \end{bmatrix}$$

- Now can compute using the DFT!

General Case for Convolution Sum

- Given: $\vec{h} \in \mathrm{R}^M$ $\vec{x} \in \mathrm{R}^N$
- Zeropad both to M+N-1 $\vec{h}_{\mathrm{zp}} \in \mathrm{R}^{N+M-1}$ $\vec{x}_{\mathrm{zp}} \in \mathrm{R}^{N+M-1}$
- Compute: $\vec{H} = F^* \vec{h}_{\mathrm{zp}}$ $\vec{X} = F^* \vec{x}_{\mathrm{zp}}$

$$\vec{Y} = \vec{H} \cdot \vec{X}$$

- Finally: $\vec{y} = F\vec{Y}$