EE16B
Designing Information Devices and Systems II

Lecture 6A
Stability of Linear State Models

Last Time

- Described linearization about an equilibrium point using Taylor approximation
  - Continuous time
  - Discrete time
- Conditions for stability of linear systems
  - Covered:
    - Discrete, First order and scalar $|a|<1$

Stability of Linear State Models

Previously, the scalar case:

$$x(t+1) = ax(t) + bu(t)$$

$|a| < 1 \quad \Rightarrow$ stable

$|a| \geq 1 \quad \Rightarrow$ unstable

Vector case:

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

Solve with recursion:

$$\vec{x}(1) = A\vec{x}(0) + Bu(0)$$

$$\vec{x}(2) = A^2\vec{x}(0) + ABu(0) + Bu(1)$$

$$\vec{x}(t) = A^t\vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k}Bu(k)$$

Stability – The Vector Case

$$\vec{x}(t) = A^t \vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k} B u(k)$$

Q: How do we determine stability?

A (partial): Not simple as in the scalar case – the state variables and inputs are coupled.

Approach: Let's change variables, to decouple them

Change of Variables - Diagonalization

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

$$\vec{z}(t) = T\vec{x}(t)$$

$$\vec{z}(t+1) = T\vec{x}(t+1)$$

$$= TA\underbrace{\vec{x}(t)}_{T^{-1}\vec{z}(t)} + TBU(t)$$

$$\underbrace{A_{\text{new}} = TAT^{-1} \qquad B_{\text{new}} = TB}_{\text{Similarity transformation}}$$

Q: What T to choose?

A: Choose T s.t. A_{new} is diagonal

Remember eigen values of new system same as original!

Diagonalization

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\vec{z}(t+1) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \vec{z}(t) + \underbrace{B_{\text{new}} u(t)}$$

$$\overrightarrow{v(t)} = B_{\text{new}} u(t)$$

$$z_1(t+1) = \lambda_1 z_1(t) + v_1(t)$$

Diagonalization

Diagonalization = decoupling!

$$z_1(t+1) = \lambda_1 z_1(t) + v_1(t)$$
$$z_2(t+1) = \lambda_2 z_2(t) + v_2(t)$$
$$\vdots$$
$$z_n(t+1) = \lambda_n z_n(t) + v_n(t)$$

Stable if: $|\lambda_i| < 1, \qquad i = 1, 2, \cdots, n$

unstable if: $|\lambda_i| \geq 1, \qquad i = 1, 2, \cdots, n$

Remember eigen values of new system same as original!

Non-Diagonalizable Systems

Q: What if A is not diagonalizable?

A: Transform to upper diagonal form (always possible)

beyond 16B material

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \cdots & \star \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Stable if: $|\lambda_i| < 1, \qquad i = 1, 2, \cdots, n$

unstable if: $|\lambda_i| \geq 1, \qquad i = 1, 2, \cdots, n$

Non-Diagonalizable Proof

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & \star & \cdots & \star \\ 0 & \lambda_2 & \cdots & \star \\ & & \ddots \lambda_{n-1} & \star \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Show stability for z_n :

$$z_n(t+1) = \lambda_n z_n(t) + v_n(t) \qquad |\lambda_n| < 1$$

z_n is bounded, show stability for z_{n-1} :

$$z_{n-1}(t+1) = \lambda_{n-1} z_{n-1}(t) + \star z_n(t) + v_{n-1}(t)$$

treat as bounded input

Bounded if $|\lambda_{n-1}| < 1$

show stability for z_i recursively !

Example of non-diagonalizable:

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Stability of Cont.-Time Linear Systems

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

Start with scalar x(t):

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b\int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Stability of Cont.-Time Linear Systems

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Q: When is the system stable?

Stability of Cont.-Time Linear Systems

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$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b\int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Q: When is the system stable?

A: For $a < 0$

Proof outline:

Show: $e^{at} \to 0, \quad t \to \infty$

$$\text{if} \quad |u(s)| \leq M \quad \forall s \Rightarrow \int \{\} < \text{Const}$$

Stability of Cont.-Time Linear Systems

$$x(t) = \underbrace{e^{at}x(0)}_{\text{Initial condition}} + \underbrace{b\int_0^t e^{a(t-s)}u(s)ds}_{\text{Due to input}}$$

Q: When is the system unstable?

A: For $a \geq 0$

Proof : choose x(0)≠0 and u(t) = M

either “due to input” or “due to initial condition” explodes

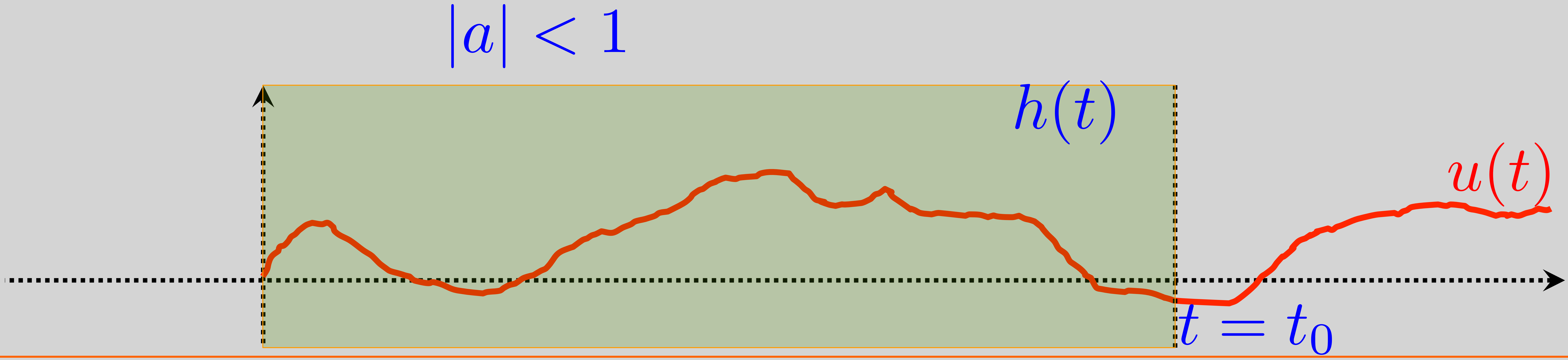
A few words…..

$$\int_0^t e^{a(t-s)} u(s) ds = \int_{-\infty}^{\infty} h(t-s) u(s) ds$$

This integral equation is also called: Convolution

$$h(t) = \begin{cases} e^{at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$u(t) = 0 \quad \forall \quad t < 0$$



A few words.....

$$\int_0^t e^{a(t-s)} u(s)ds = \int_{-\infty}^{\infty} h(t-s)u(s)ds$$

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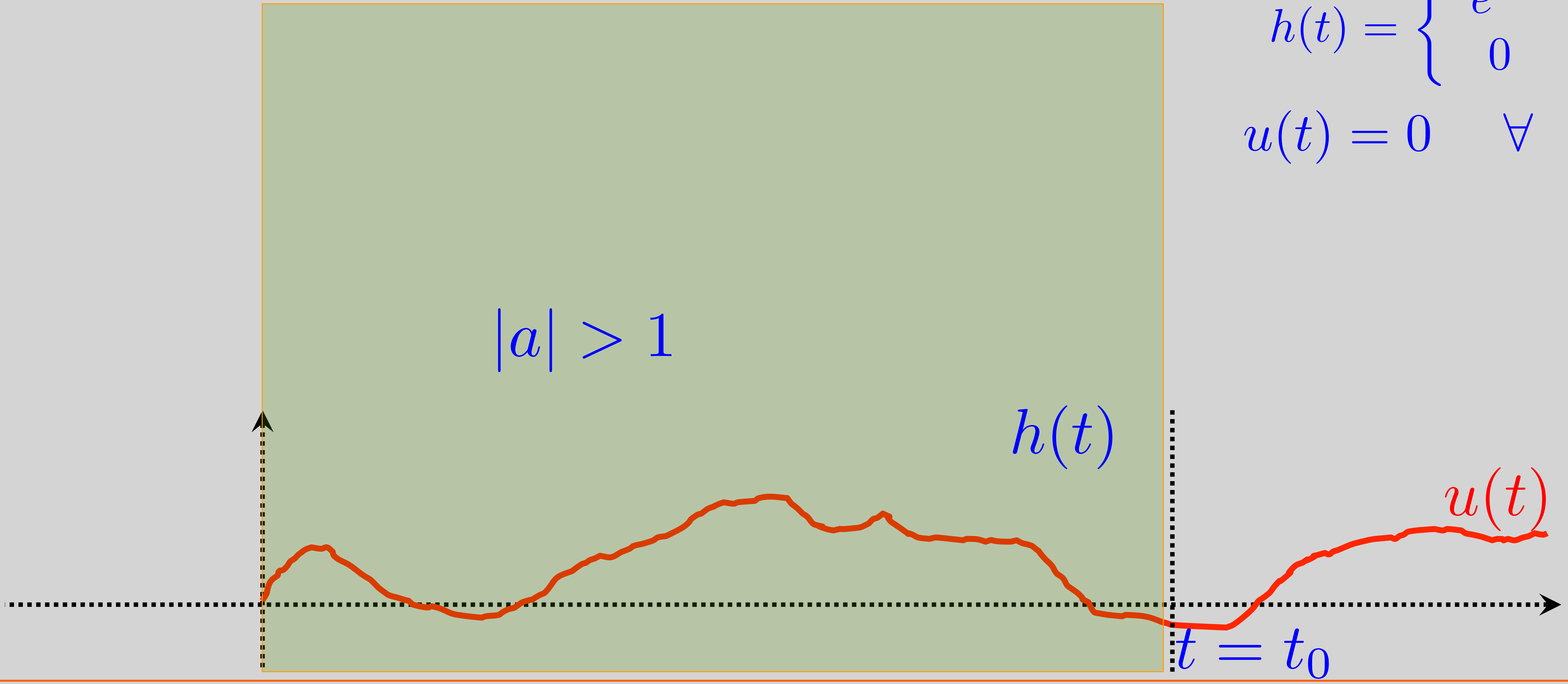


Stability of Cont.-Time Linear Systems

Summary:

$a < 0 \quad \Rightarrow$ stable

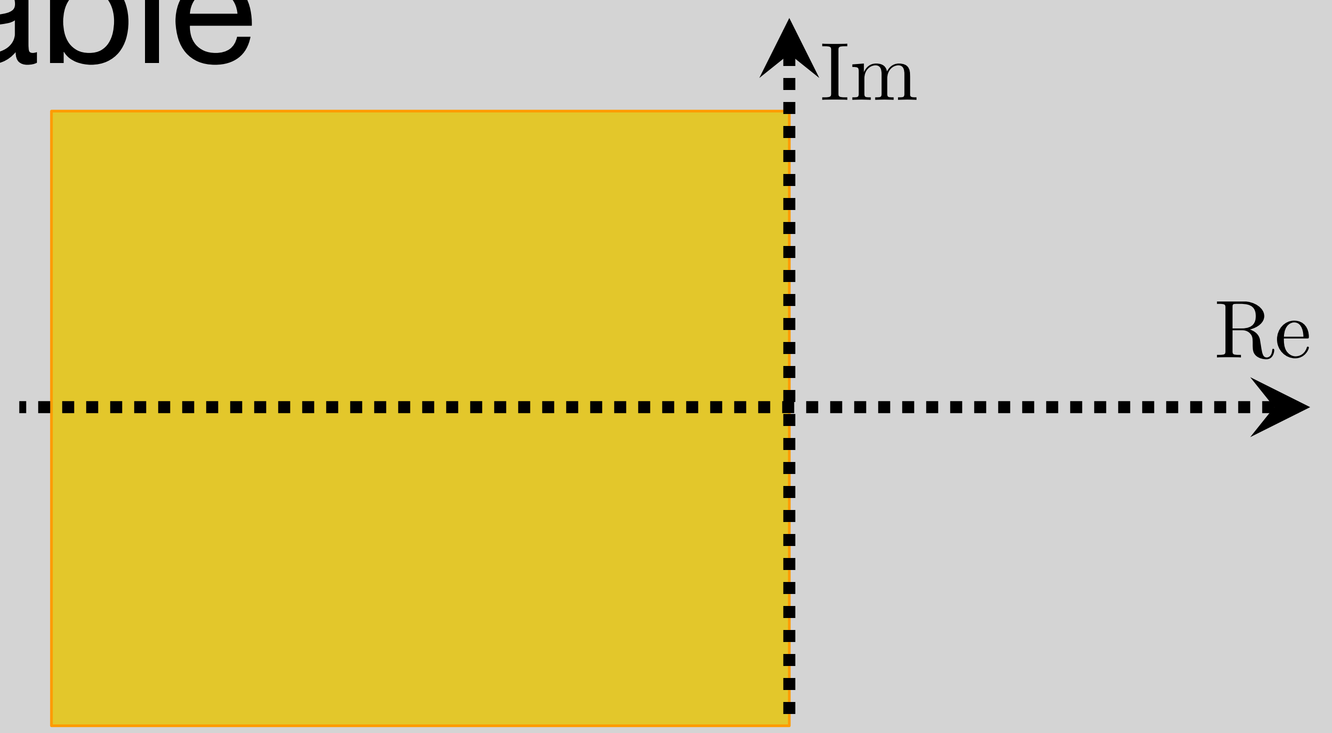
$a \geq 0 \quad \Rightarrow$ unstable

If a is complex, then:

$\text{Re}\{a\} < 0 \quad \Rightarrow$ stable

$\text{Re}\{a\} \geq 0 \quad \Rightarrow$ unstable

$$|e^{a_r + i a_i}| = |e^{a_r}| \cdot |e^{i a_i}| = |e^{a_r}|$$



Stability of Cont.-Time Linear Systems

Vector Case:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

Diagonalize:

$$\vec{z}(t) = T\vec{x}(t)$$

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\frac{d}{dt}z_i(t) = \lambda_i z_i(t) + v_i(t)$$

Stability of Cont.-Time Linear Systems

Stability test for

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$$

$$\text{Re}\{\lambda_i(A)\} < 0 \quad \forall i | \quad i = 1, 2, \cdots, n$$ ⇒ stable

$$\text{Re}\{\lambda_i(A)\} \geq 0 \quad \exists i | \quad i = 1, 2, \cdots, n$$ ⇒ unstable

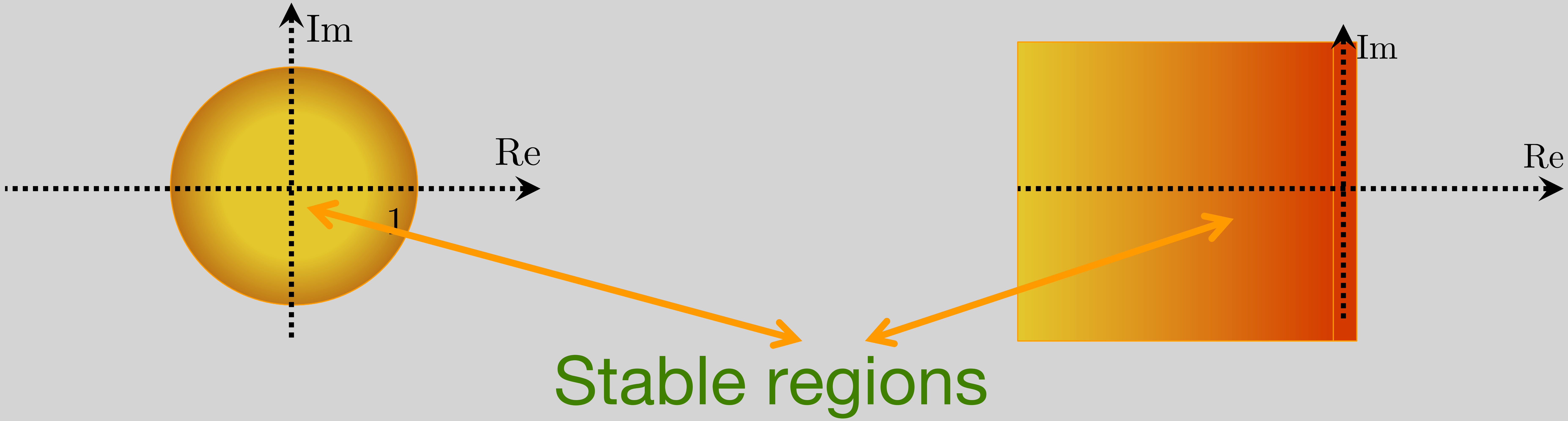
Stability -- Summary

Discrete-Time

$$|\lambda_i(A)| < 1$$

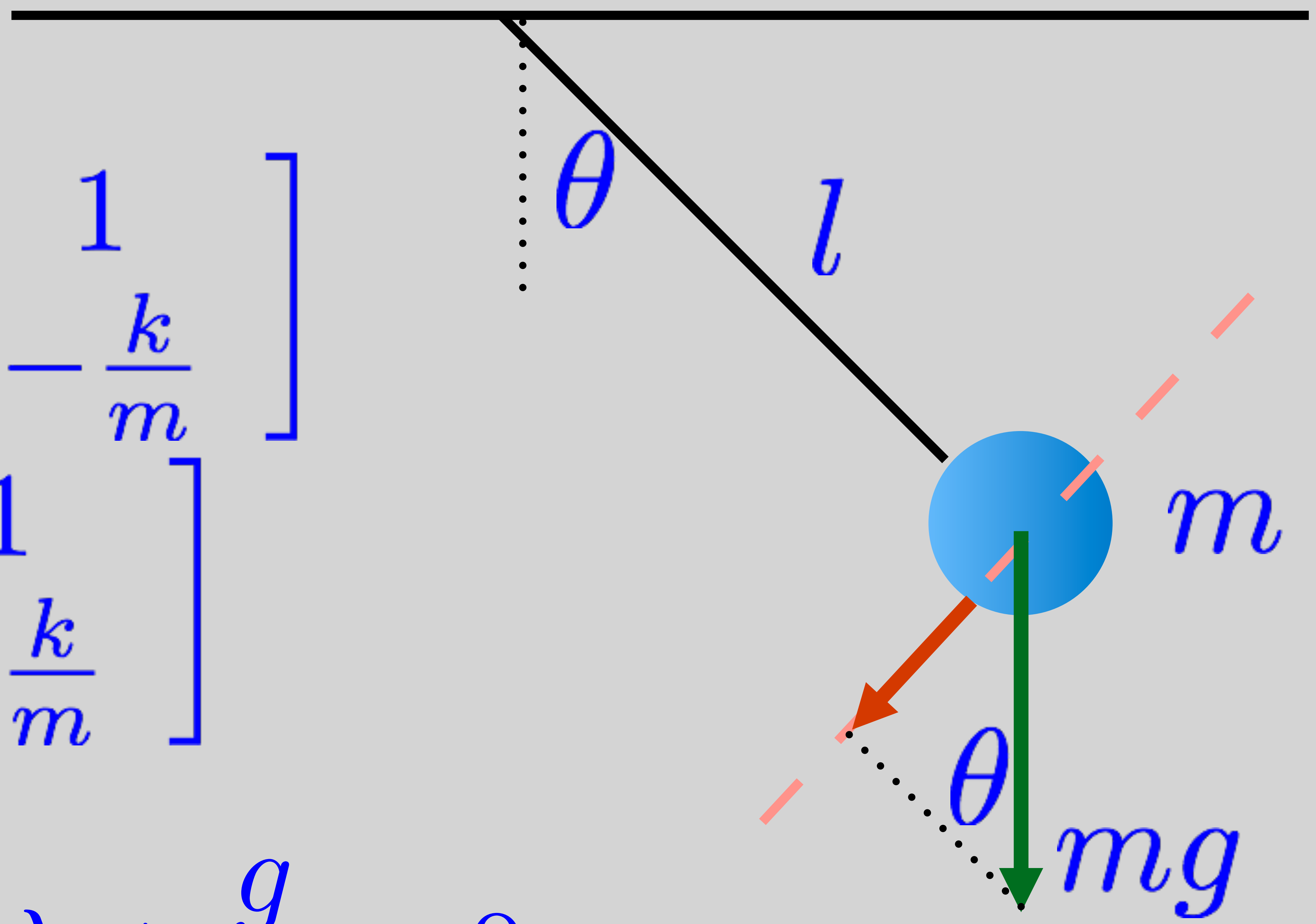
Continuous-Time

$$\text{Real}\{\lambda_i(A)\} < 0$$



Stay away from boundaries! System uncertainty can Move you over to unstable region

Back to the Pendulum



$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

$$|\lambda I - A_{\text{down}}| = \begin{bmatrix} \lambda & -1 \\ \frac{g}{l} & \lambda + \frac{k}{m} \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda + \frac{g}{l} = 0$$

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$$

Back to the Pendulum

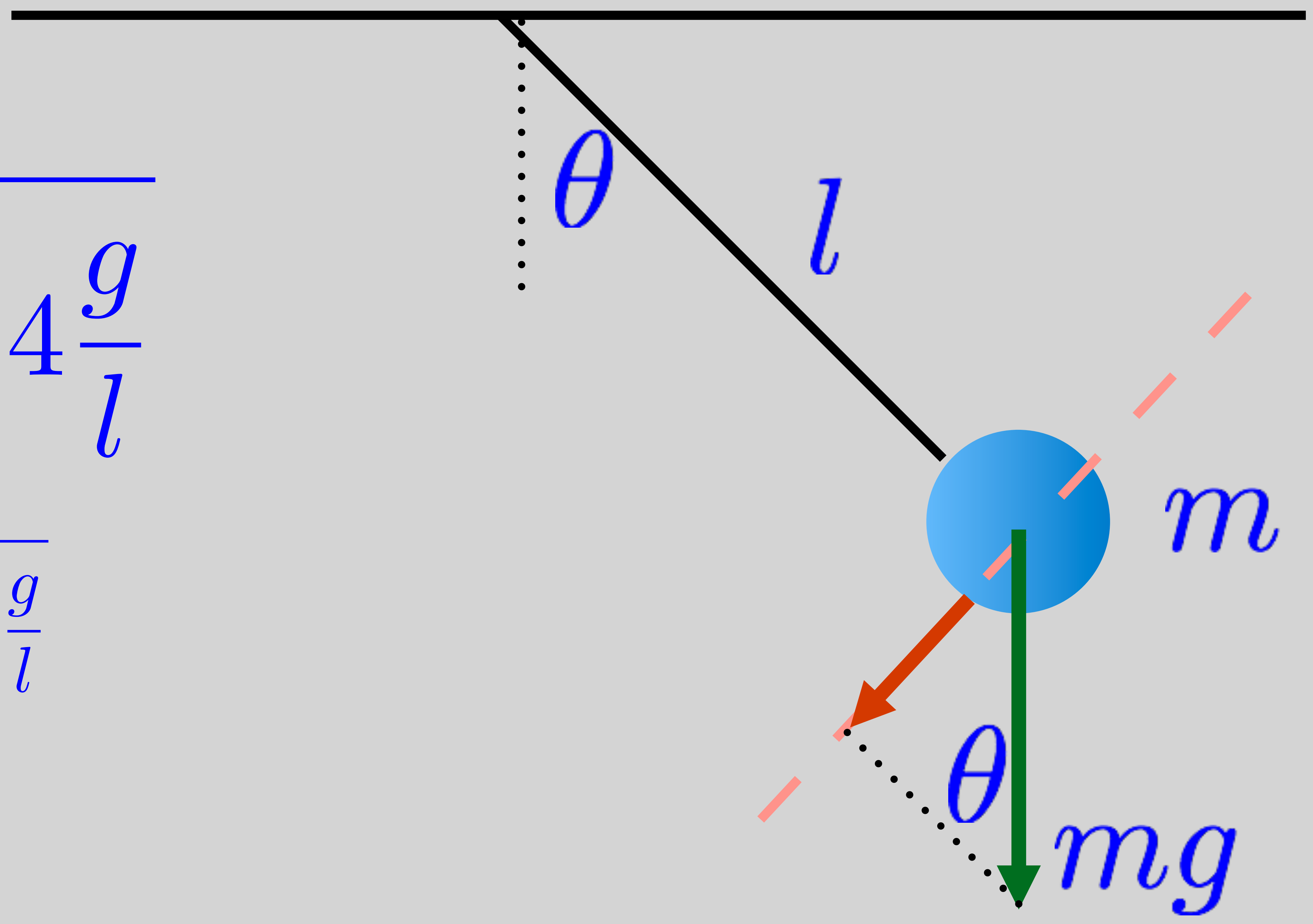
$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$$

If $\frac{k^2}{m^2} \geq 4\frac{g}{l}$, i.e, sqrt is real, then $\frac{k}{2m} \geq \frac{1}{2}\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l}}$

So, $\lambda_{1,2}$ always negative -- stable!

If $\frac{k^2}{m^2} < 4\frac{g}{l}$, i.e, sqrt is imaginary, then Re{$\lambda_{1,2}$} $= -\frac{k}{2m}$

So, Re{$\lambda_{1,2}$} always negative -- stable!



Back to the Pendulum

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix}$$

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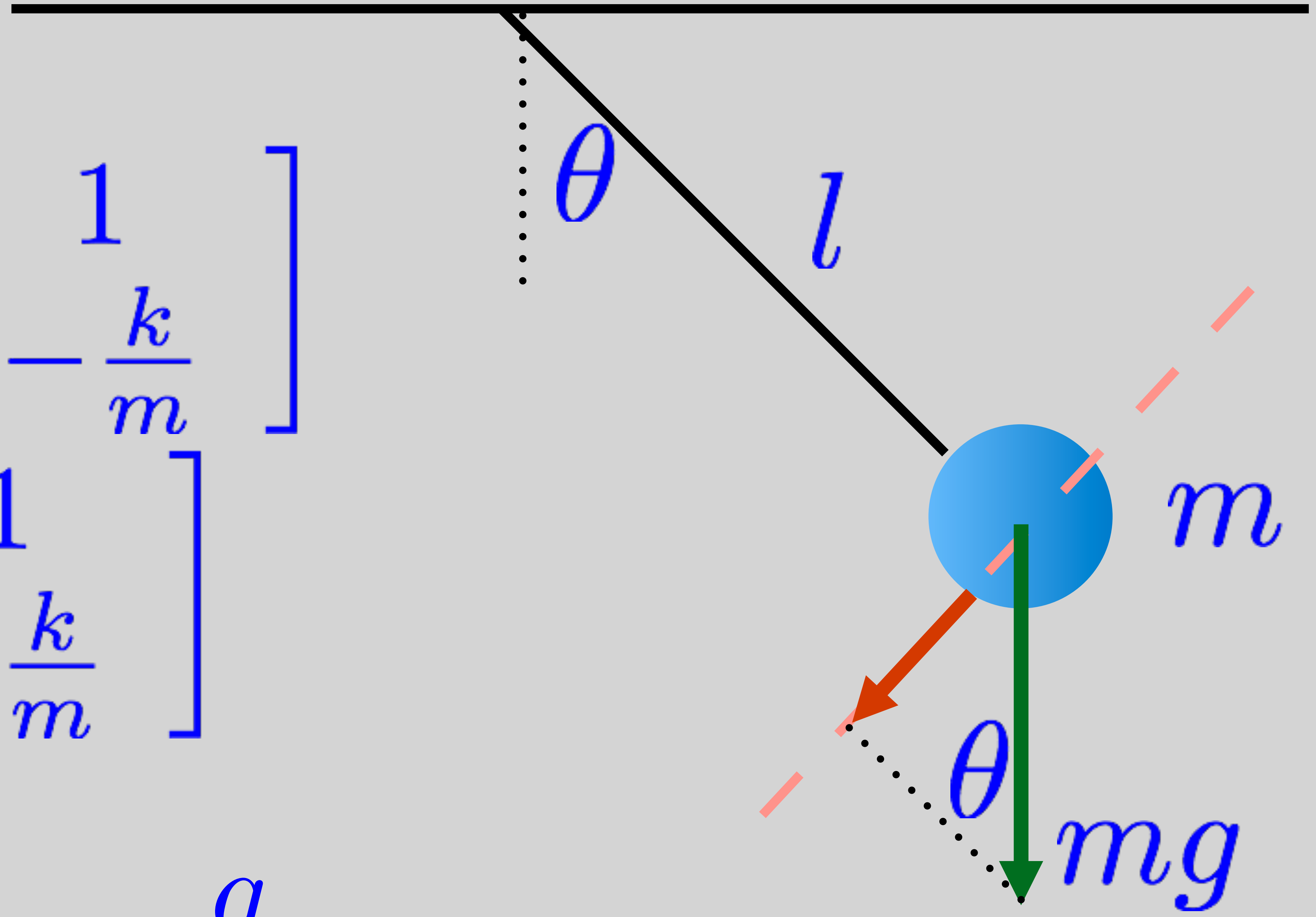


$$|\lambda I - A_{\text{up}}| = \begin{bmatrix} \lambda & -1 \\ -\frac{g}{l} & \lambda + \frac{k}{m} \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda - \frac{g}{l} = 0$$

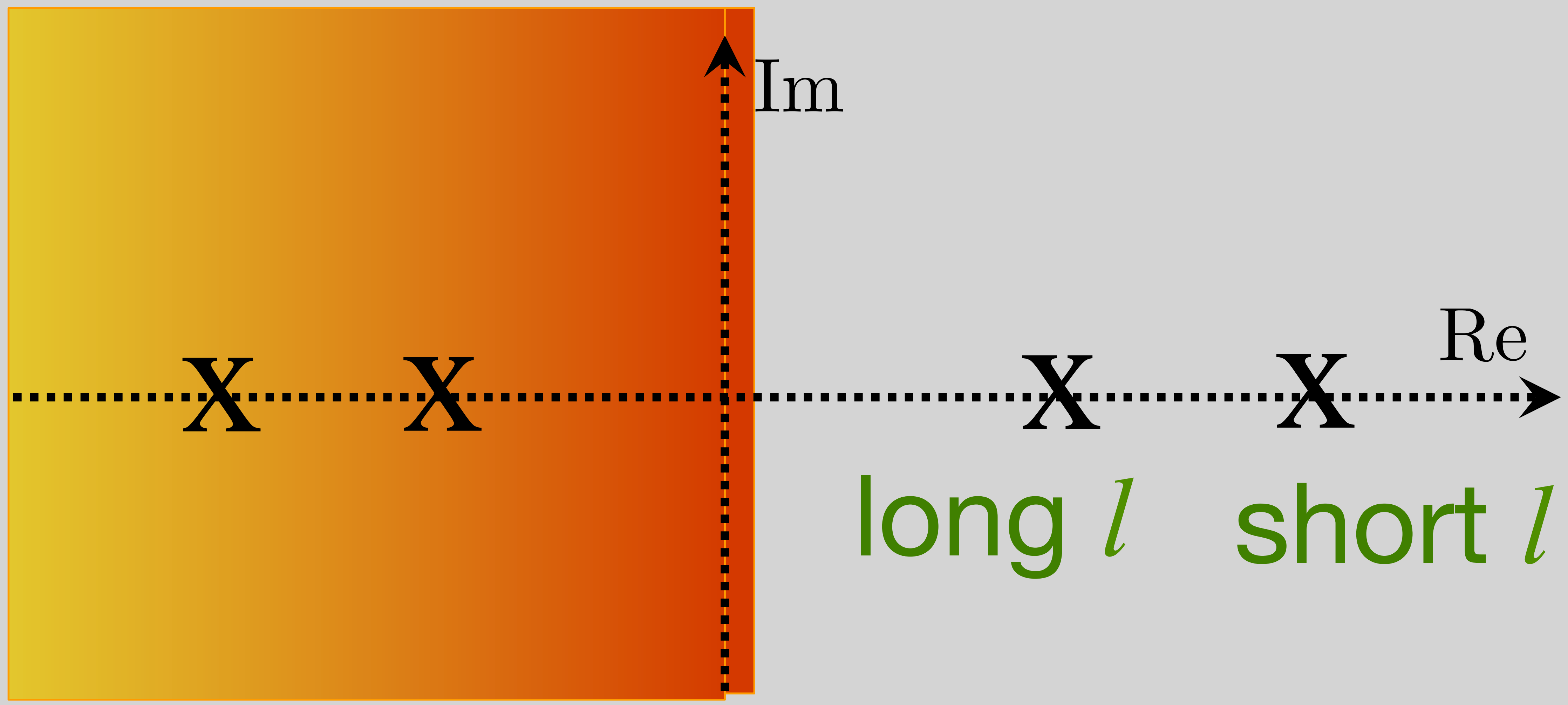
$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}$$

$$\lambda_1 > 0$$

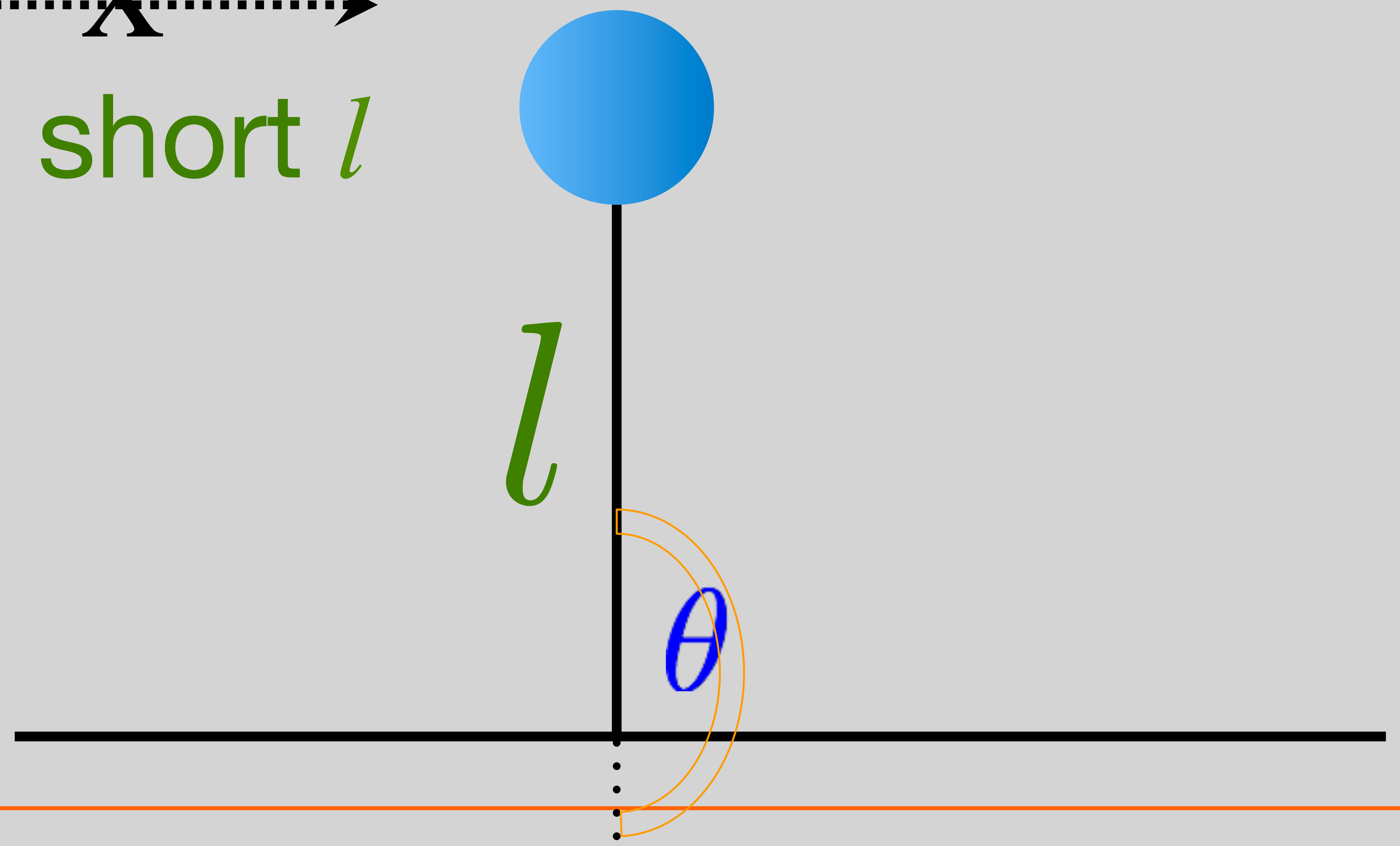
$$\lambda_2 < 0$$

Back to the Pendulum

$$\lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\frac{k^2}{m^2} + 4\frac{g}{l}}$$



$\lambda_1 > 0$
$\lambda_2 < 0$

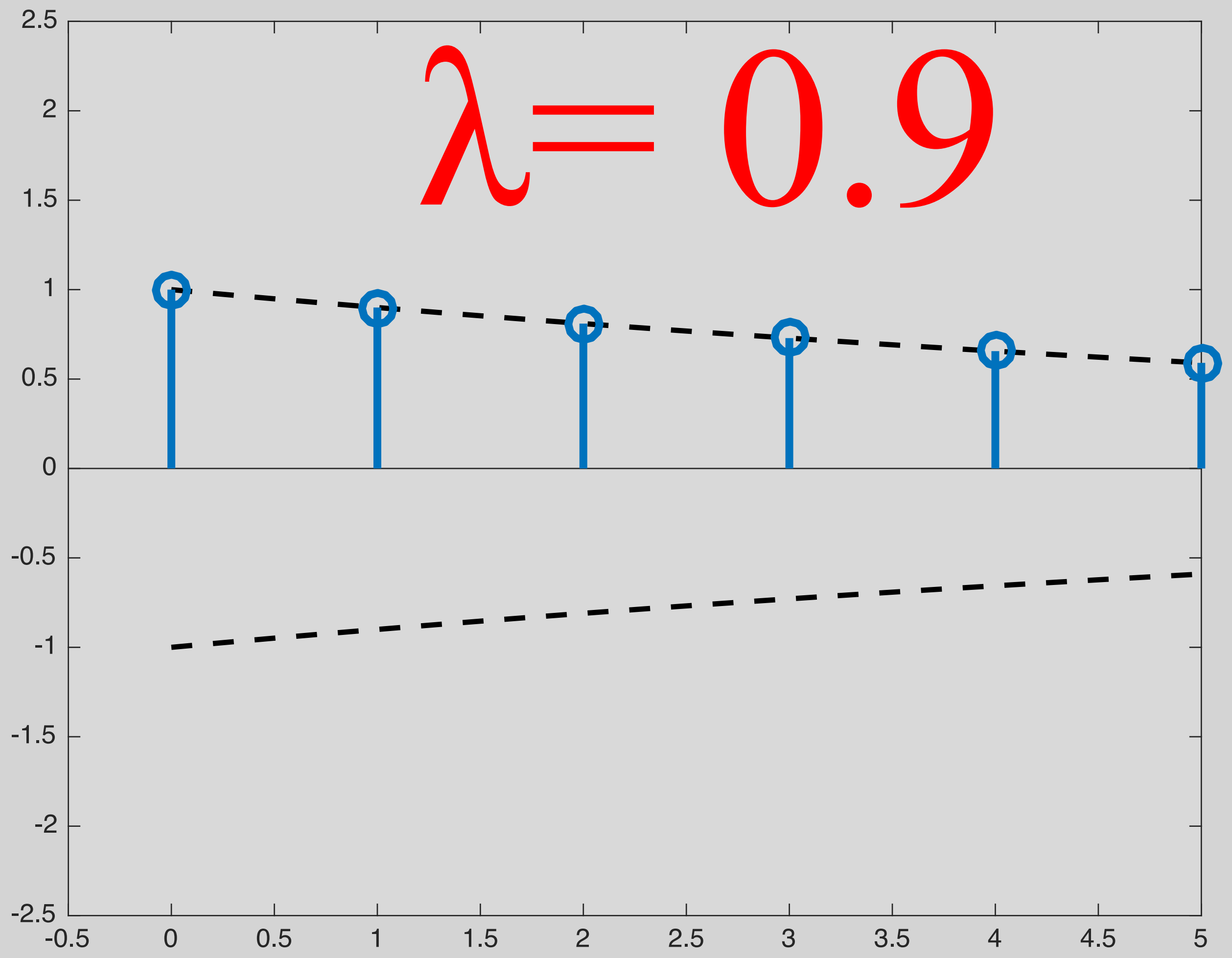


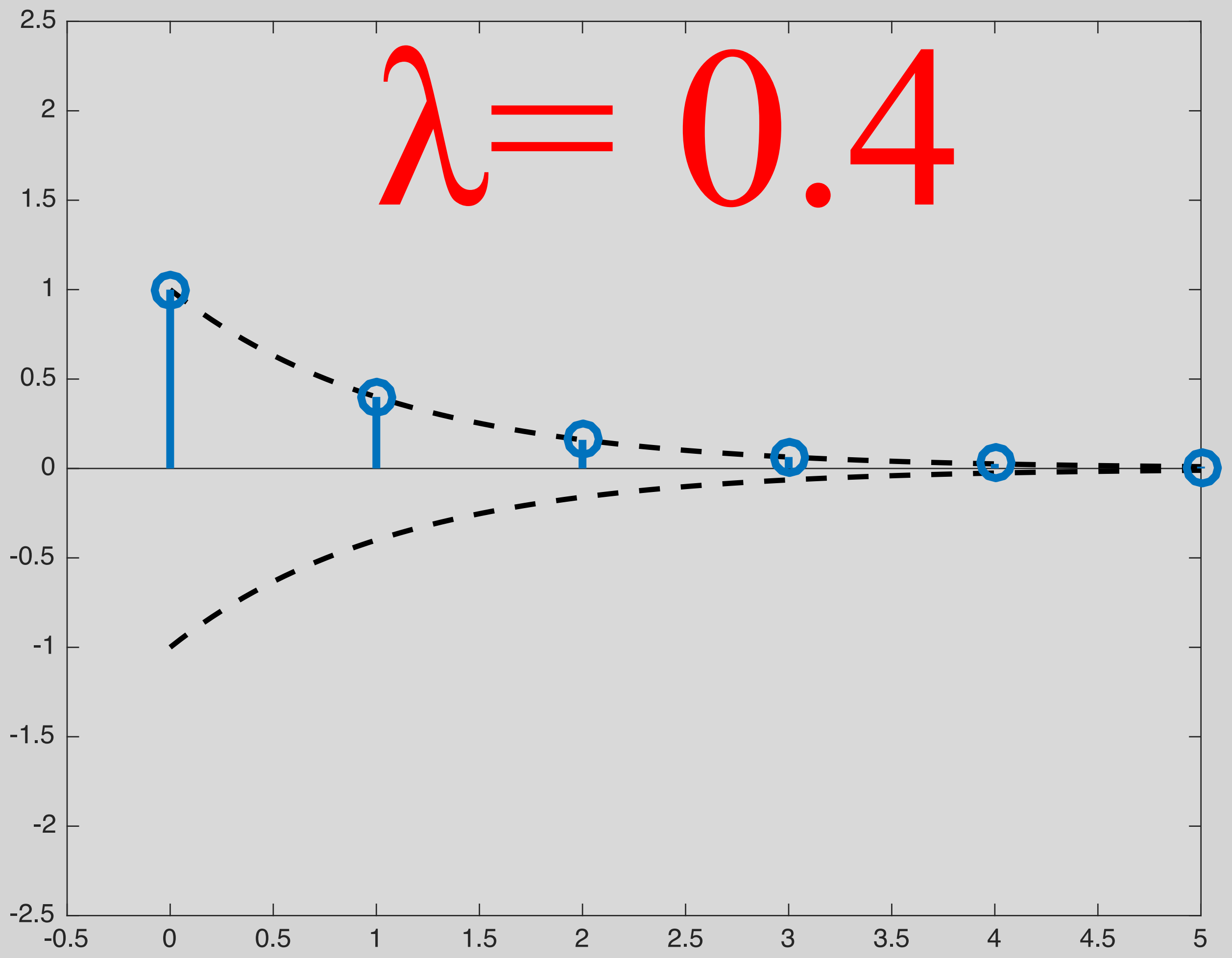
Predicting System Behavior

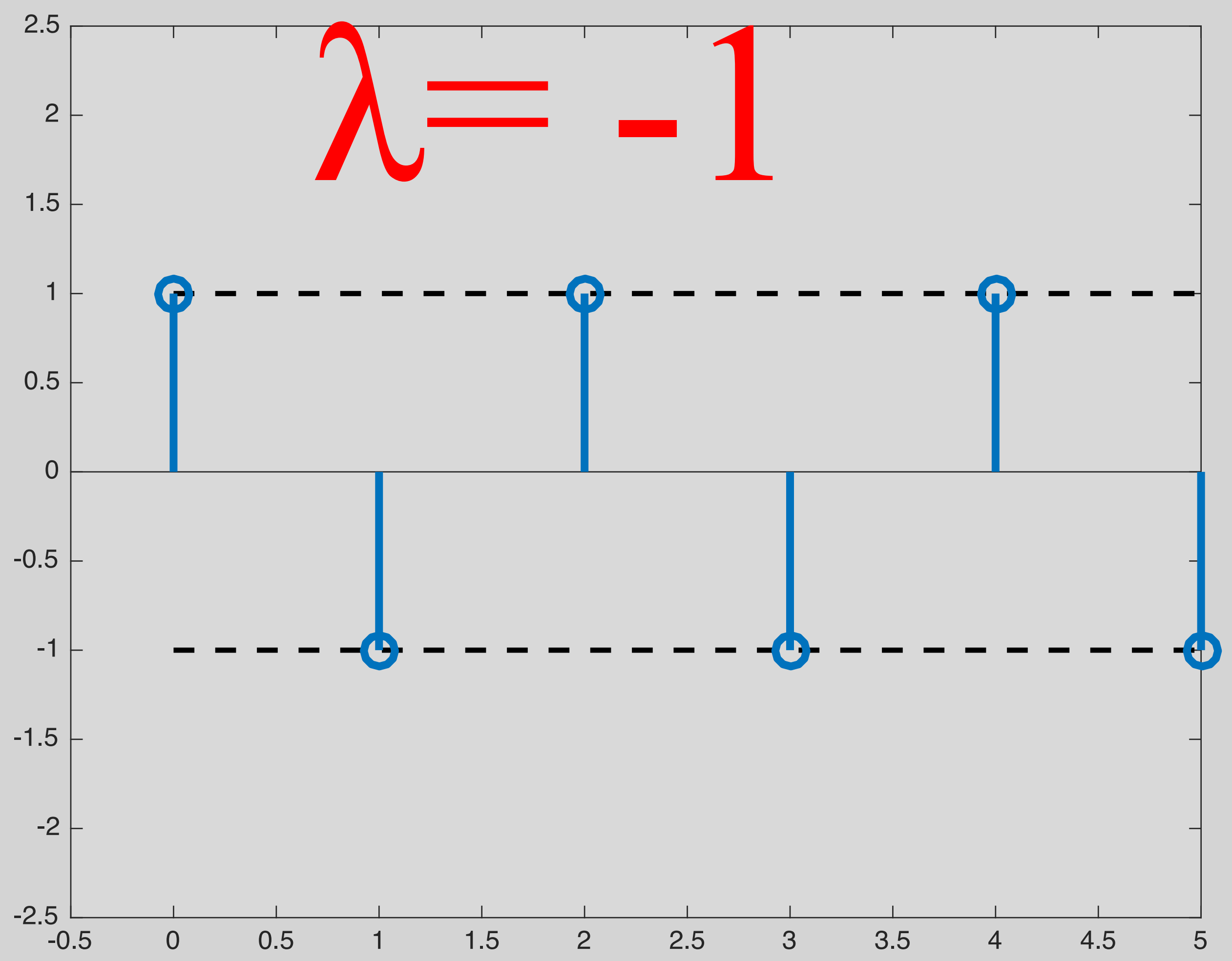
$$z(t+1) = \lambda_i z(t)$$
$$\text{Soln}: \quad \lambda_i^t z(0)$$

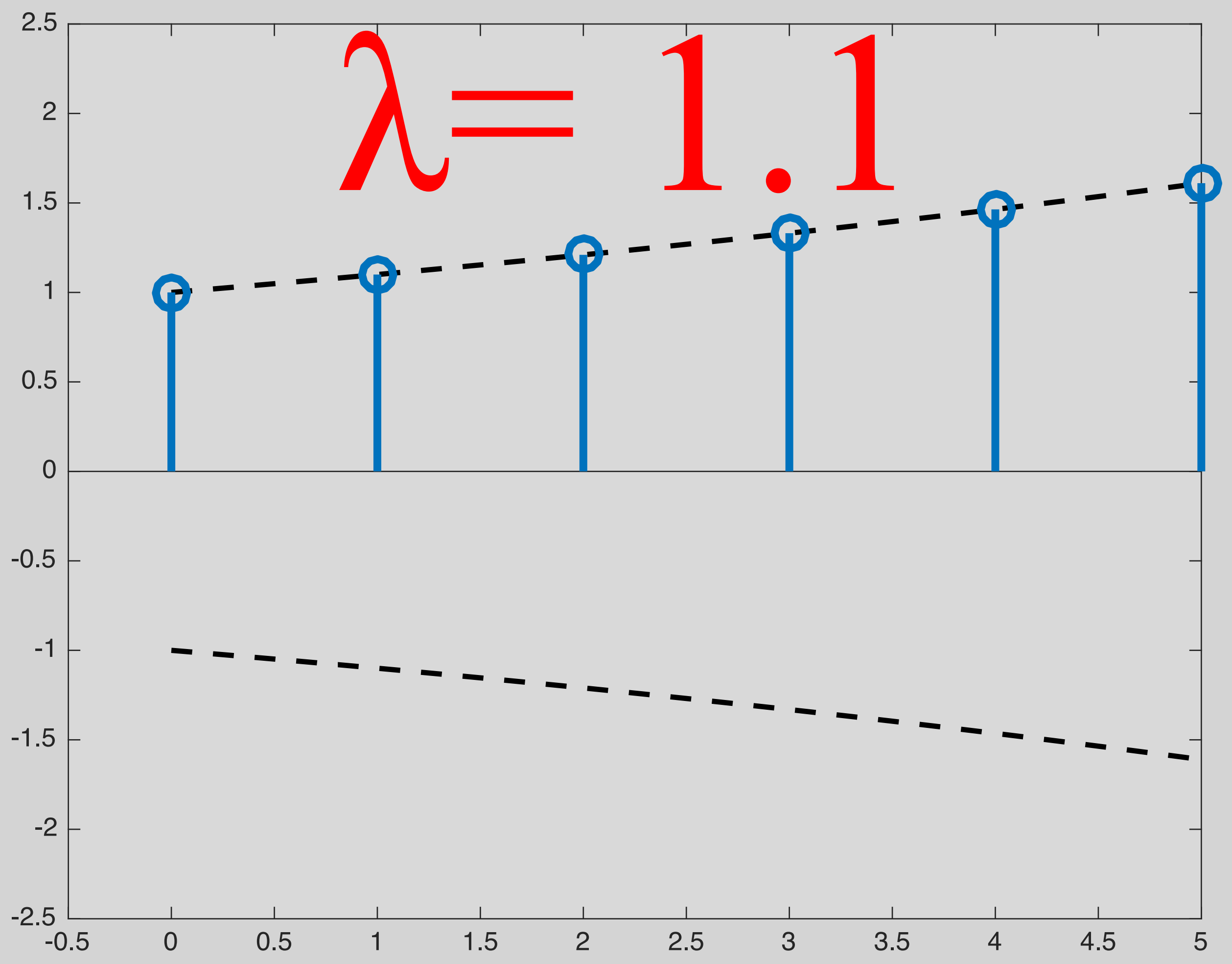
Discrete Time

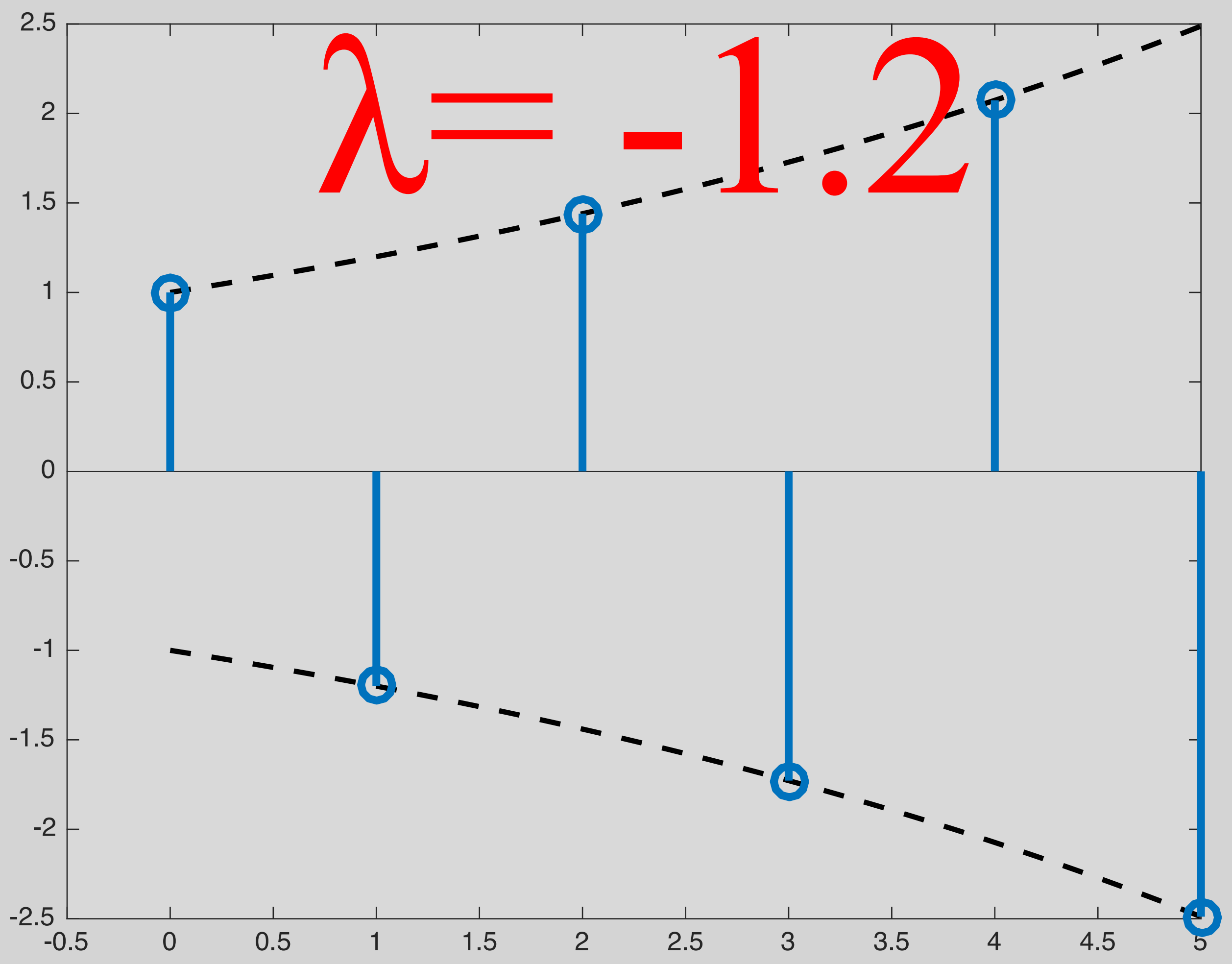
$$\lambda^t$$

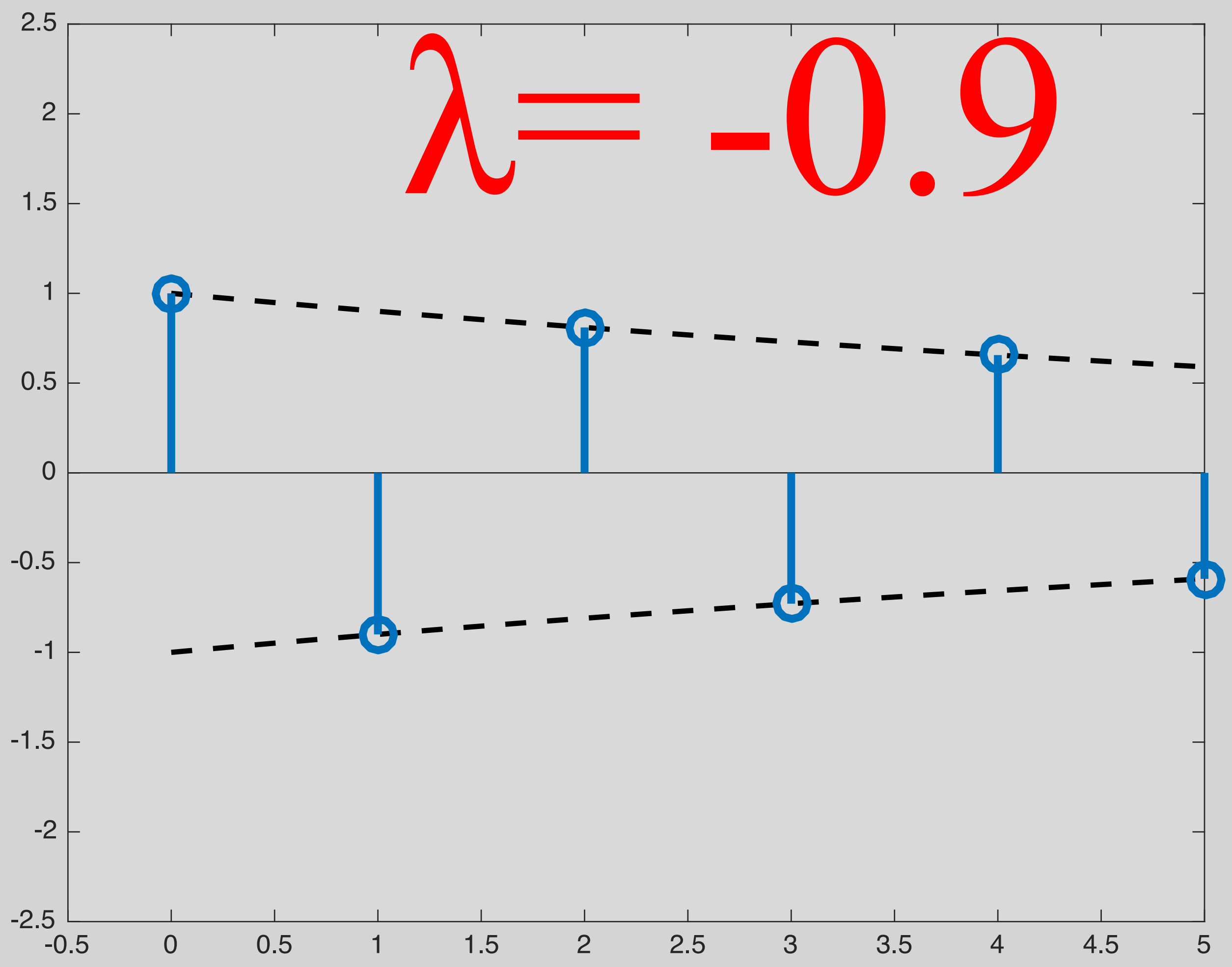








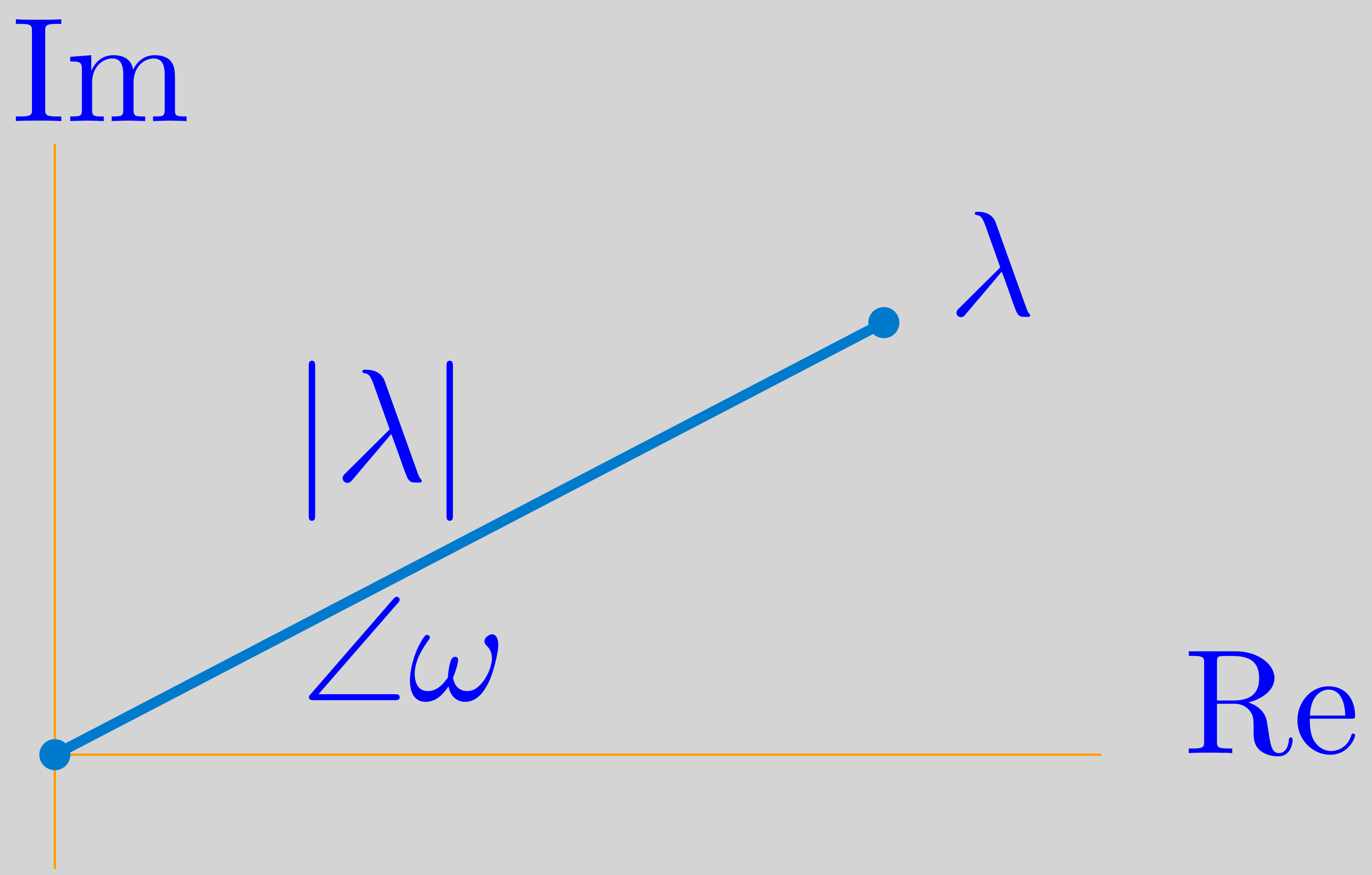




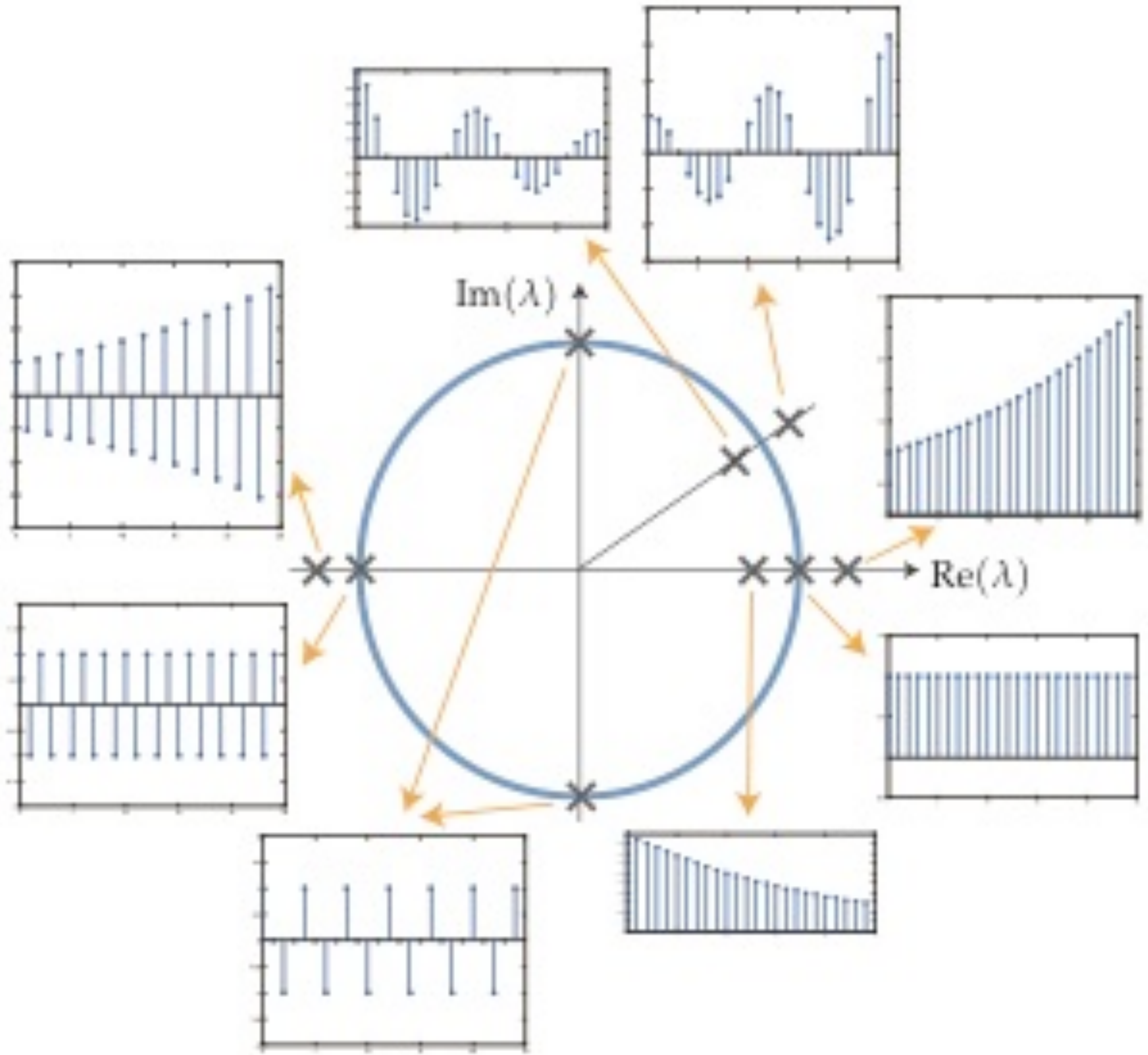
- If λ is complex

$$\lambda^t = \left(|\lambda| e^{j\omega}\right)^t$$

$$= |\lambda|^t e^{j\omega t}$$

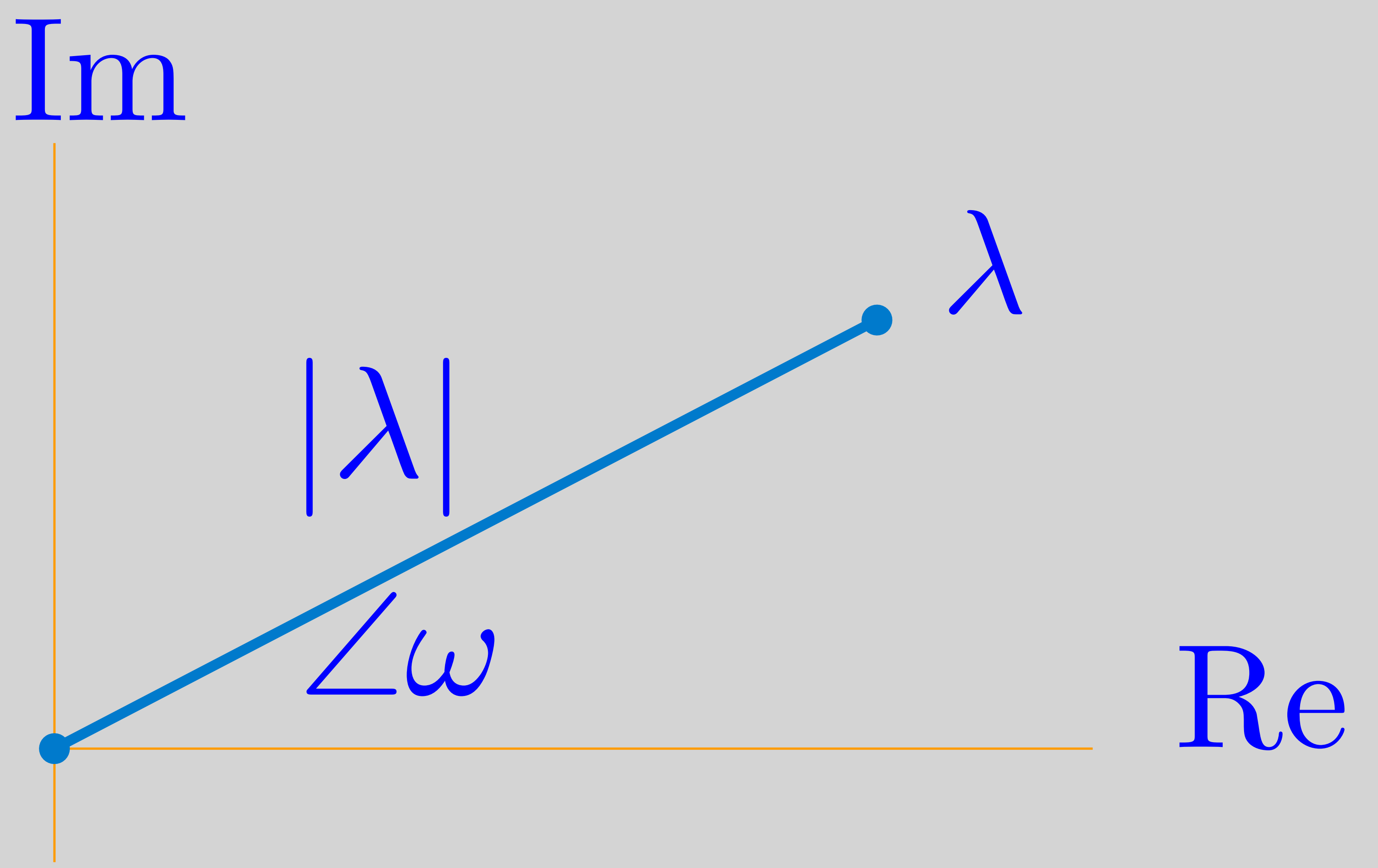


$\lambda = |\lambda| e^{j\omega}$ (Euler)


Im(λ)
Re(λ)

- If λ is complex

$$\lambda^t = \left(|\lambda|e^{j\omega}\right)^t$$
$$= |\lambda|^t e^{j\omega t}$$



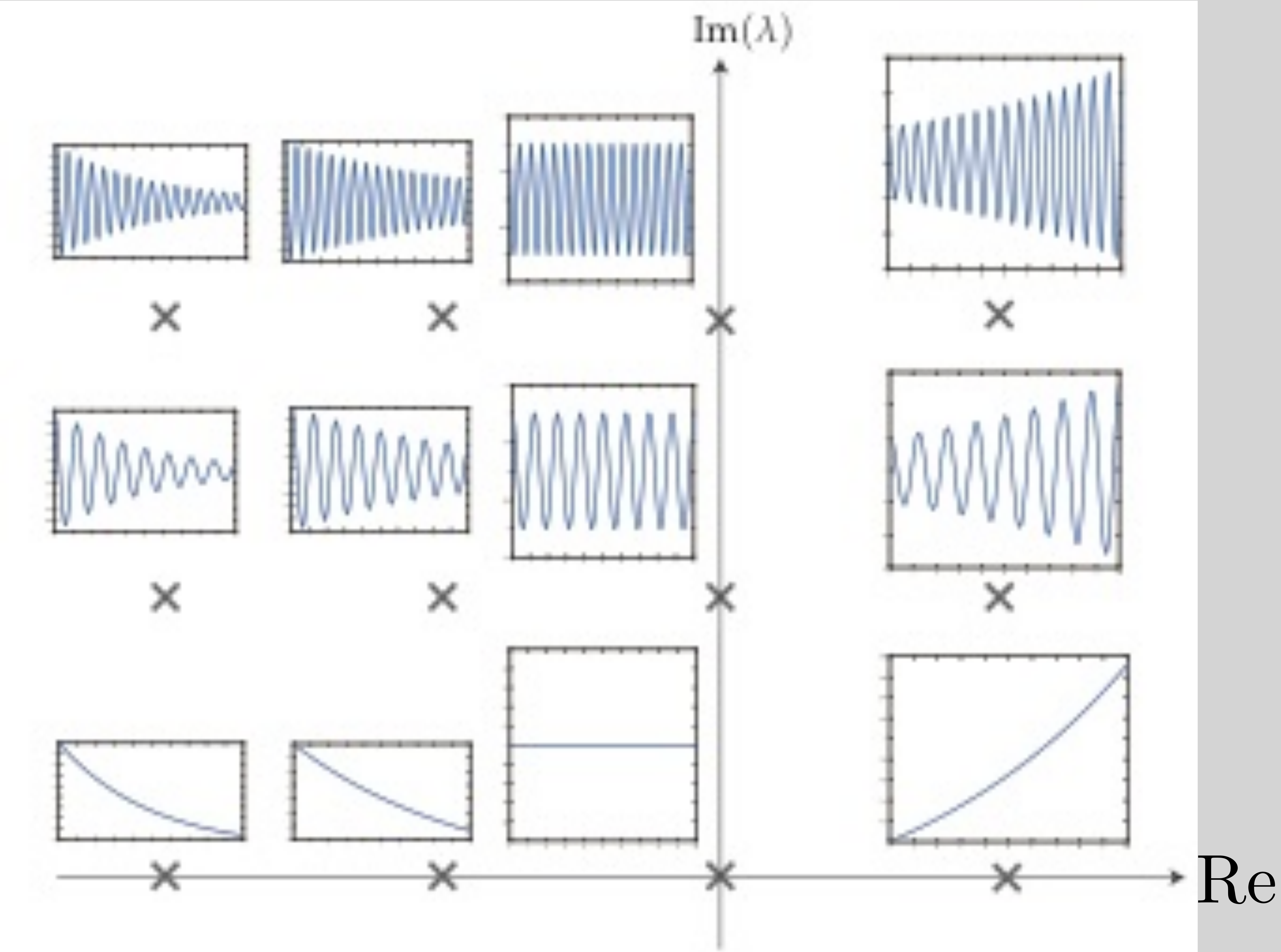
$\lambda = |\lambda|e^{j\omega}$ (Euler)

- Continuous time:

$$\frac{d}{dt}Z_i(t) = \lambda_i Z_i(t) \Rightarrow e^{\lambda_i t}Z_i(0)$$

Q) What does $e^{\lambda t}$ look like for different choices of λ?

A) $\lambda = v + j\omega \quad \Rightarrow e^{\lambda t} = e^{vt}e^{j\omega t}$


Im(λ)
Re

Summary

- Derived stability conditions for vector discrete and continuous systems
- Showed that it is easy to analyze with change of variables!
- Prediction of system behaviour for different eigen-values
  - For discrete – Phase (angle) determines frequency and magnitude determines relaxation
  - For continuous – Real part = relaxation, imaginary = frequency of oscillations
- Next time: Control design – putting the eigenvalues where we want them!