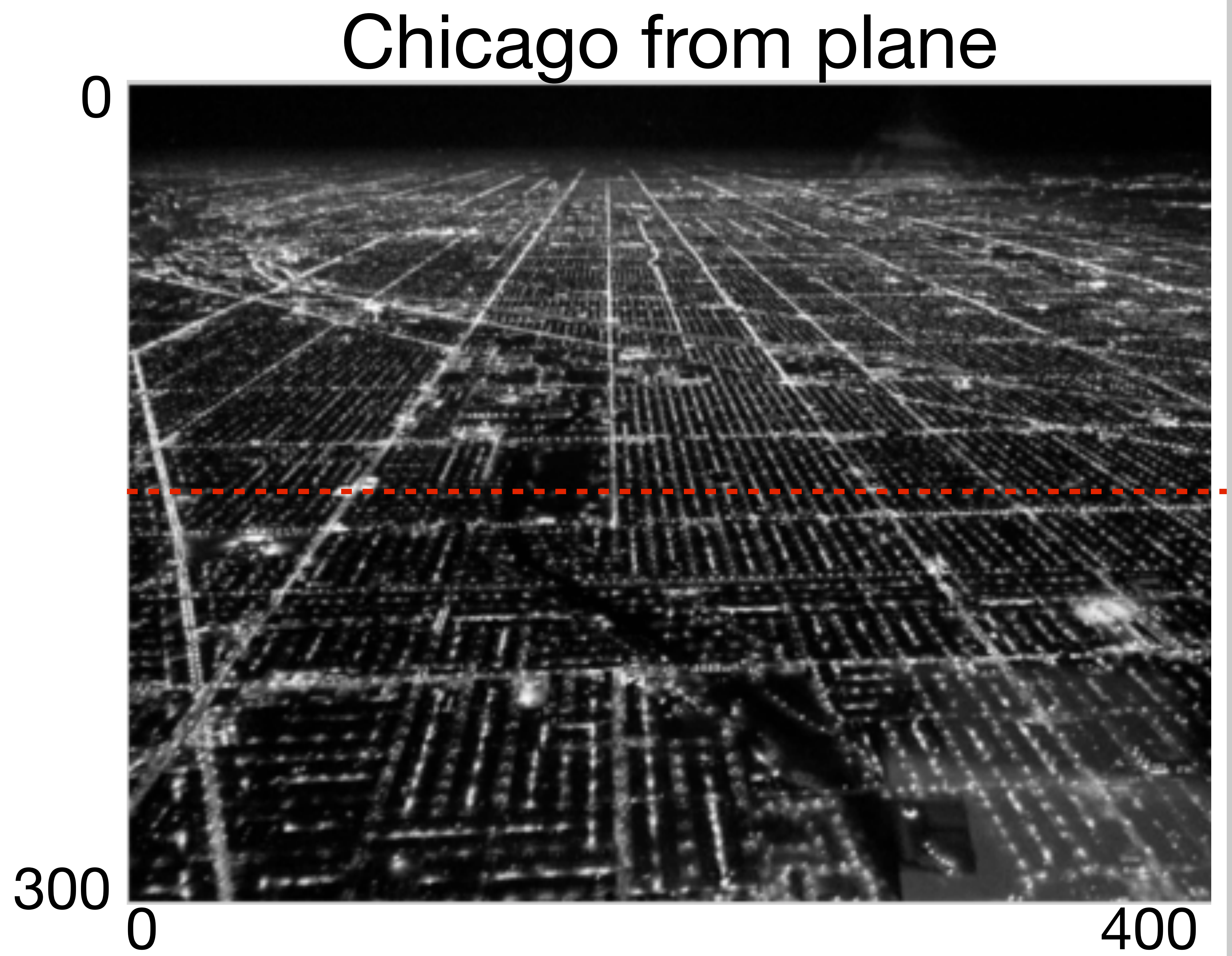
EE16B
Designing Information Devices and Systems II

Lecture 14B
Convolutions using the DFT


Chicago from plane
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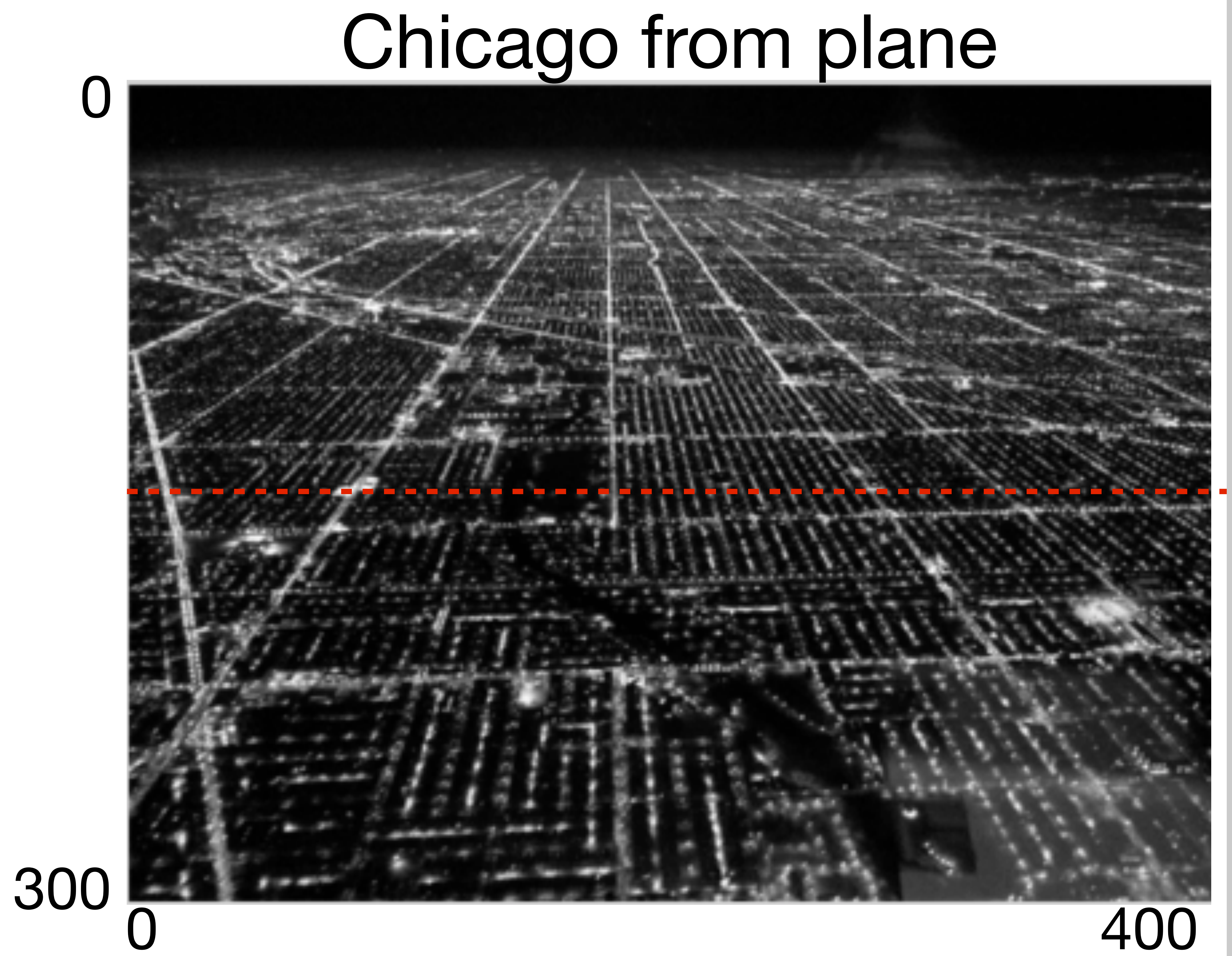
Chicago from plane
0
300
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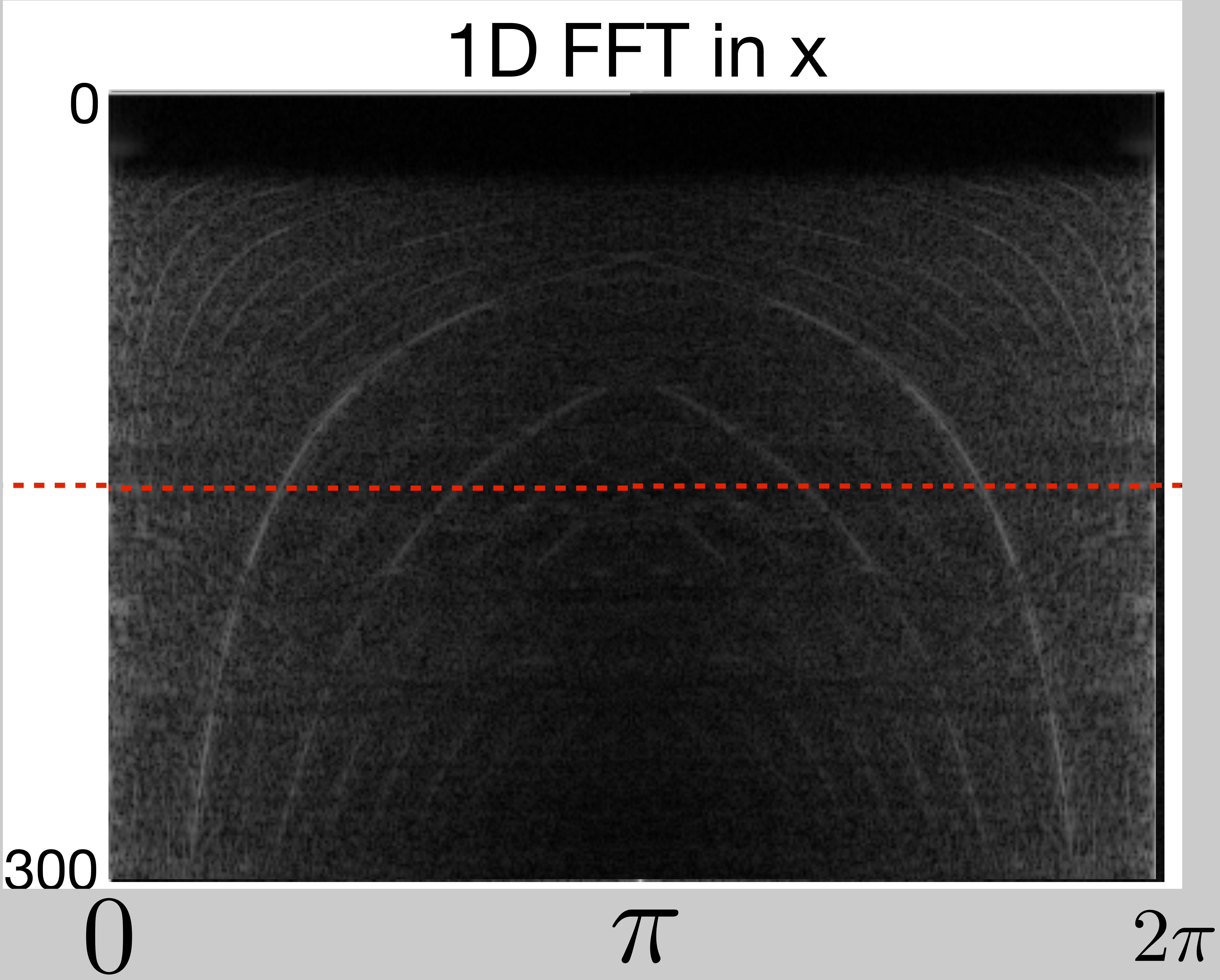
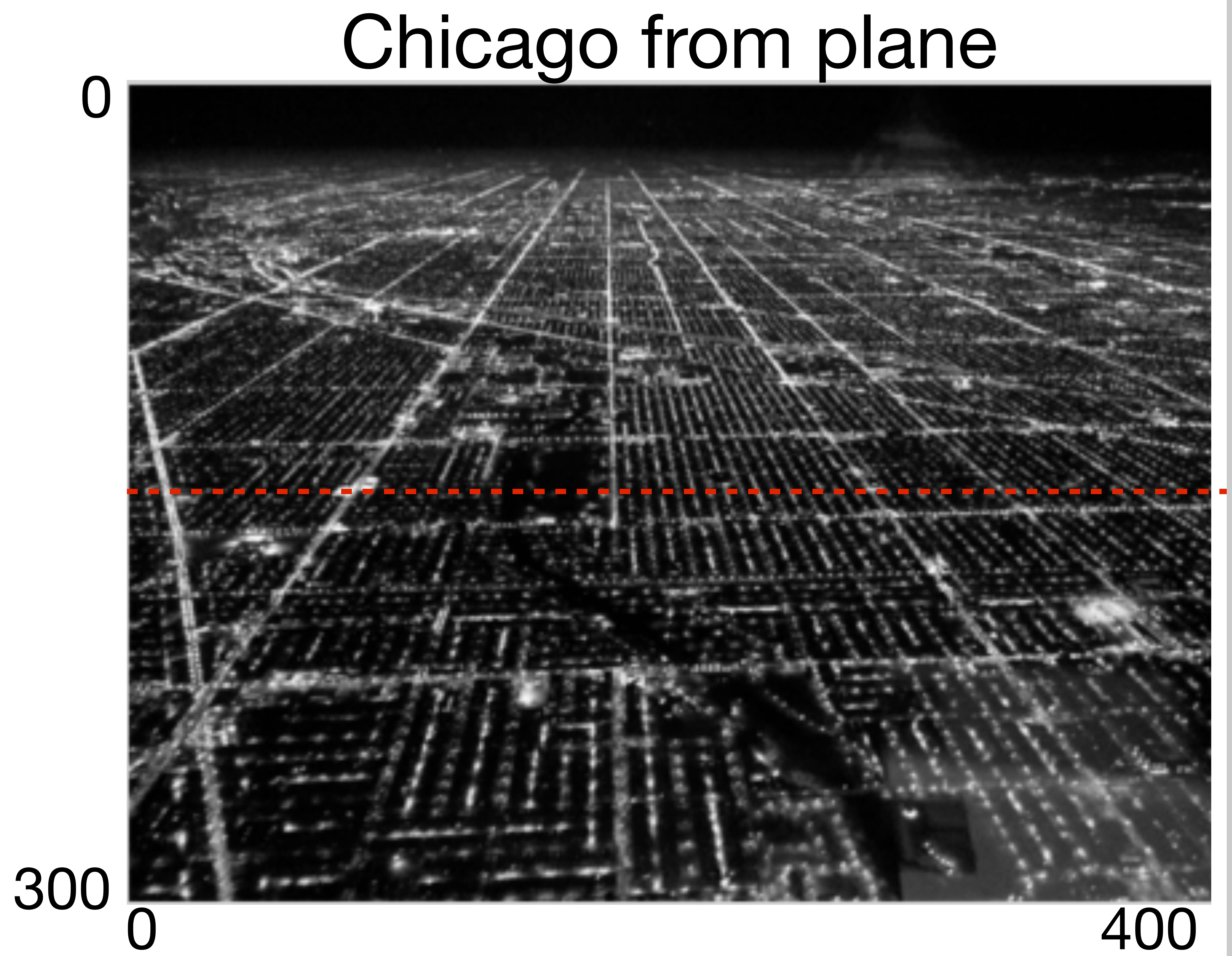
400


1D FFT in x
0
300
0
π
2π


Chicago from plane
0
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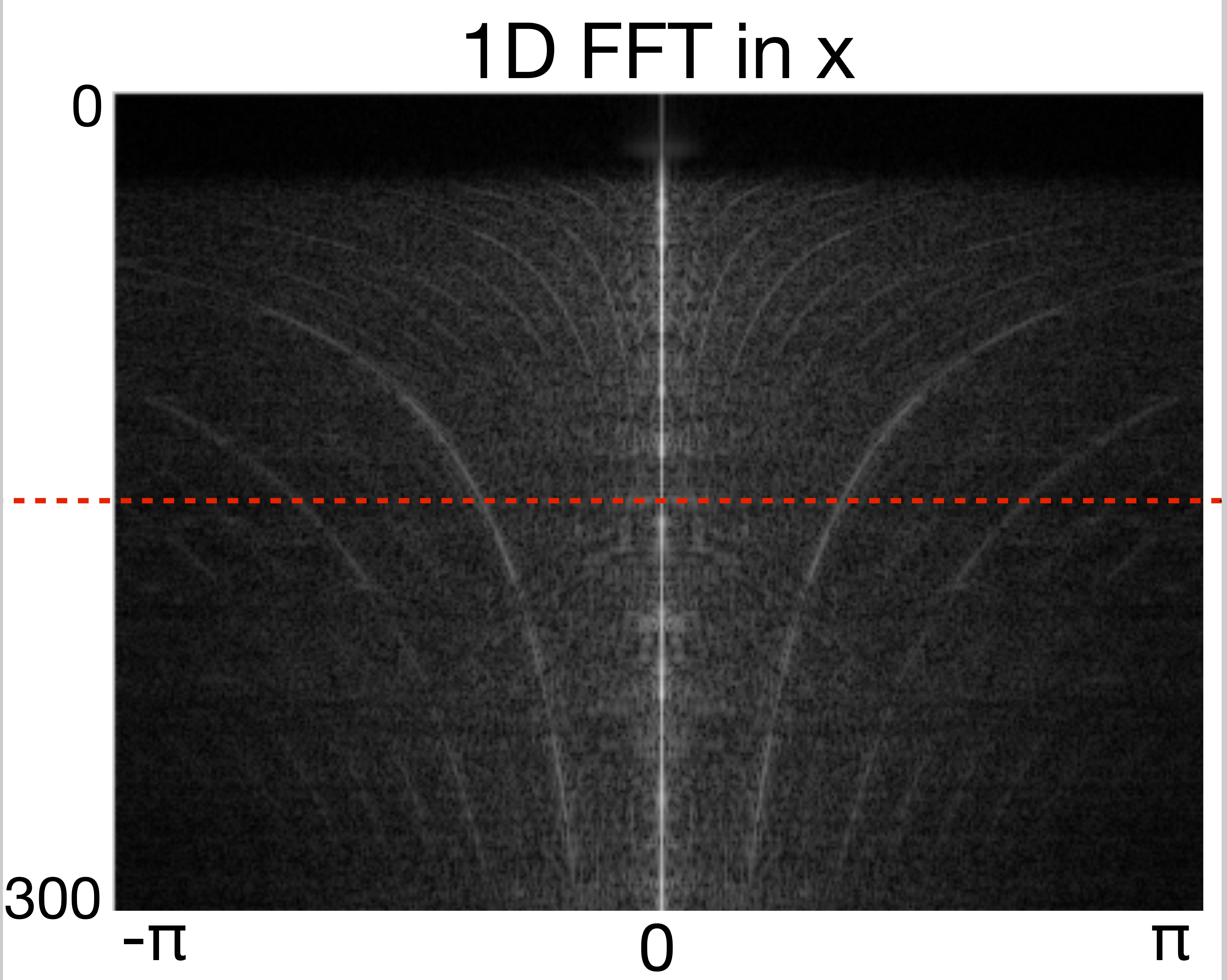
0
400


1D FFT in x
0
300
-π
0
π

Modulation and Circular shift

Modulation – Circular shift

$$x[n]e^{j\frac{2\pi n}{N}k_0} = x[n]W_N^{nk_0} \Rightarrow X[\text{mod}_N(k-k_0)]$$

$$\Rightarrow \text{DFT}_N\{x[n]W_N^{nk_0}\} = \sum_{n=0}^{N-1} x[n]W_N^{nk_0}W_N^{-nk}$$

$$= \sum_{n=0}^{N-1} x[n]W_N^{-n(k-k_0)} = X[\text{mod}_N(k-k_0)]$$

Similarly, circular shift - modulation

$$x[\text{mod}_N(n-n_0)] \Rightarrow X[k]W_N^{-kn_0}$$

DFT Matrix and Circulant Matrices

- DFT diagonalizes Circulant matrices:

$$C = \begin{bmatrix} c[0] & c[N-1] & \cdots & c[2] & c[1] \\ c[1] & c[0] & c[N-1] & & c[2] \\ \vdots & c[1] & c[0] & \ddots & \vdots \\ c[N-2] & \vdots & \ddots & \ddots & c[N-1] \\ c[N-1] & c[N-2] & \cdots & c[1] & c[0] \end{bmatrix}$$

$$F^*CF = \sqrt{N} \begin{bmatrix} C[0] & & & \\ & C[1] & & \\ & & \ddots & \\ & & & C[N-1] \end{bmatrix} \qquad \text{where, } \vec{C} = F^*\vec{c}$$

DFT Matrix and Circulant Matrices

$$F^* \begin{bmatrix} c[0] & c[N-1] & \cdots & c[2] & c[1] \\ c[1] & c[0] & c[N-1] & & c[2] \\ \vdots & c[1] & c[0] & \ddots & \vdots \\ c[N-2] & \vdots & \ddots & \ddots & c[N-1] \\ c[N-1] & c[N-2] & \cdots & c[1] & c[0] \end{bmatrix} F = \begin{bmatrix} C[0] & C[0] & & C[0] \\ C[1] & W_N^{-1\cdot 1}C[1] & & W_N^{-(N-1)1}C[1] \\ C[k] & W_N^{-1\cdot k}C[k] & \cdots\vdots\cdots & W_N^{-(N-1)k}C[k] \\ C[N-1] & W_N^{-1\cdot(N-1)}C[N-1] & \vdots & W_N^{-(N-1)(N-1)}C[N-1] \end{bmatrix} F$$

$$= \begin{bmatrix} C[0] & & & 0 \\ & C[1] & & \\ & & \ddots & \\ 0 & & & C[N-1] \end{bmatrix} \begin{bmatrix} 1 & 1 & & 1 \\ 1 & W_N^{-1\cdot 1} & & W_N^{-(N-1)1} \\ 1 & W_N^{-1\cdot k} & \cdots\vdots\cdots & W_N^{-(N-1)k} \\ 1 & W_N^{-1\cdot(N-1)} & \vdots & W_N^{-(N-1)(N-1)} \end{bmatrix} F$$

$$= \sqrt{N} \begin{bmatrix} C[0] & & & 0 \\ & C[1] & & \\ & & \ddots & \\ 0 & & & C[N-1] \end{bmatrix} F^* F = \sqrt{N} \begin{bmatrix} C[0] & & & 0 \\ & C[1] & & \\ & & \ddots & \\ 0 & & & C[N-1] \end{bmatrix}$$

Fast Circulant Matrix Vector Multiplication

- Given : $\vec{X} = F^*\vec{x}$ $\quad \vec{C} = F^*\vec{c}$ $\quad \vec{Y} = F^*\vec{y}$

$$C \leftarrow \text{circulant}$$

- If, $\vec{y} = C\vec{x}$ then, $\vec{Y} = \sqrt{N}(\vec{C} \cdot \vec{X})$

$$F^*\vec{y} = F^*C\vec{x}$$

$$F^*\vec{y} = F^*CFF^*\vec{x}$$

$$\vec{Y} = \sqrt{N} \begin{bmatrix} C[0] & & & 0 \\ & C[1] & & \\ & & \ddots & \\ 0 & & & C[N-1] \end{bmatrix} \vec{X}$$

Fast Circulant Matrix Vector Multiplication

- Why bother?
- Option I, compute: $\vec{y} = C\vec{x} \quad \Rightarrow O(N^2)$
- Option II, compute: $\vec{y} = F\big((F^*\vec{c}) \cdot (F^*\vec{x})\big) \quad \Rightarrow O(N^2)$

Using the fast Fourier Transform (FFT)
calculation of the DFT (and inverse) is O(N log N)

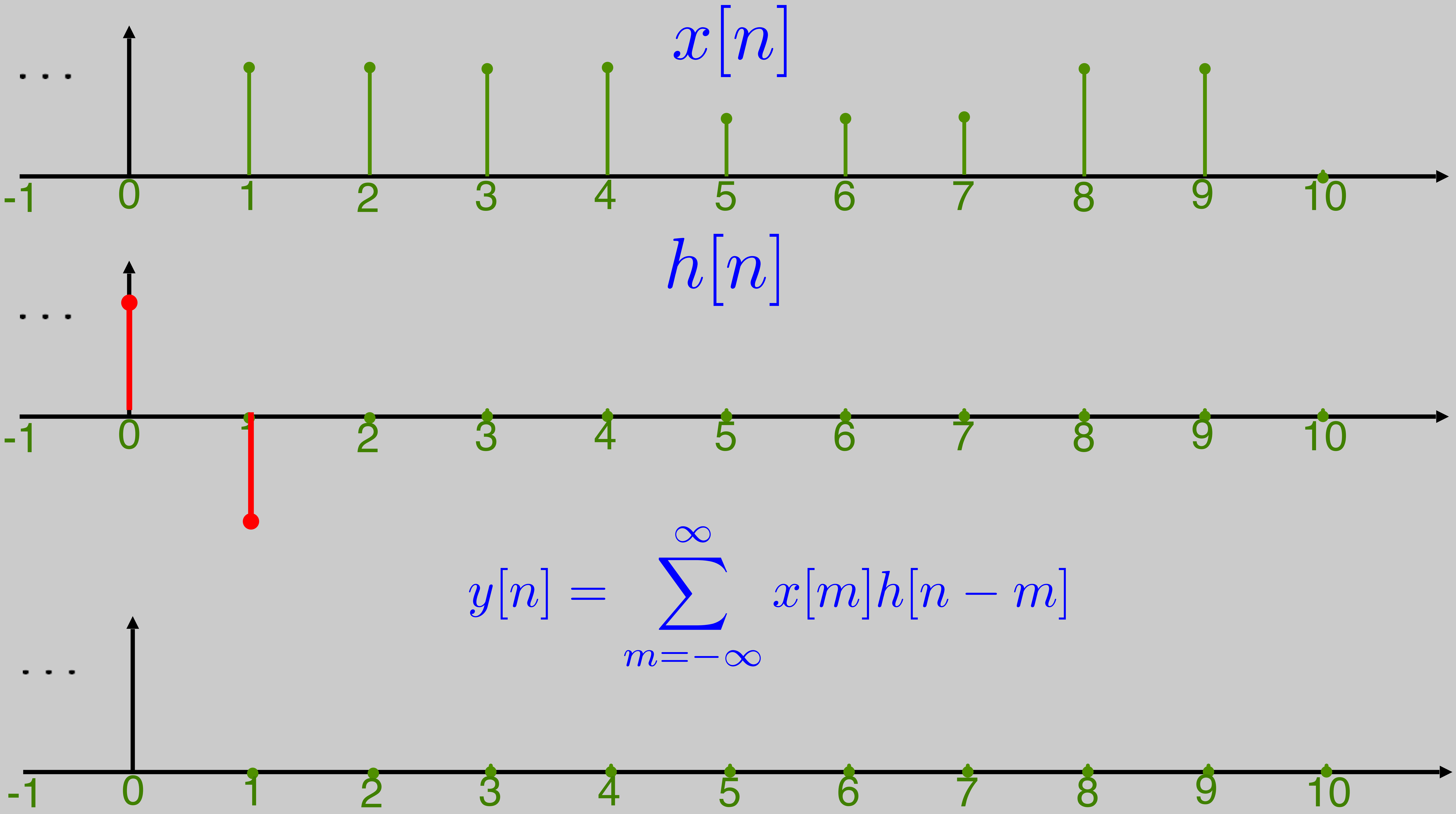
For N = 1000: N^2 = 1,048,576 whereas, N log N = 10240

Fast Convolution Sum using the DFT

- We can write linear operators on finite sequences as matrix vector multiplication

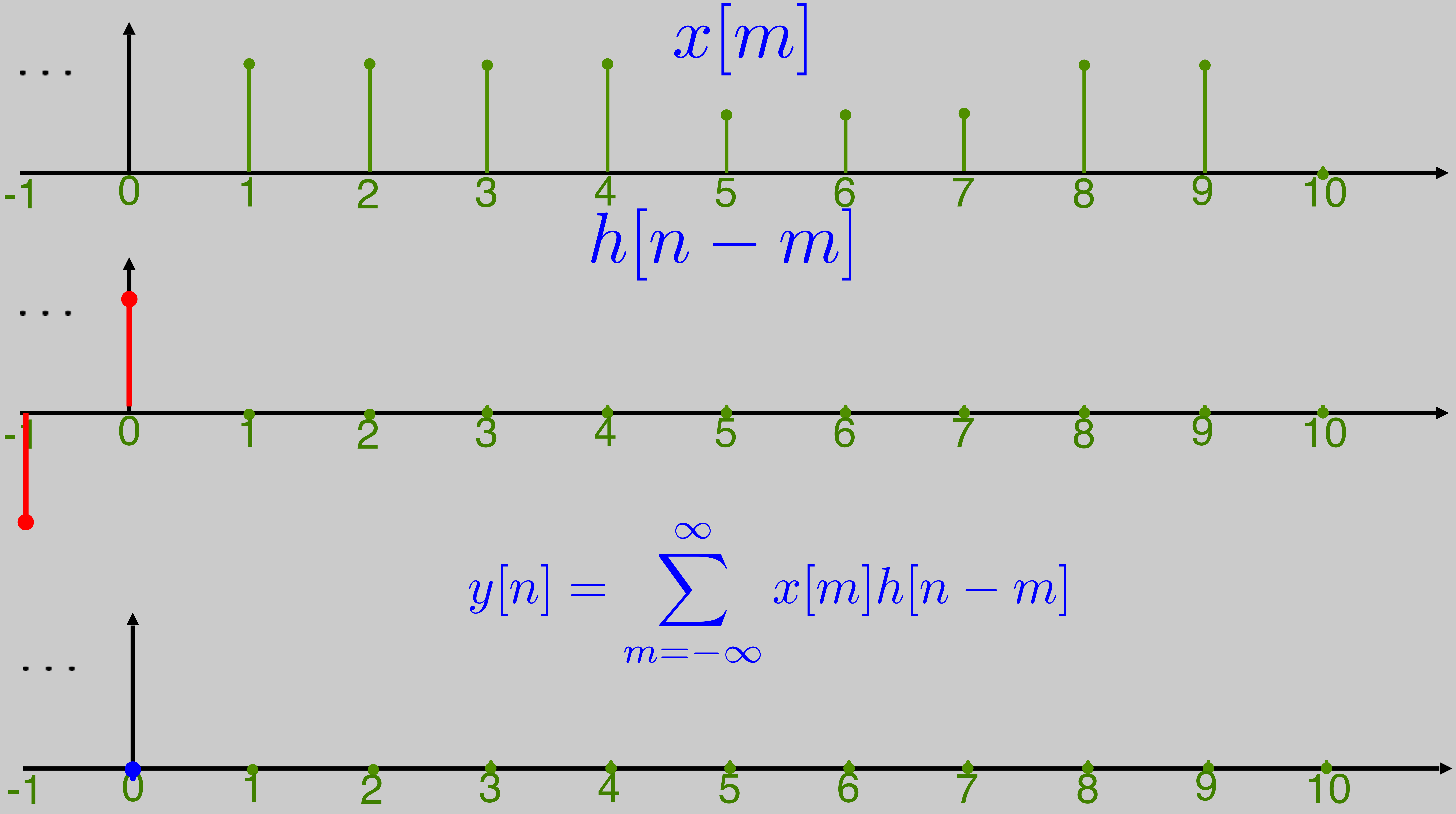
- Recall… convolution sum.....

Graphical Example of Convolution



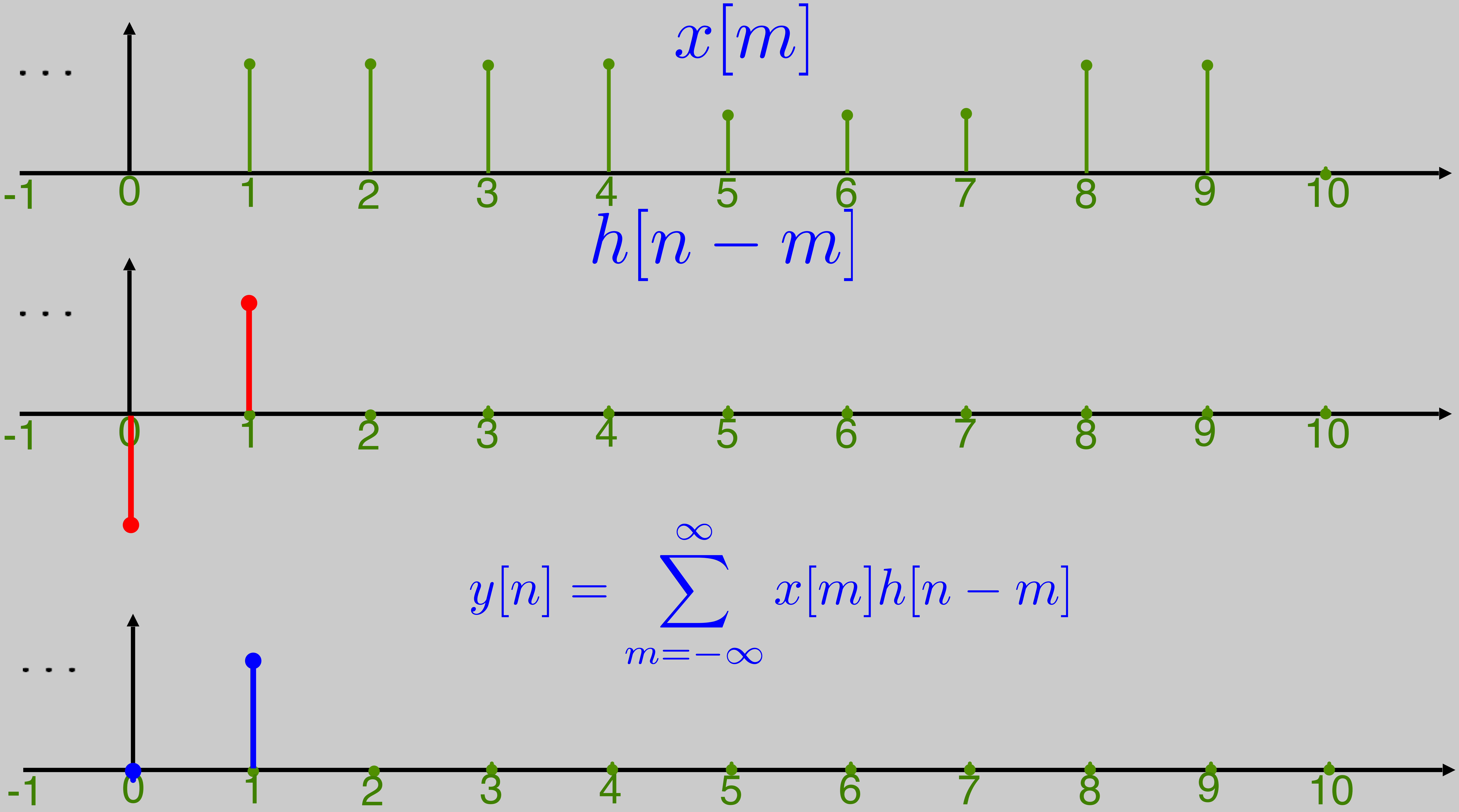
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



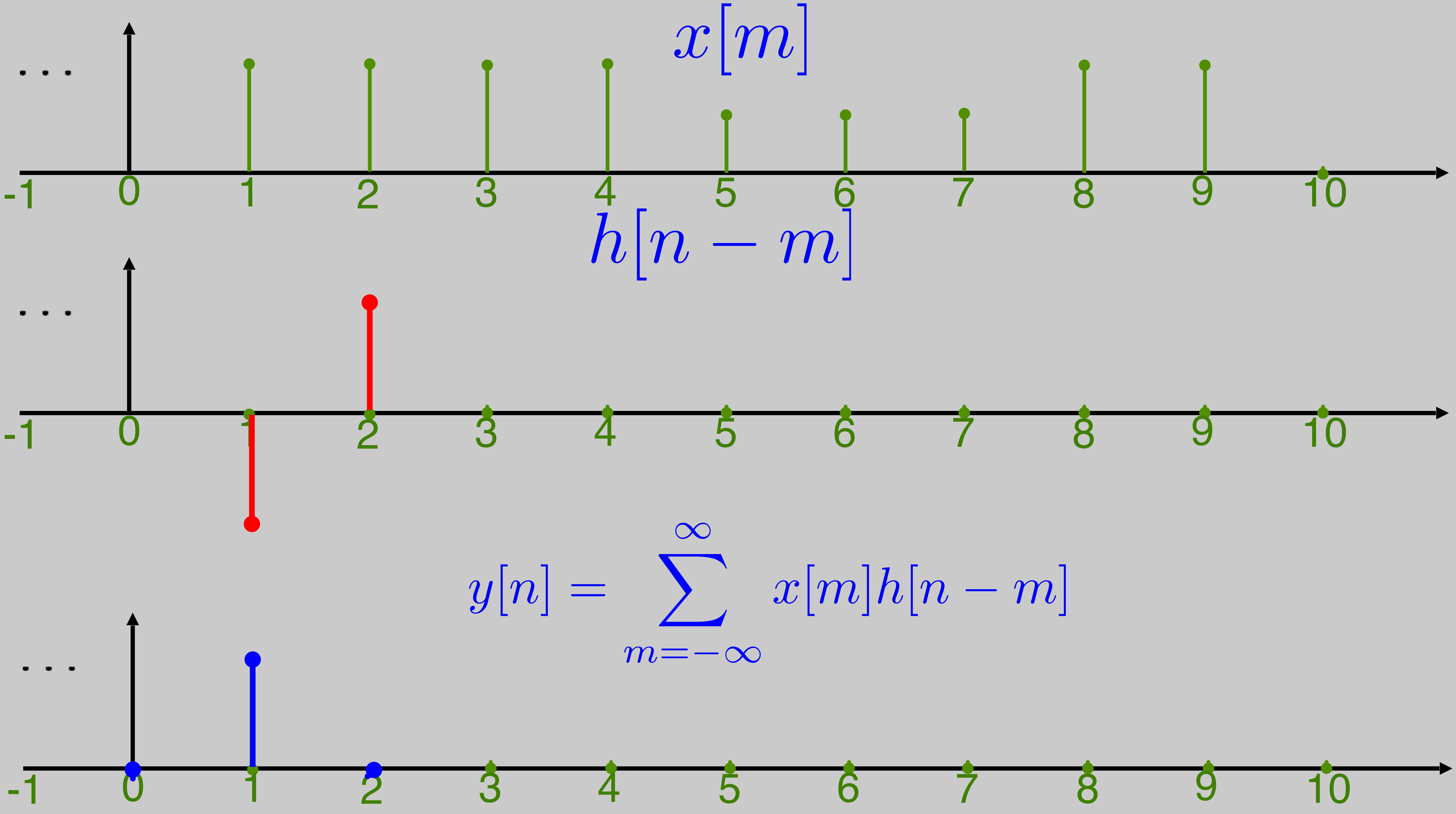
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



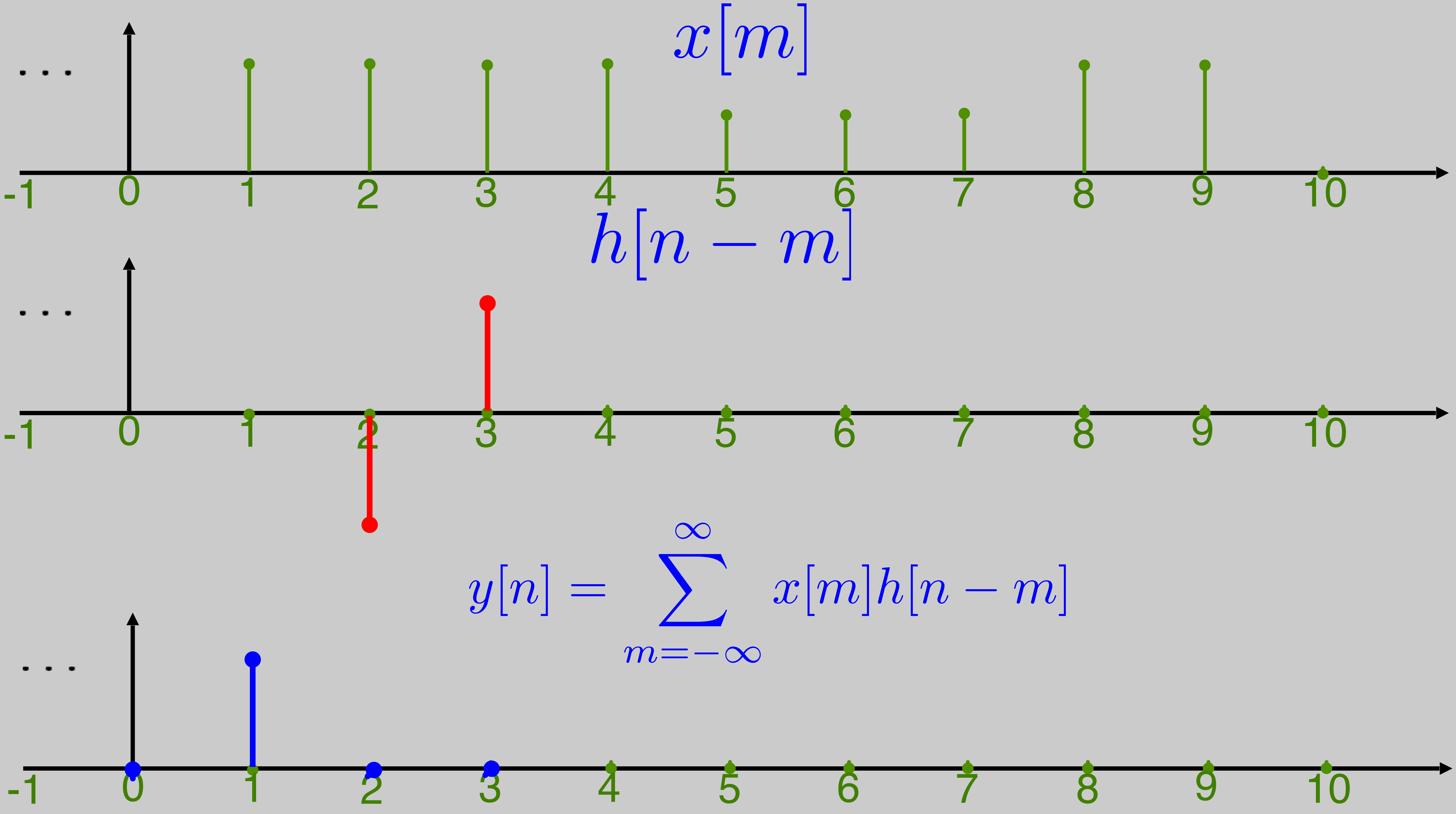
$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

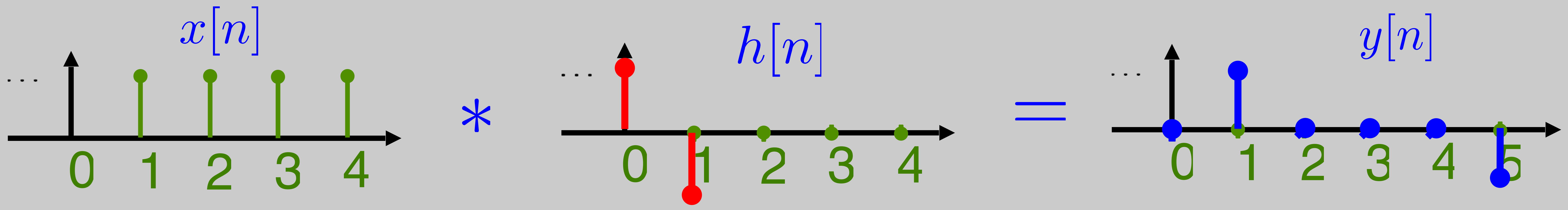
Graphical Example of Convolution



$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

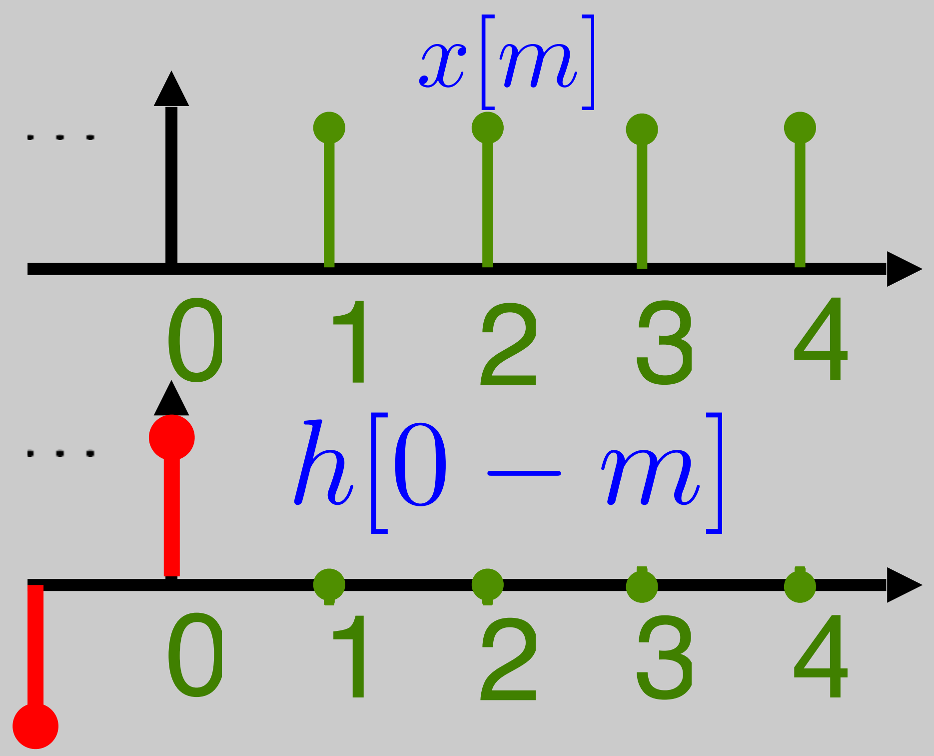
Example:

If h[n] is length 2 and x[n] is length 5, what is the length of their convolution sum?



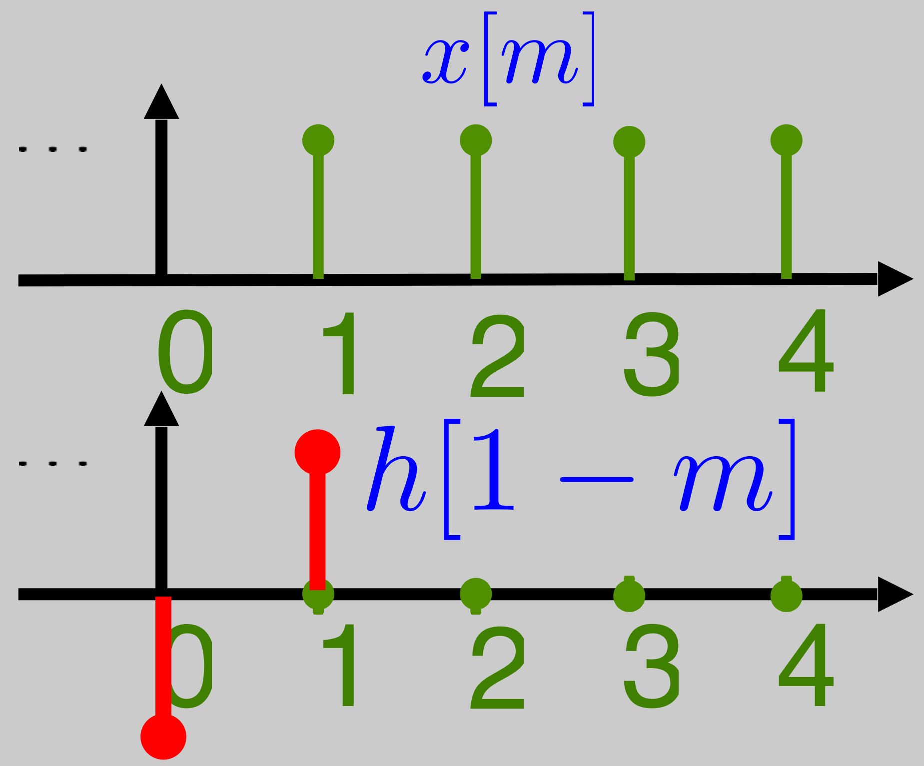
Example

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$



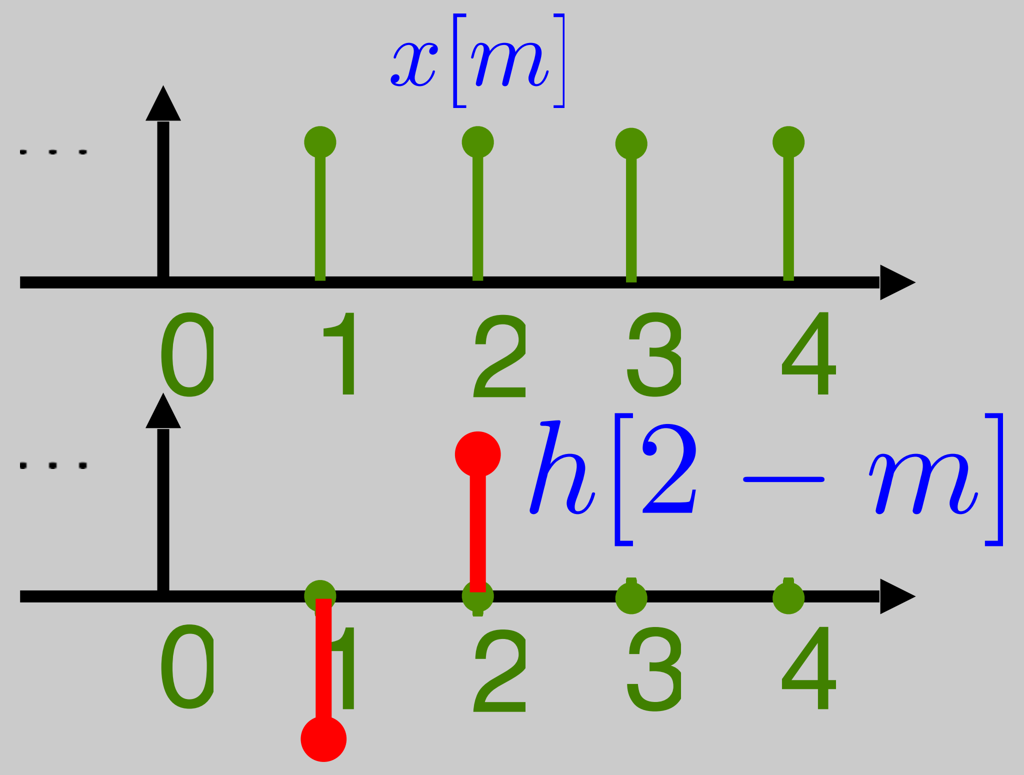
Example

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$



Example

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$



Example

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix}$$

- This matrix is called a Toeplitz matrix
  - But.. Not square… not circulant....

Example

- Convert system to be square circulant by zero-padding

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ 0 \end{bmatrix}$$

- Now can compute using the DFT!

General Case for Convolution Sum

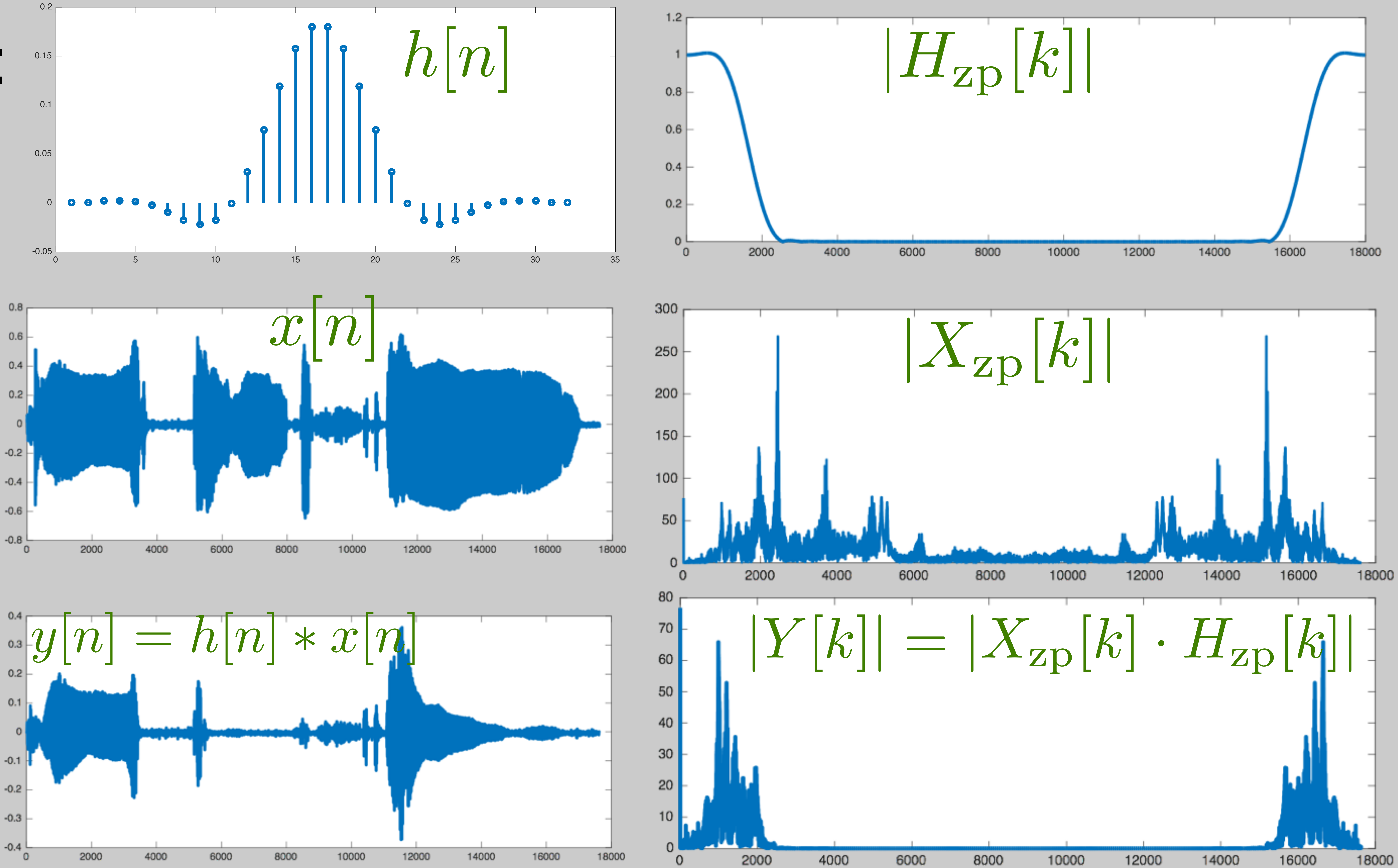
- Given: $\vec{h} \in \mathrm{R}^M$ $\vec{x} \in \mathrm{R}^N$
- Zeropad both to M+N-1 $\vec{h}_{\mathrm{zp}} \in \mathrm{R}^{N+M-1}$ $\vec{x}_{\mathrm{zp}} \in \mathrm{R}^{N+M-1}$
- Compute: $\vec{H} = F^* \vec{h}_{\mathrm{zp}}$ $\vec{X} = F^* \vec{x}_{\mathrm{zp}}$

$$\vec{Y} = \vec{H} \cdot \vec{X}$$

- Finally: $\vec{y} = F\vec{Y}$

Spectrum of filtering?

- Example:

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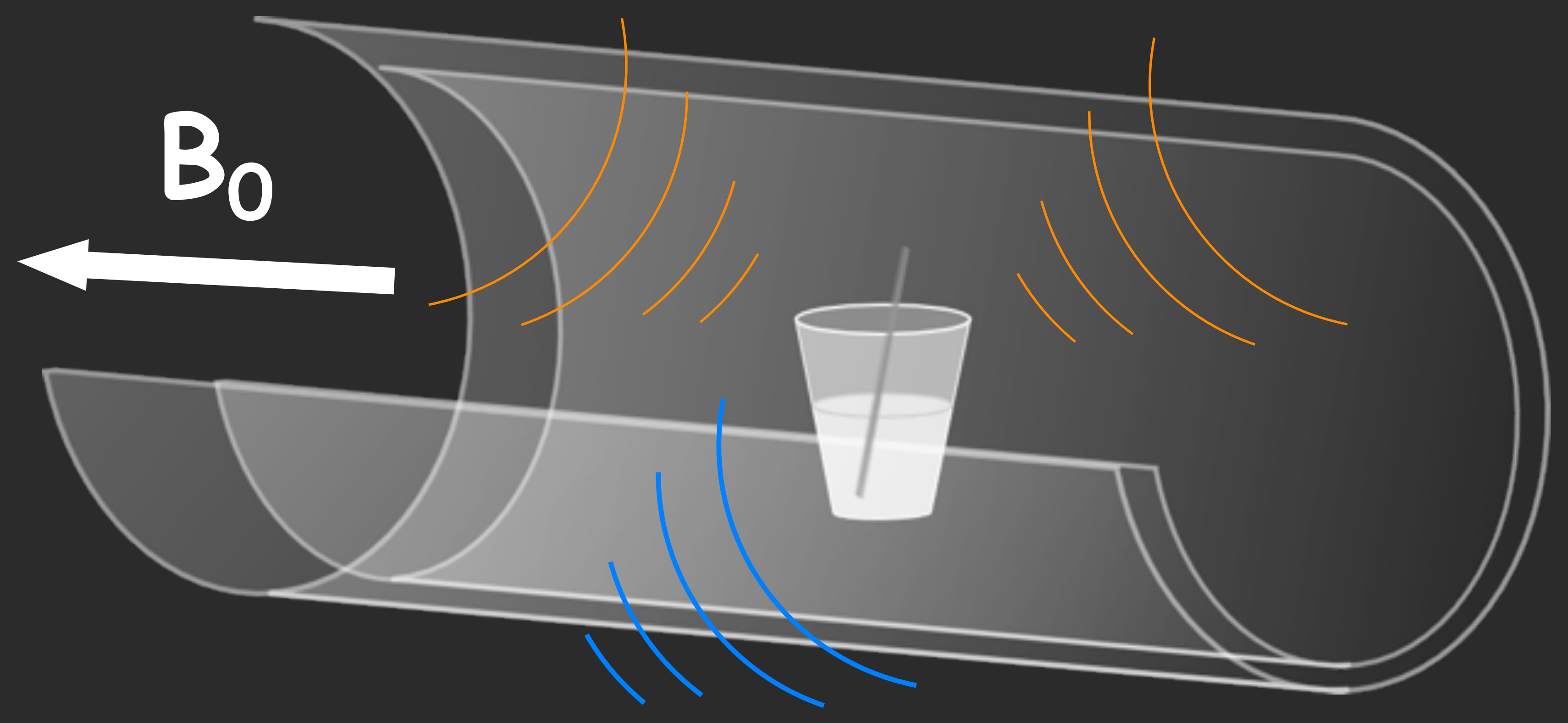
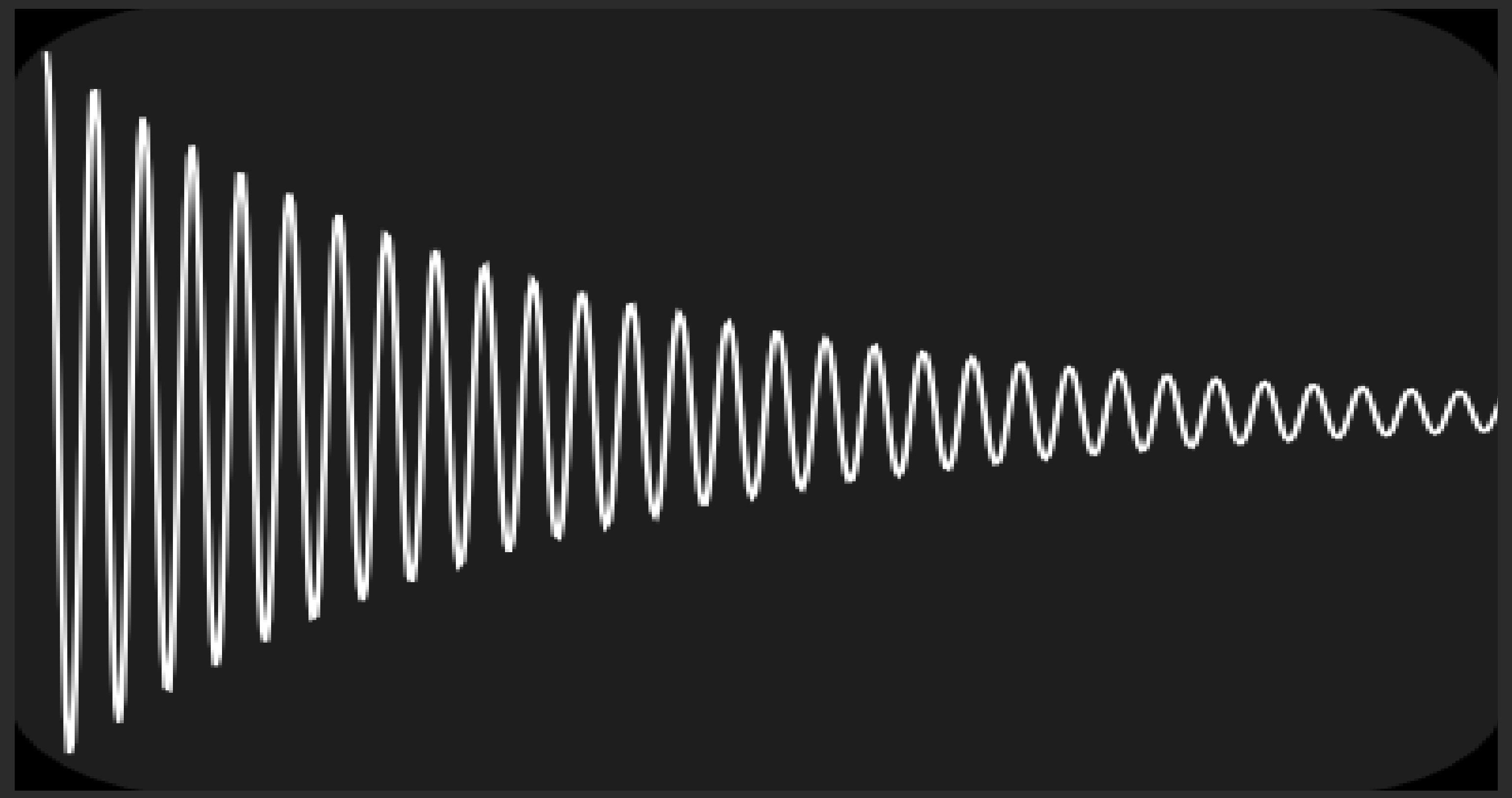
Intro to MRI - The NMR signal

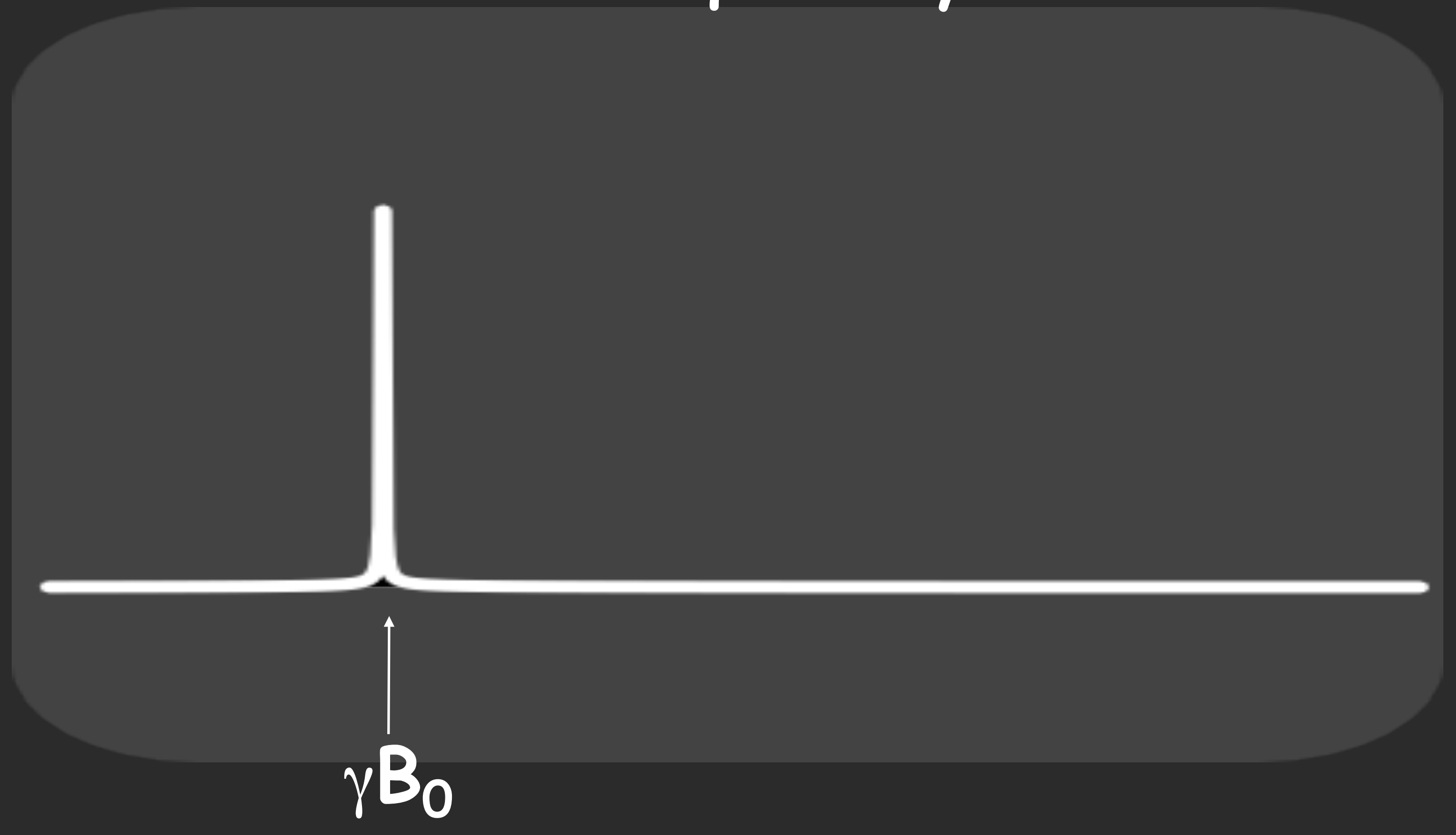
- Signal from ^{1}H (mostly water)
- Magnetic field ⇒ Magnetization
- Radio frequency ⇒ Excitation
- Frequency ∝ Magnetic field



time

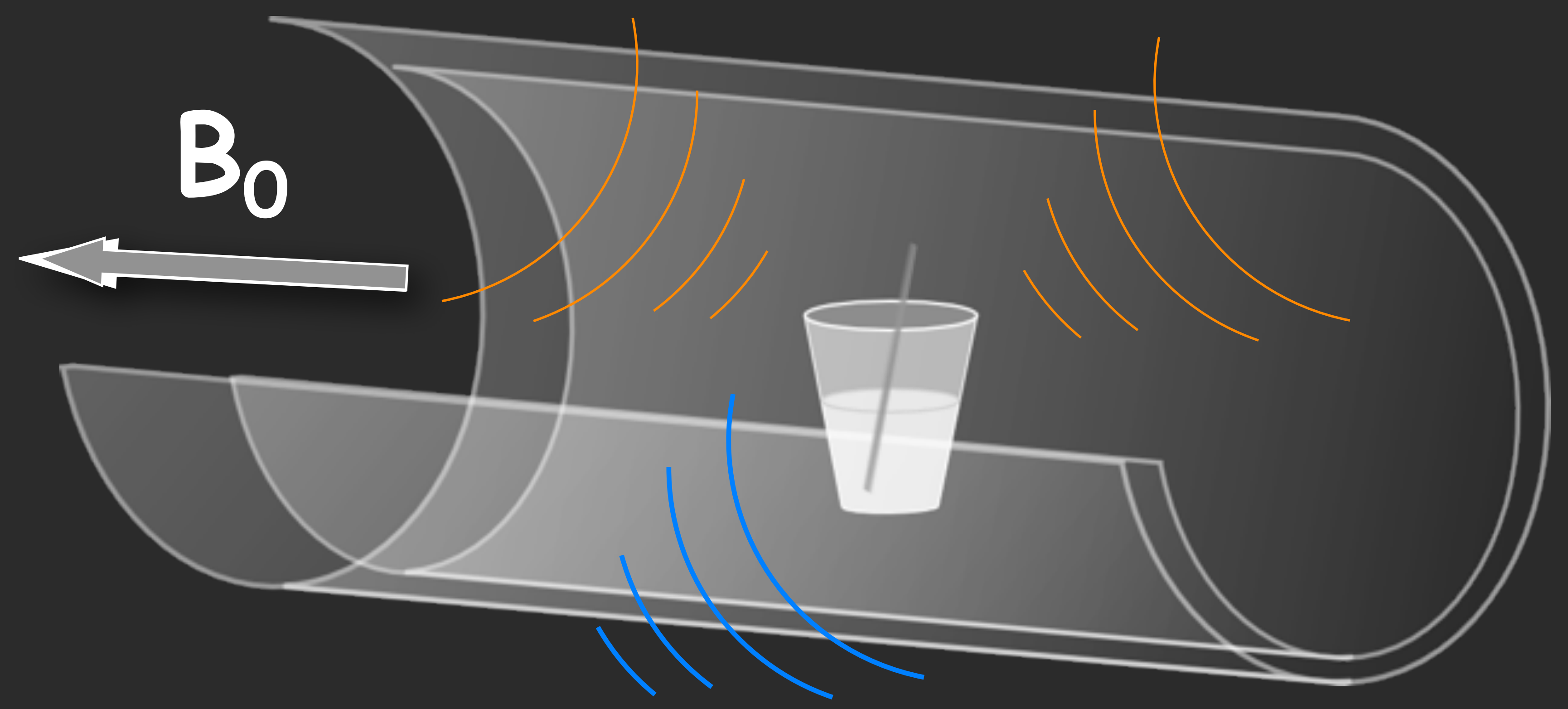


frequency

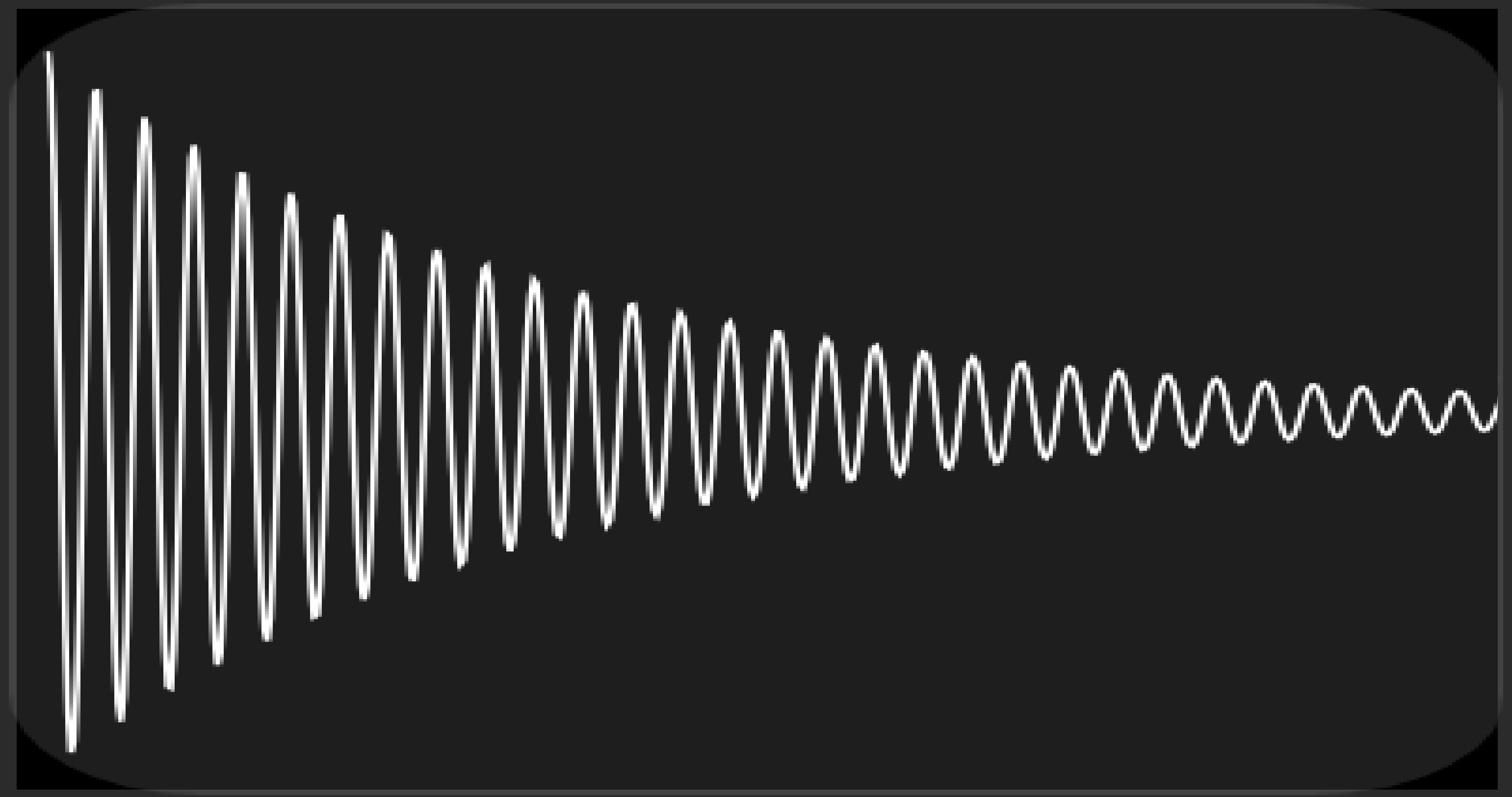


Intro to MRI - The NMR signal

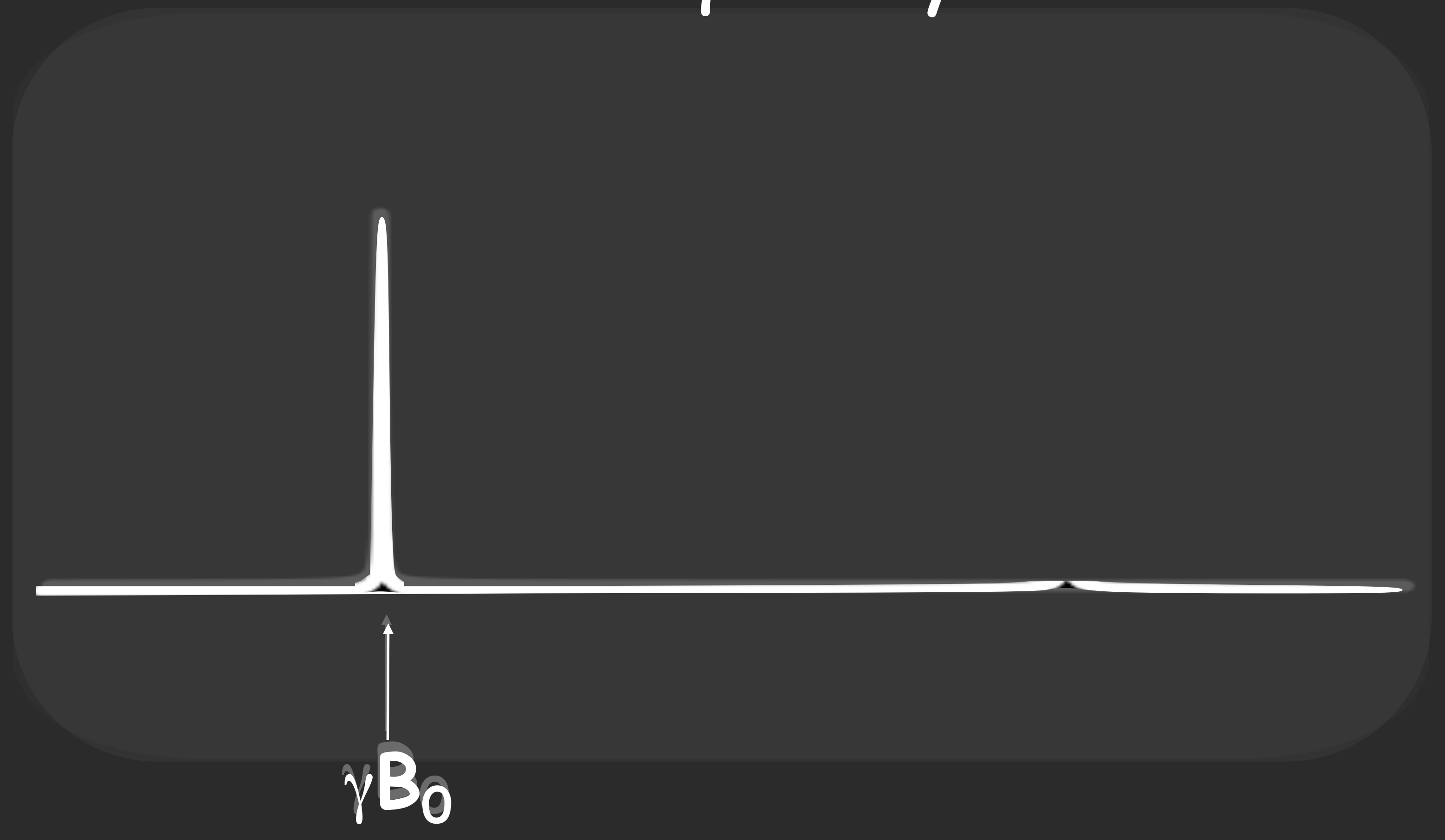
- Signal from ^1H (mostly water)
- Magnetic field $\Rightarrow$ Magnetization
- Radio frequency $\Rightarrow$ Excitation
- Frequency $\propto$ Magnetic field



time

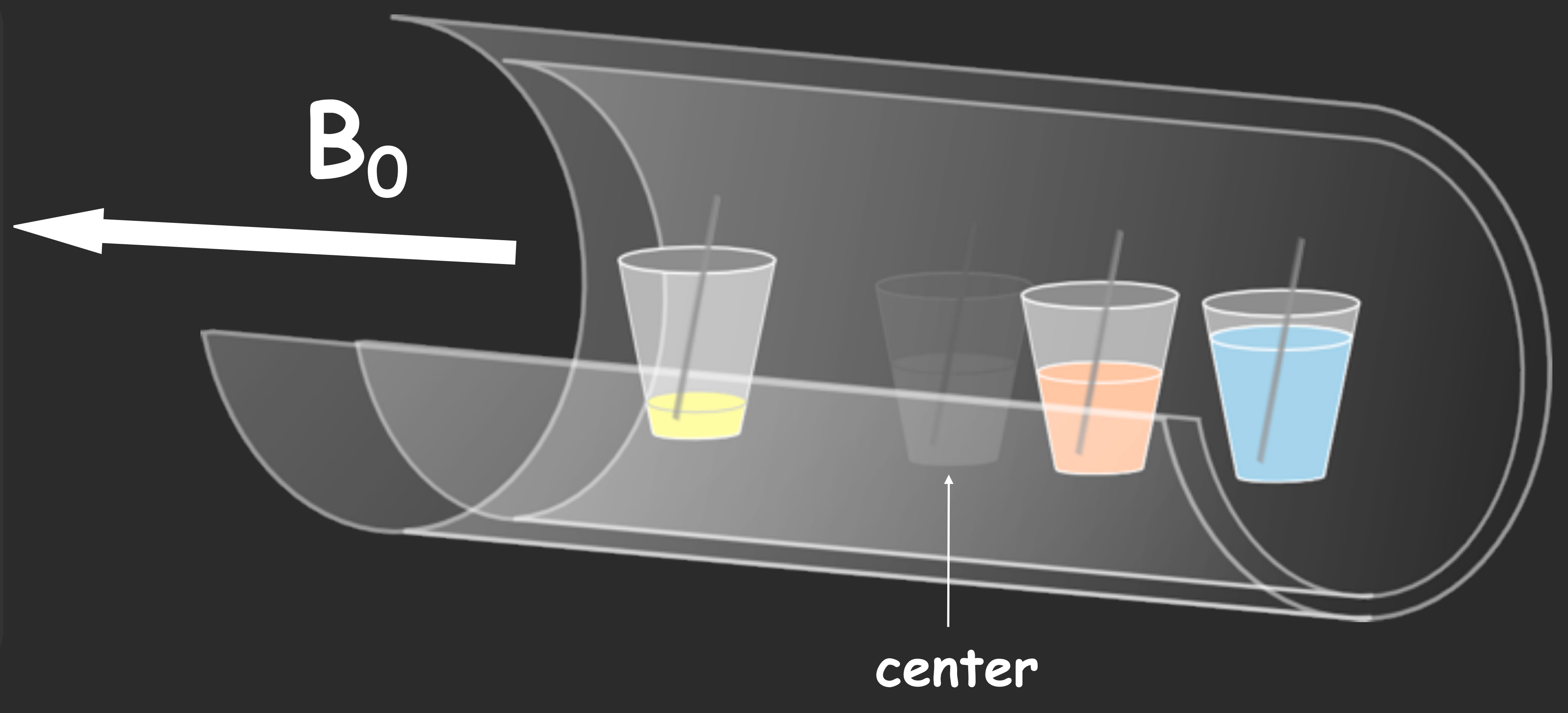


frequency

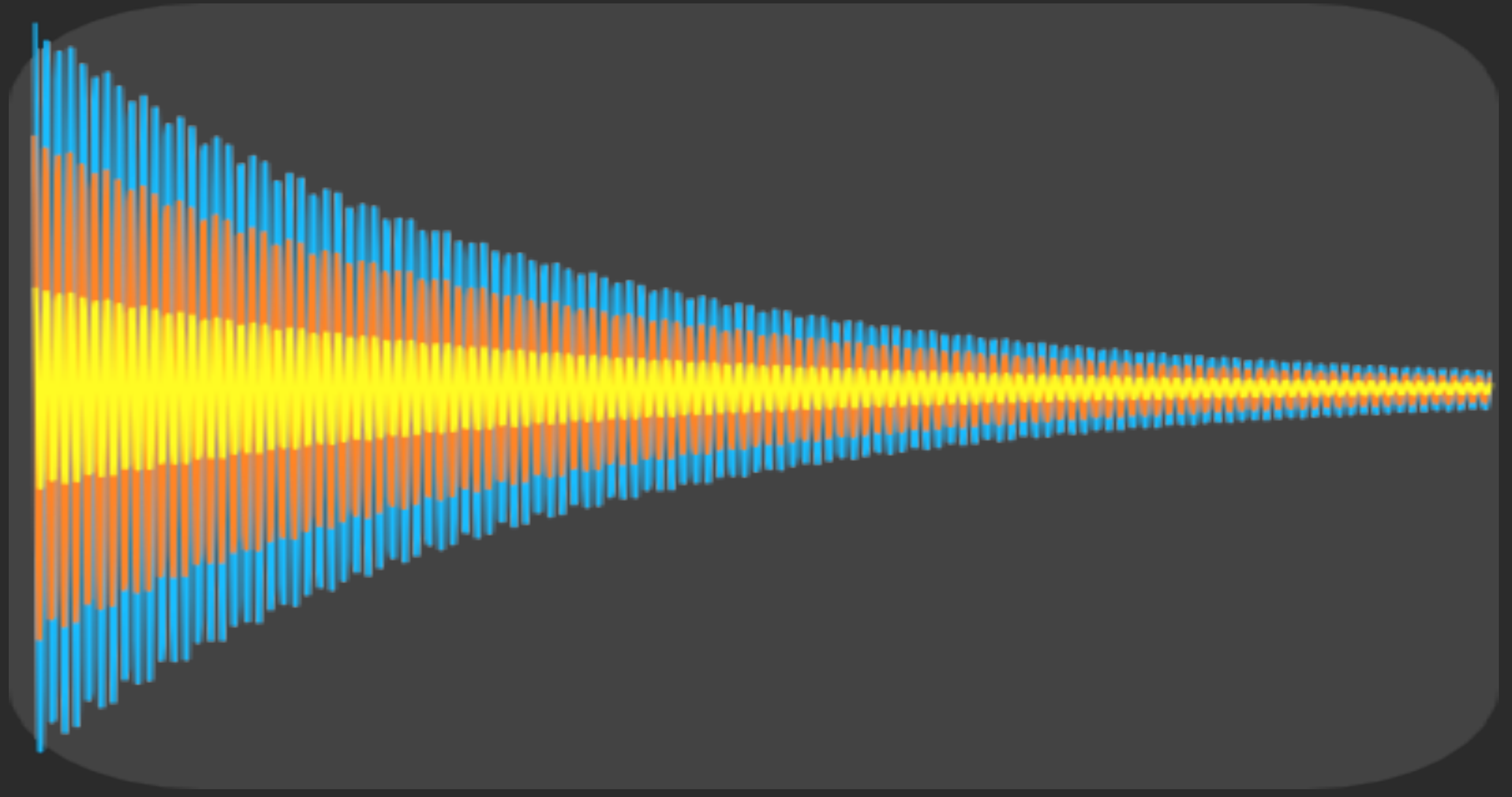


Intro to MRI - Imaging

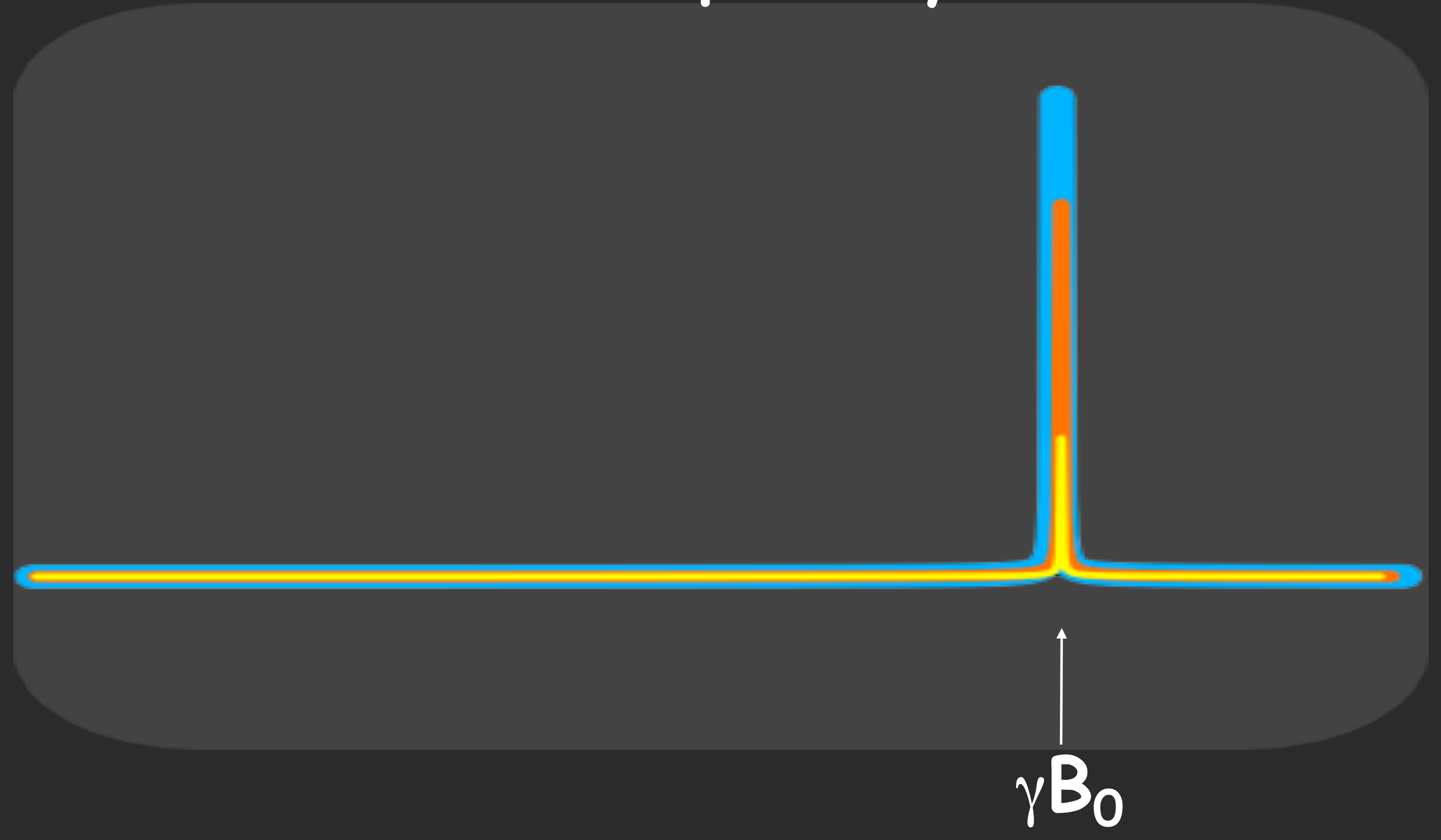
- B_0 Missing spatial information



time



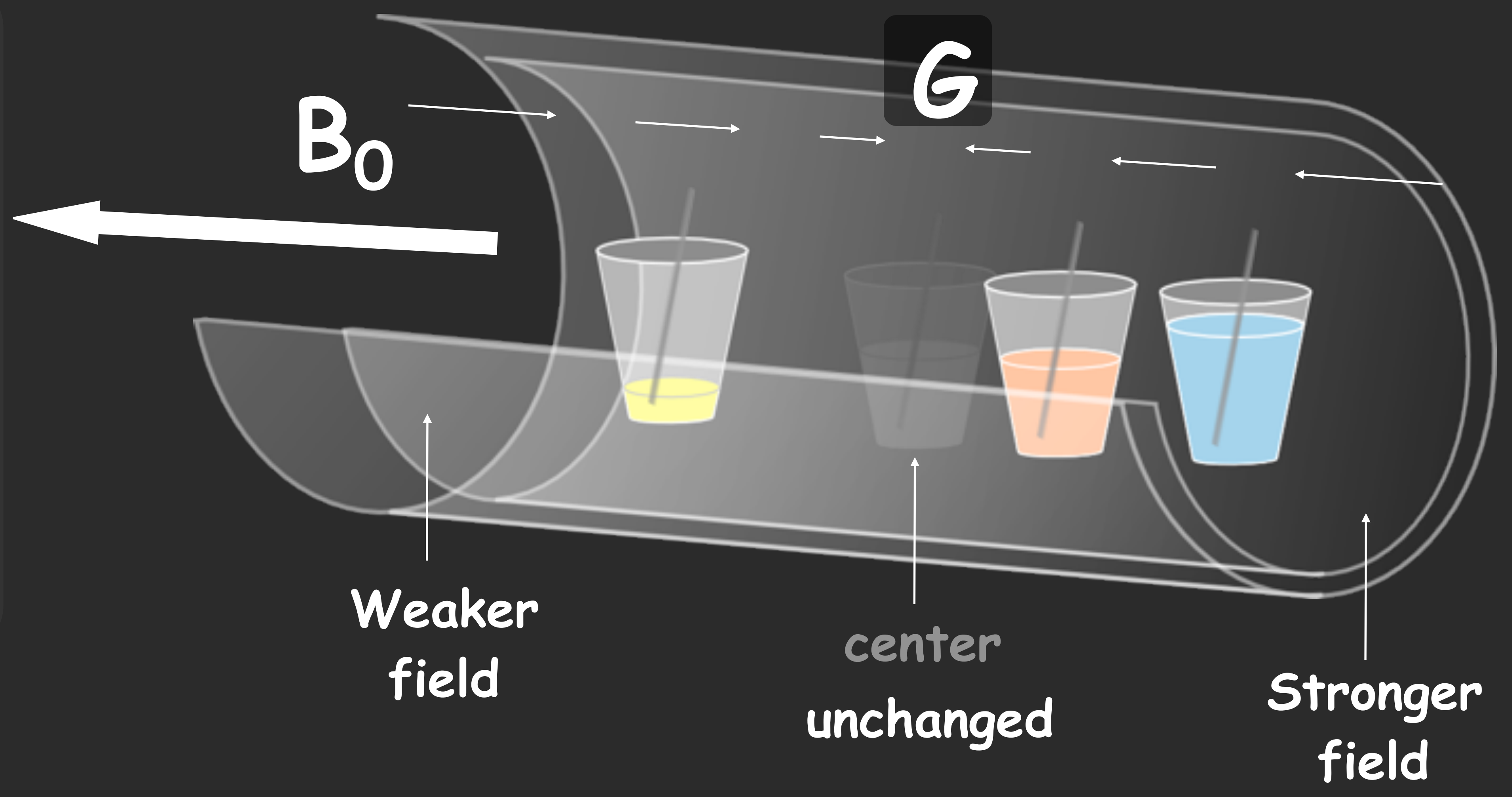
frequency

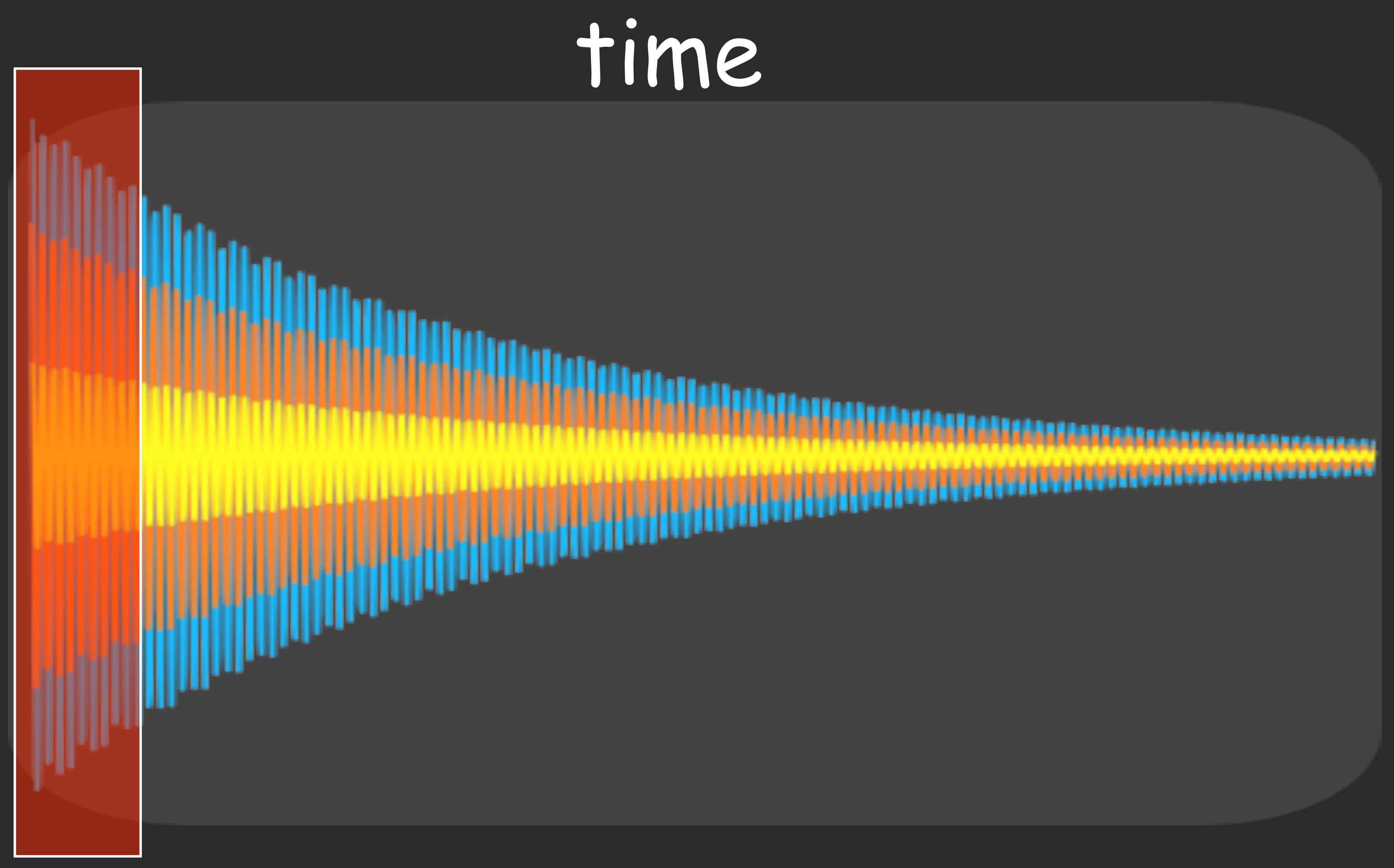


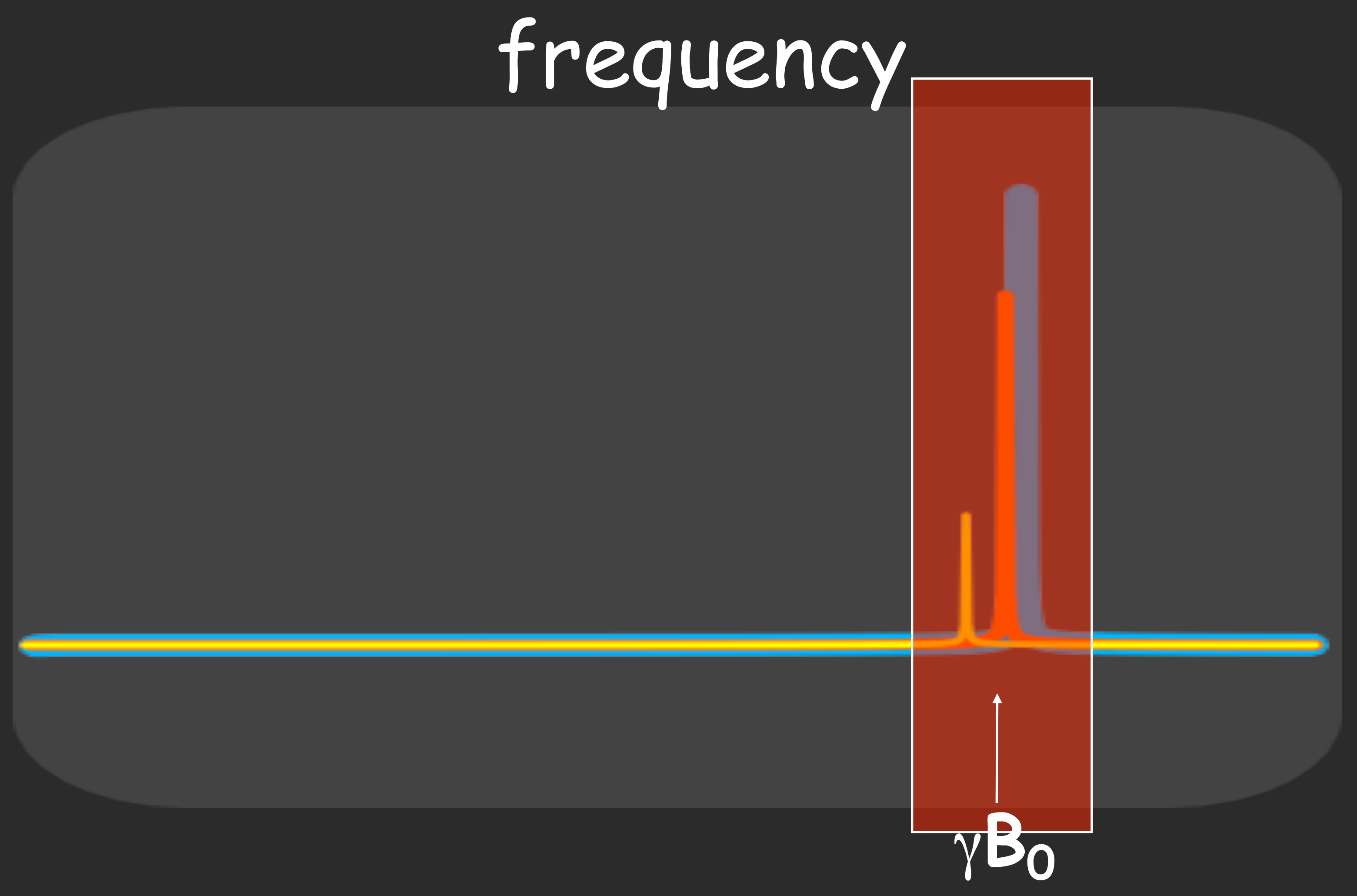
Phone Imaging I

Intro to MRI - Imaging

- B_0 Missing spatial information
- Add gradient field, G

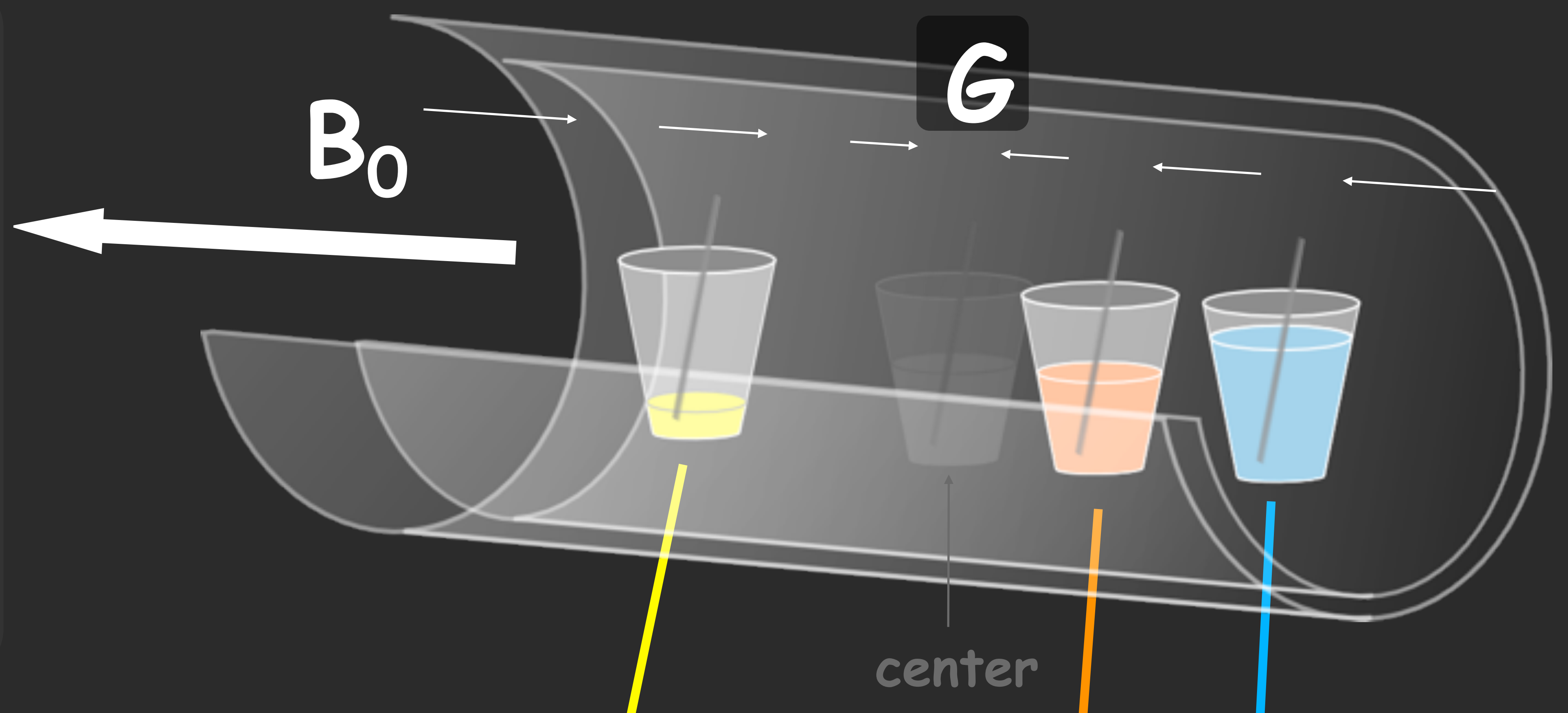


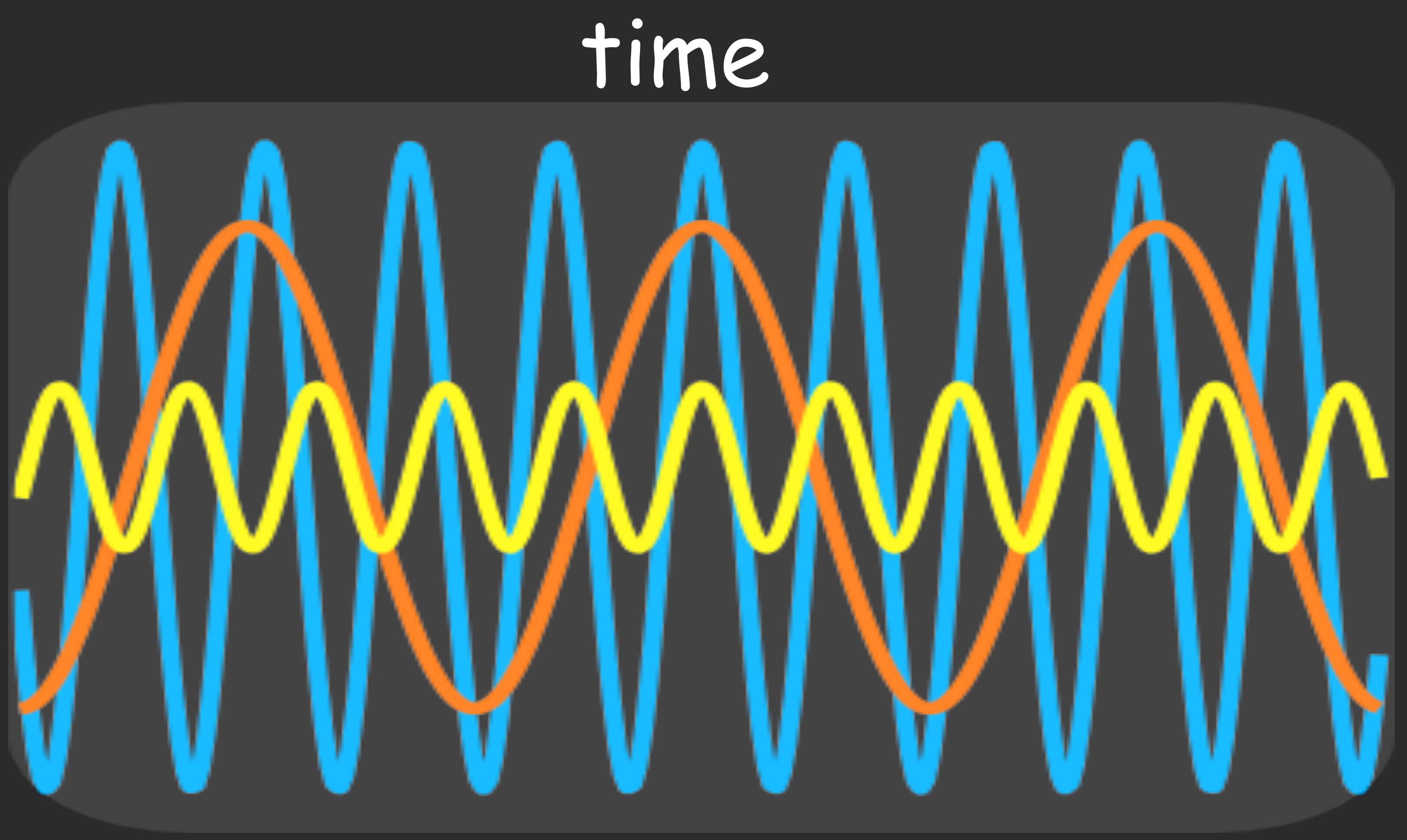


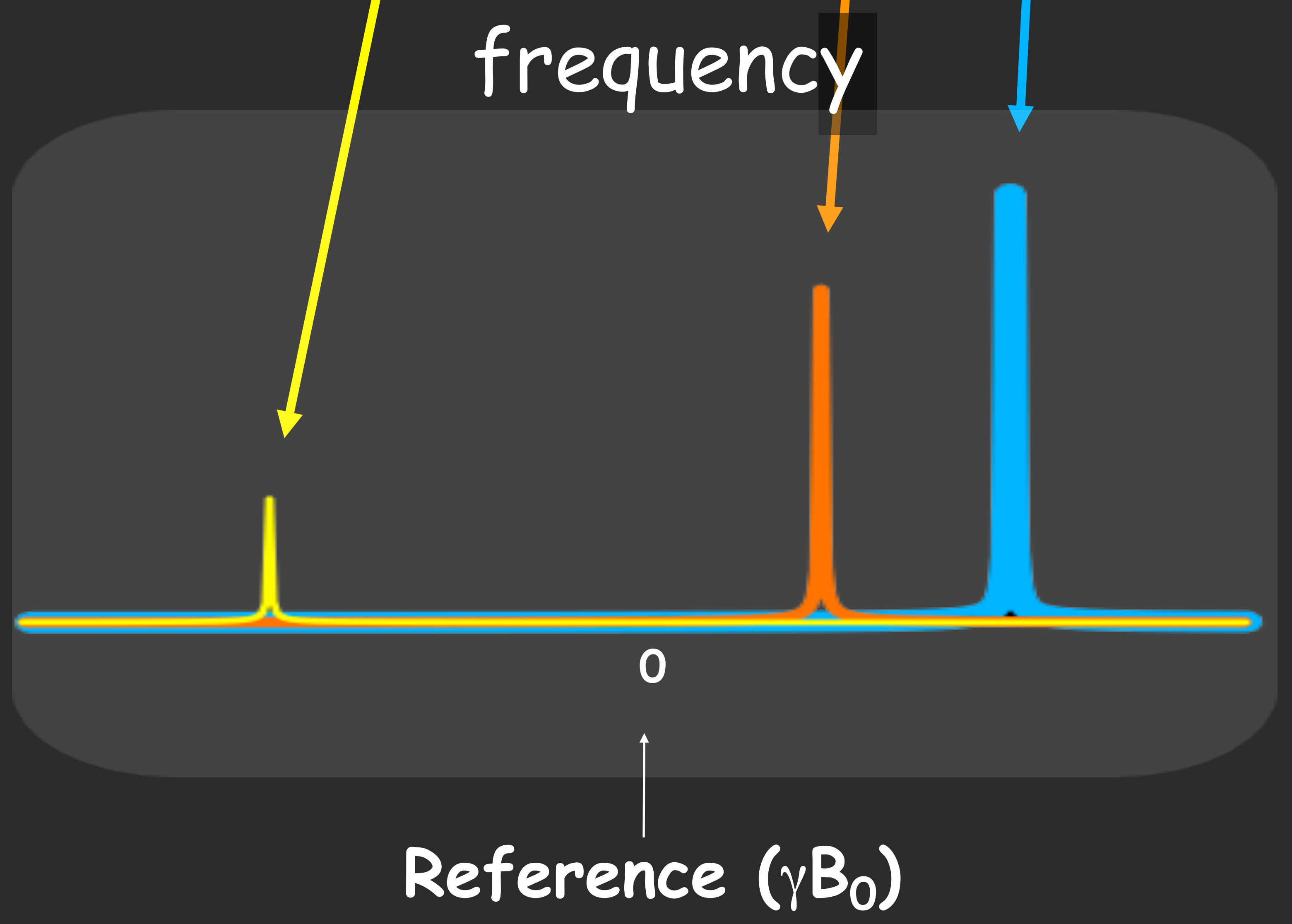


Intro to MRI - Imaging

- B_0 Missing spatial information
- Add gradient field, G
- Mapping:
  spatial position ⇒ frequency







Phone Imaging II

MR Imaging

Fourier

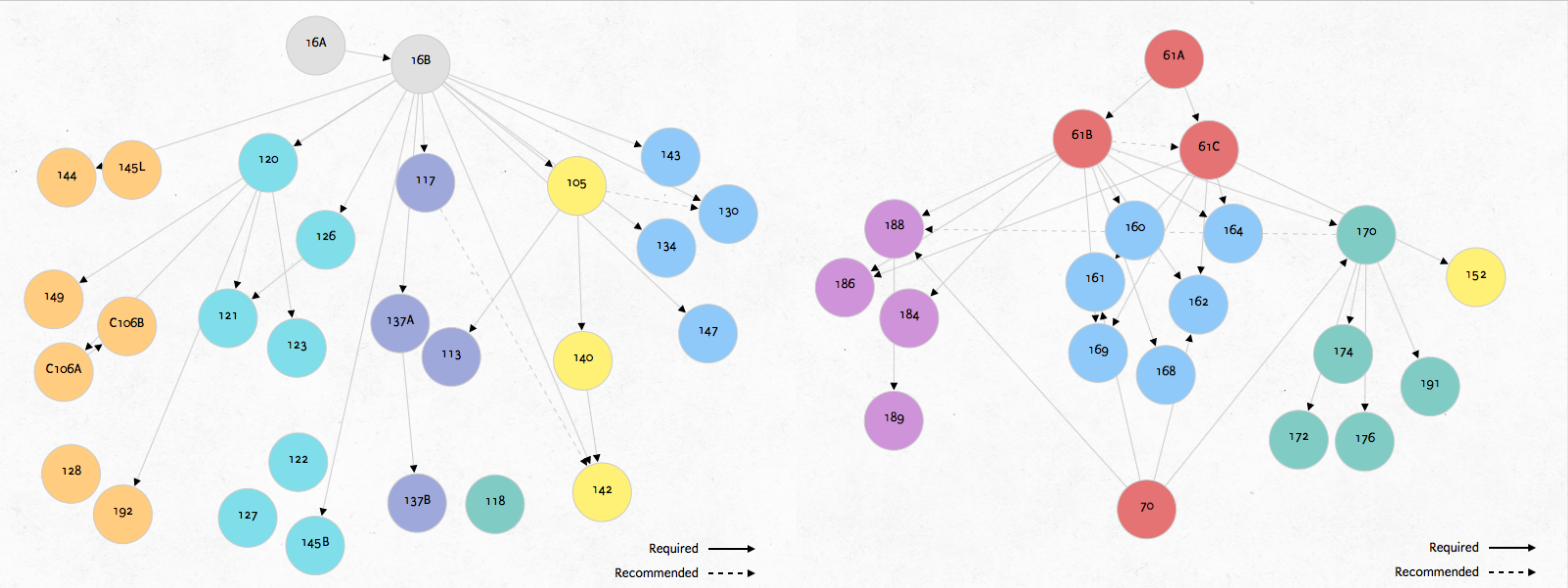
magnitude k-space (Raw Data)

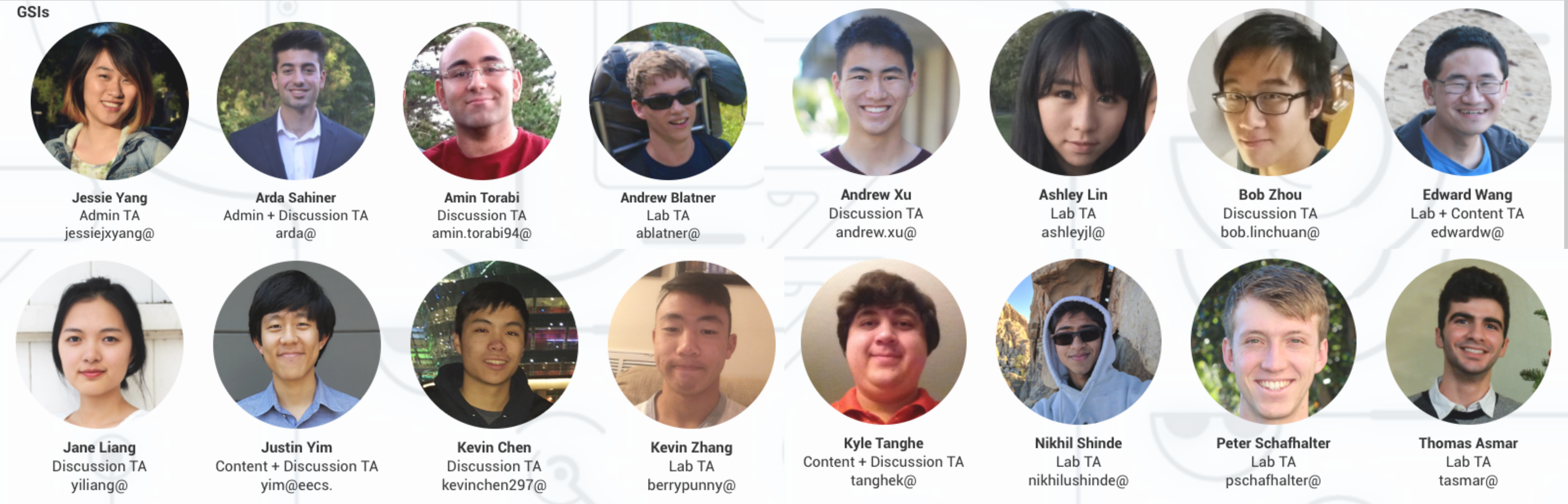
Image

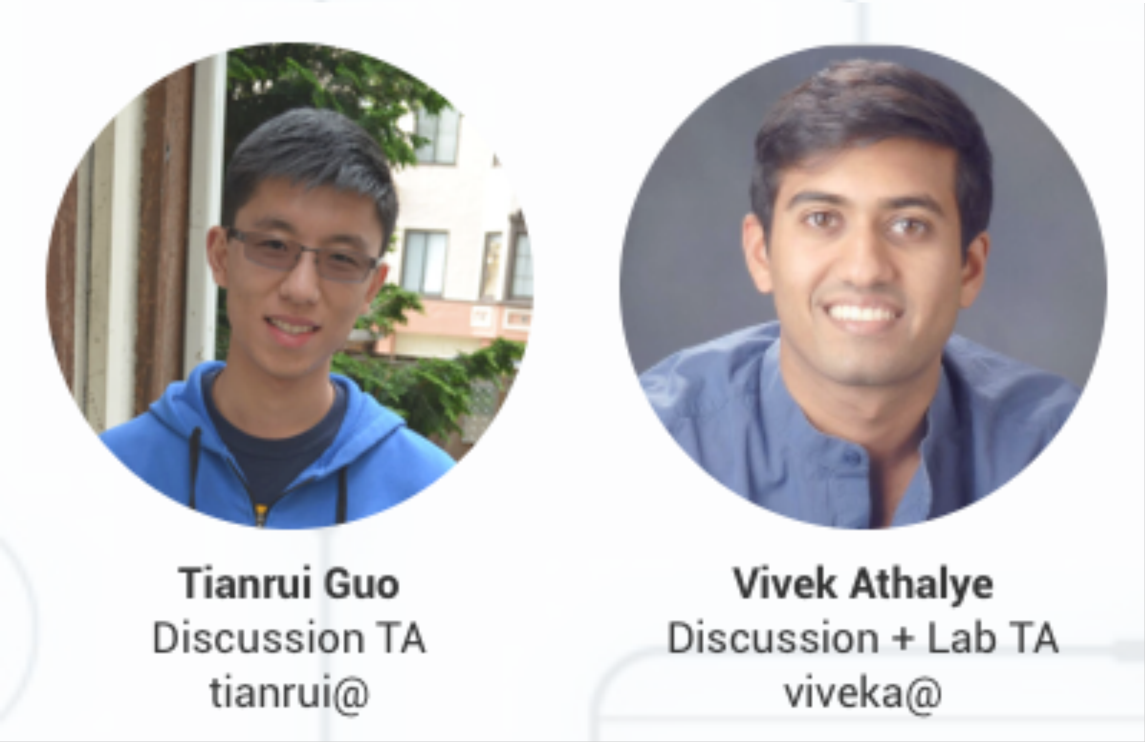
Fourier transform

Video courtesy Brian Hargreaves

Where from here….







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