EE16B
Designing Information Devices and Systems II

Lecture 6B
Controllability

Administration

- Lecture slides (Miki Lustig):
  - Alpha version will be posted a few days before
  - Beta version will be posted same day before noon
    - Includes 4 slides per page for printing
  - version 1.0 will be posted affter student's corrections are integrated.

- Lecture notes (Murat Arcak):
  - Class Reader: http://inst.eecs.berkeley.edu/~ee16b/fa17/note/16Breader.pdf

Today

- Last time:
  - Derived stability conditions for disc. and cont. systems
  - Easy to analyze using eigenvalues
  - Eigenvalues can predict system behaviour
    - Envelope (decay) and Oscillation (frequency)
- Today:
  - Controllability of systems


HOW **NOT** TO LAND AN ORBITAL ROCKET BOOSTER

Controllability

Discrete-time: $$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

From last time:

$$\vec{x}(t) = A^t\vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k}Bu(t)_{\mathrm{k}}$$

$$= A^t\vec{x}(0) + A^{t-1}Bu(0) + A^{t-2}Bu(1) + \cdots + Bu(t-1)$$

$$\vec{x}(t) - A^t\vec{x}(0) = \begin{bmatrix} & & ? & & \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

Controllability

$$\vec{x}(t) - A^t\vec{x}(0) = A^{t-1}Bu(0) + A^{t-2}Bu(1) + \cdots + Bu(t-1)$$

$$\vec{x}(t) - A^t\vec{x}(0) = \underbrace{\begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}}_{R_t} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

Q) Given any x(0), can we find u(t) s.t. x(t) = x_{target} for some t?

A) Depends if it is in the span of R_t

Controllability

$$R_t = \begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix}$$

Q) For t < n ? A) No

Q) At t=n, If columns are independent? A) Absolutely!

Q) If not independent, does increasing t helps? A) No!

Cayley-Hamilton Theorem: If A is n x n, then A^n can be written as a linear combination of A^{n-1},…A, 1

$$A^n = \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 1$$

So does: $$A^n B = \alpha_{n-1}A^{n-1}B + \cdots + \alpha_1 AB + \alpha_0 B$$

Controllability

$$R_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}$$

What about R_{n+1}?

$$R_{n+1} = \begin{bmatrix} A^{n}B & A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}$$

$$A^n B = \alpha_{n-1} A^{n-1} B + \cdots + \alpha_1 AB + \alpha_0 B$$

Controllability Test

If R_t doesn't have n independent columns at t=n, it never will for t > n either!

Therefore, we need only to examine R_n for controllability:

$$R_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}$$

Conclusion: $\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$

is controllable if and only if

$$\text{rank}\{R_n\} = n$$

Example 1:

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t)$$

$$R_2 = [AB \quad B] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$R_3 = [A^2B \quad AB \quad B] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank {R2} = 1, < n=2 ⇒ Not controllable!

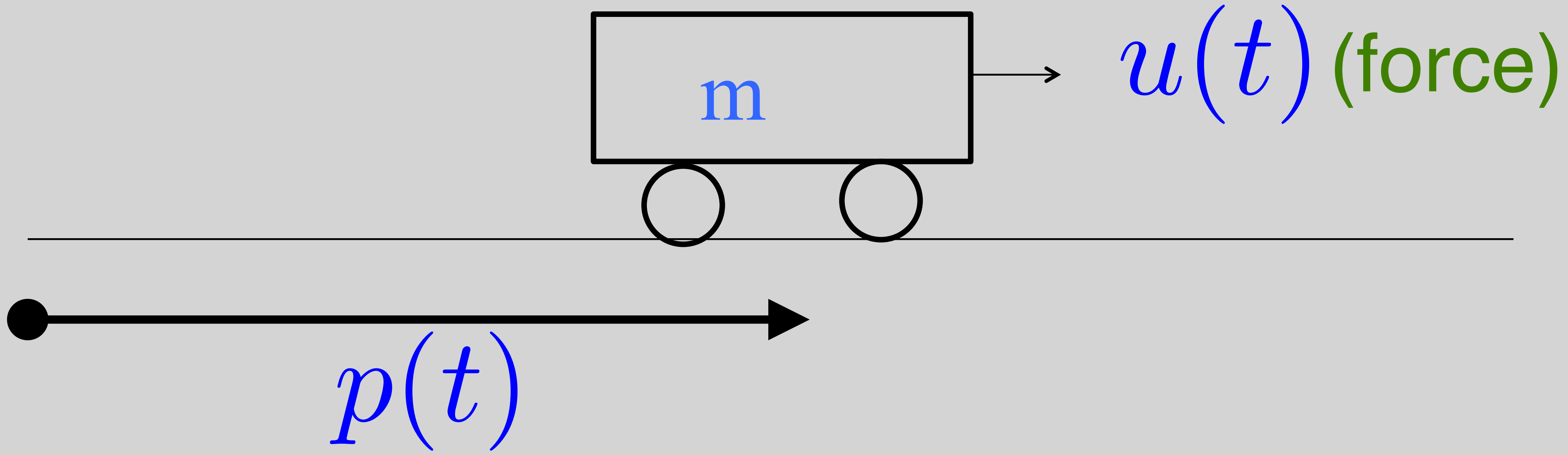
State equations:

$$x_1(t+1) = x_1(t) + x_2(t) + u(t)$$

$$x_2(t+1) = 2x_2(t) \quad \text{(not stable)}$$

Can not control x_2, not with u and not with x_1

Example 2:



$$m\frac{d^2p(t)}{dt^2} = u(t)$$

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

State variables:

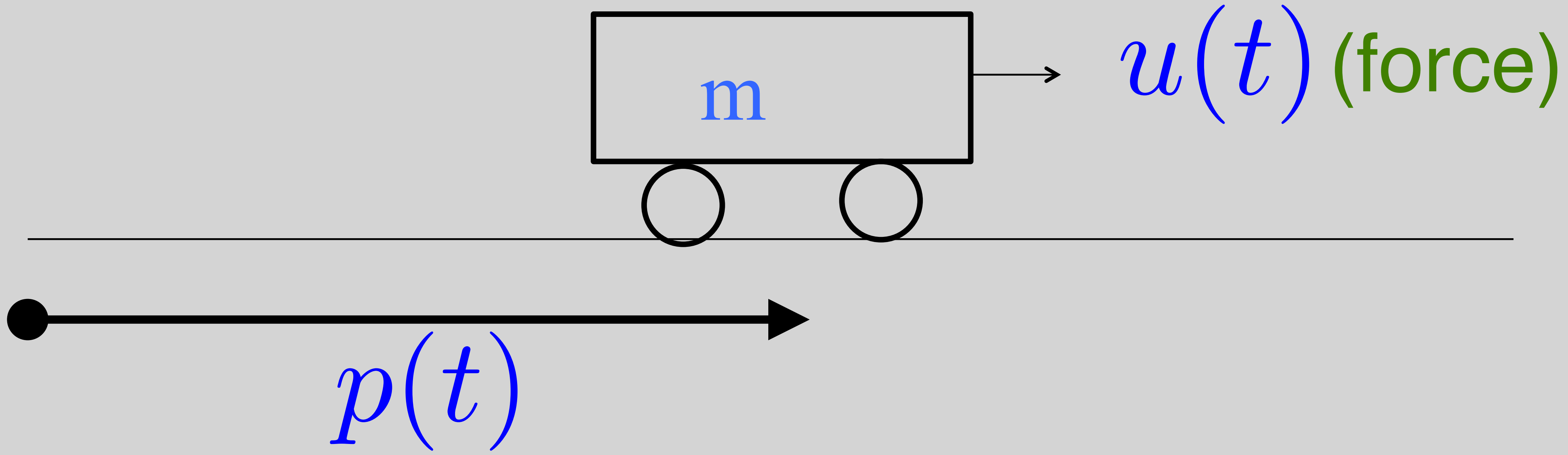
$$p(t)$$

$$\dot{p}(t) = v(t)$$

This a continuous time model!

Let's use it as a stepping stone and convert it to discrete time.

Example 2:



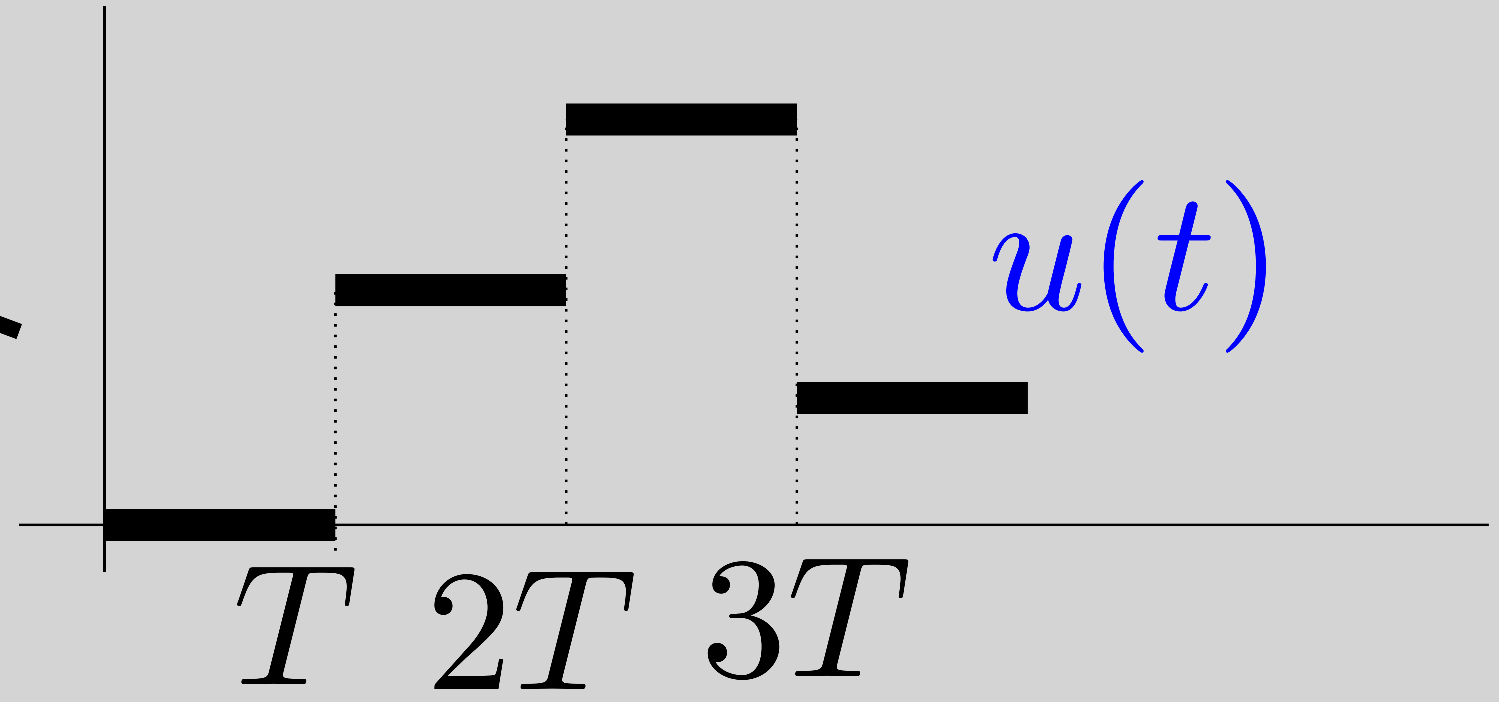
Let's convert it to discrete time:

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$\frac{d}{dt}p(t) = v(t)$$

$$\frac{d}{dt}v(t) = u(t)$$

/m



$$v(t+T) - v(t) = \int_t^{t+T} u(\tau)d\tau = Tu(t)$$

$$p(t+T) - p(t) = Tv(t) + \frac{1}{2}T^2u(t)$$ See homework!

Example 2:

$$p(t+T) = p(t) + Tv(t) + \frac{1}{2}T^2u(t)$$

$$v(t+T) = v(t) + Tu(t)$$

$$\begin{bmatrix} p(t+T) \\ v(t+T) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix}}_{B} u(t)$$

$$R_2 = [AB \quad B] = \begin{bmatrix} \frac{3}{2}T^2 & \frac{1}{2}T^2 \\ T & T \end{bmatrix}$$

Rank = 2 ⇒ Controllable!

Example 2

- Showed how to convert simple continuous time to discrete time – not always as simple!
- In Homework you will show that the continuous system is controllable

Continuous Time (no derivation here)

- The continuous-time system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

is controllable if and only if

$$R_n = \left[\begin{array}{ccccc} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{array} \right]$$

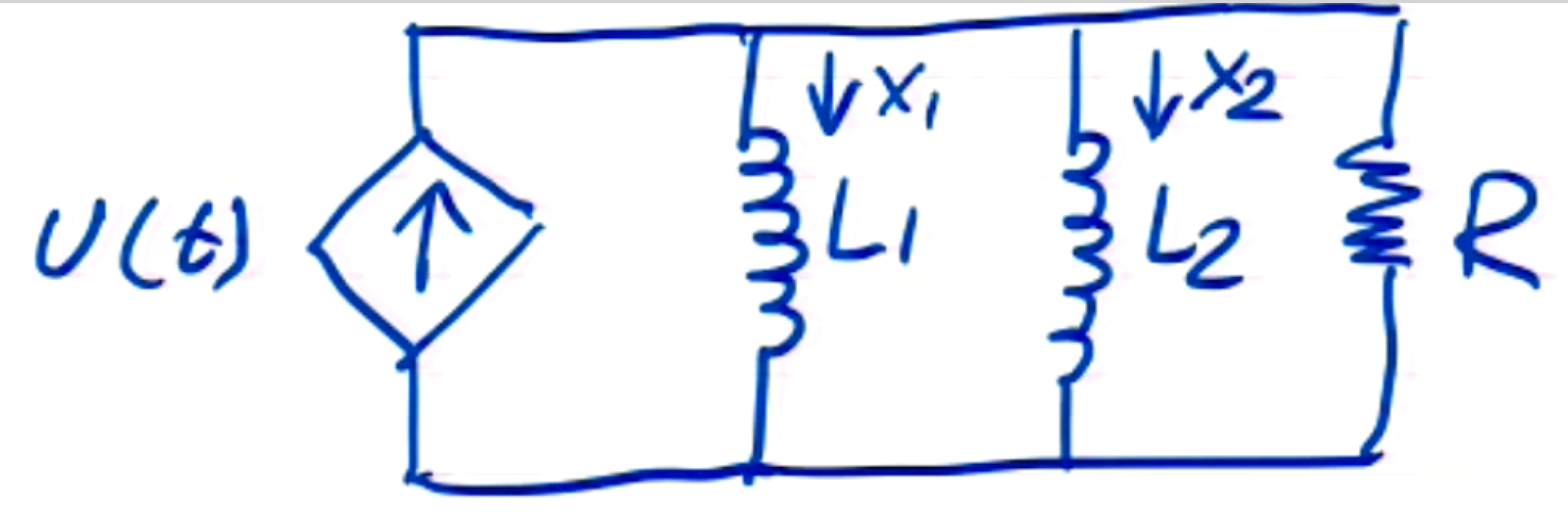
has rank = n

Example 3 + Quiz

- Write the state model:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix} u(t)$$

For:

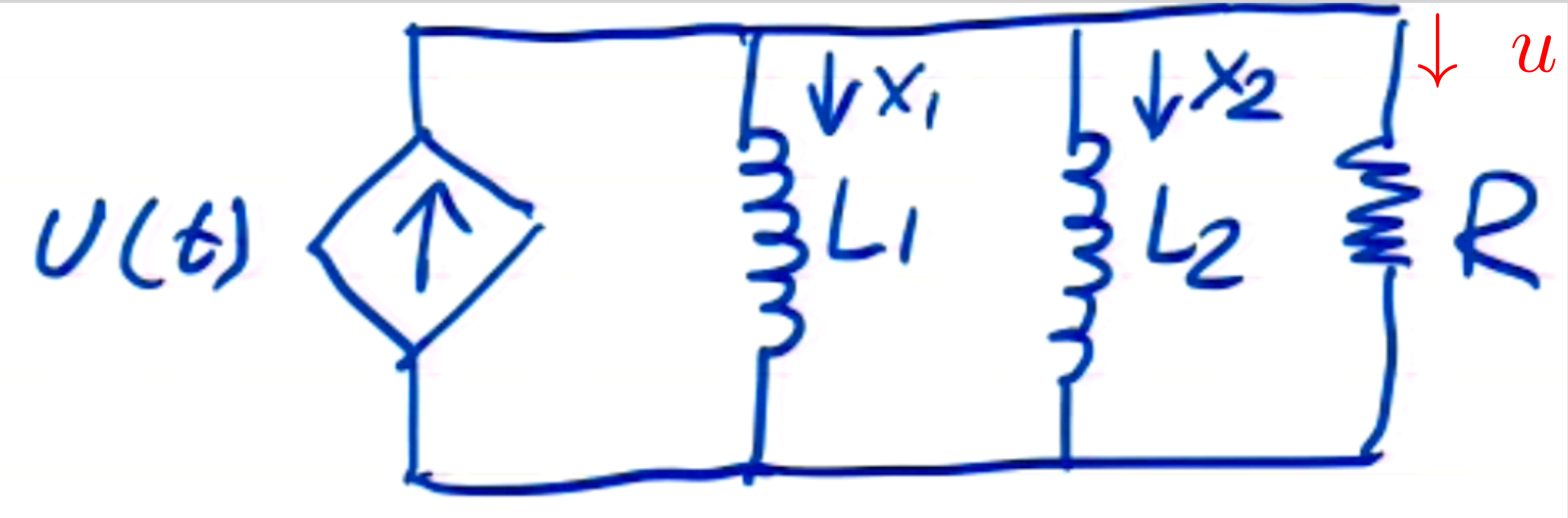


Quiz

- Write the state model:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_1} & -\frac{R}{L_1} \\ -\frac{R}{L_2} & -\frac{R}{L_2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix} u(t)$$

For:



$\downarrow\ u - x_1 - x_2$

$$V_r = R(u - x_1 - x_2) = L_1\dot{x}_1 = L_2\dot{x}_2$$

Example 3

- Controllability:

$$B = \begin{bmatrix} \frac{R}{L_1} \\ \frac{R}{L_2} \end{bmatrix}$$

$$AB = \begin{bmatrix} -\frac{R}{L_1}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \\ -\frac{R}{L_2}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \end{bmatrix}$$

$$R = [AB \quad B]$$

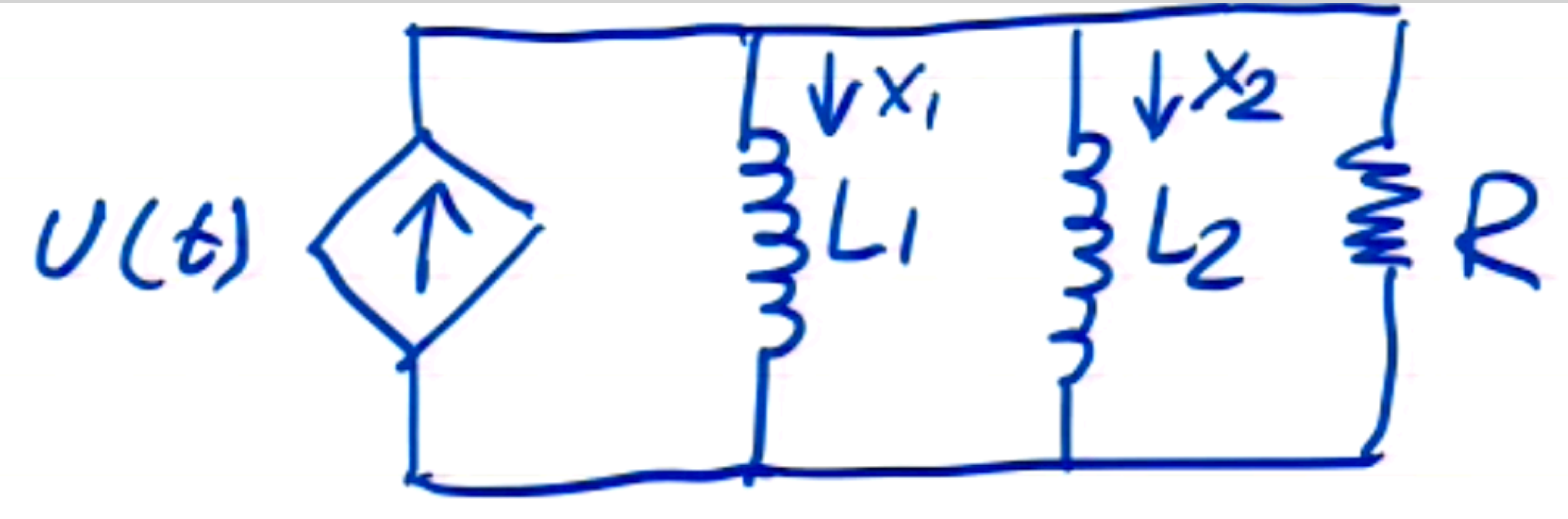
$$AB = \left(\frac{R}{L_1} + \frac{R}{L_2}\right) B$$

Rank = 1 !

Not controllable

Physical explanation

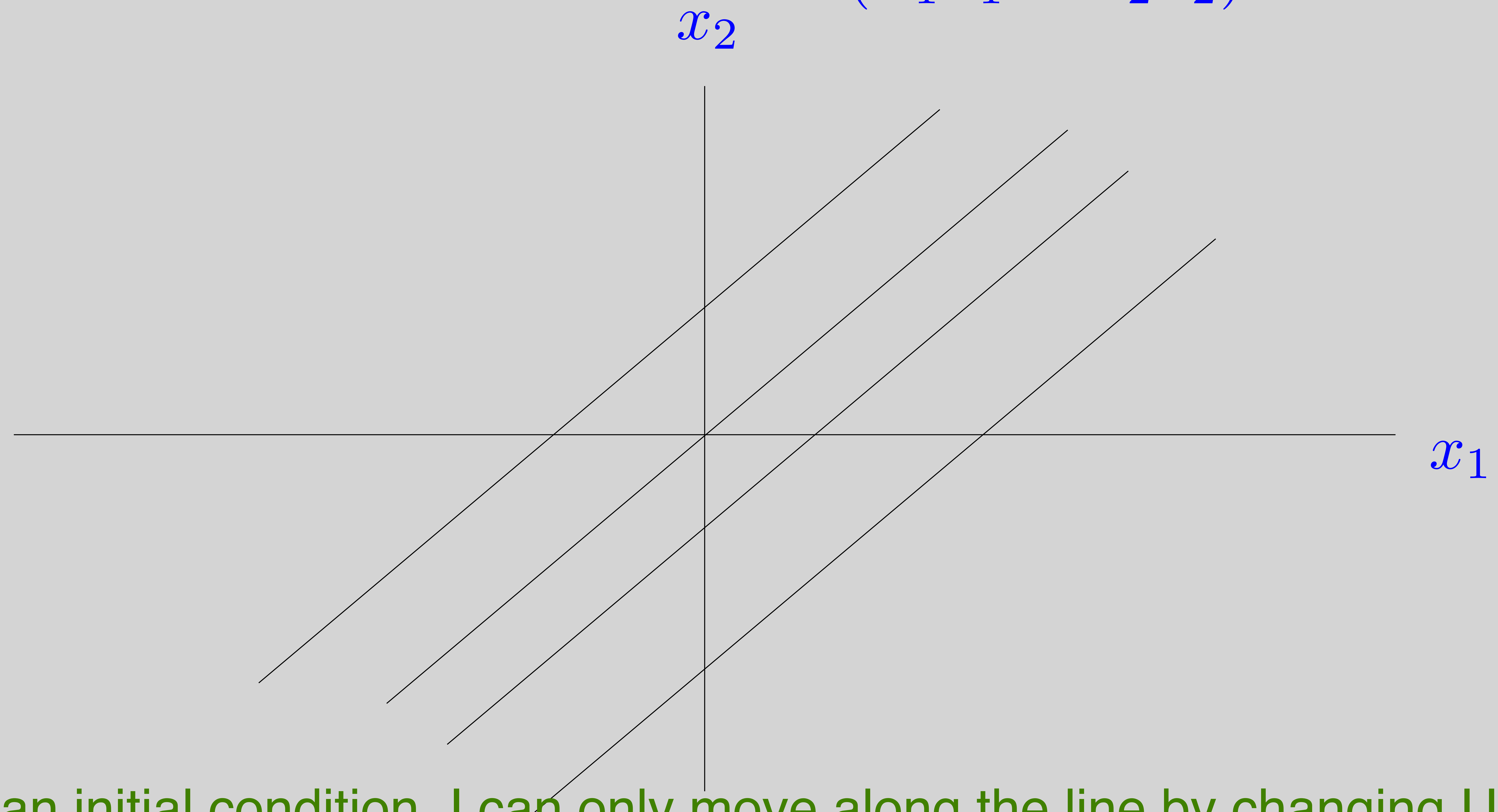
- Why can't I drive the currents x1 and x2 freely using U(t)?



$$L_1 \frac{dx_1}{dt} = L_2 \frac{dx_2}{dt} = R \cdot i_{\mathrm{R}} = V_{\mathrm{R}} \quad \Rightarrow L_1 \frac{dx_1}{dt} - L_2 \frac{dx_2}{dt} = 0$$

$$\frac{d}{dt}\left(L_1 x_1 - L_2 x_2\right) = 0 \quad \Rightarrow \left(L_1 x_1 - L_2 x_2\right) = \mathrm{Const}$$

$$(L_1 x_1 - L_2 x_2) = \text{Const}$$



Given an initial condition, I can only move along the line by changing U

Q) What if A = 0 ? Can the system be controllable?

$$\frac{d}{dt}\vec{x}(t) = Bu(t)$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & B \end{bmatrix}$$

A) Only if u(t) is a vector with the same number of elements as the number of states

Summary

- Described and derived conditions for controllability of linear state models.
  - Rank of R_n for both discrete and continuouse
- Showed how to discretize continuous systems
- Showed examples of controllable and non-controllable systems

- Next time:
  - Open loop and state feedback control
  - Controllers to make systems do what we want!