

EE16B

Designing Information Devices and Systems II

Lecture 7A

State Feedback Control

Intro

- Last time:
 - Described and derived conditions for controllability of linear state models.
 - Rank of R_n for both discrete and continuous
 - Showed how to discretize continuous systems
 - Showed examples of controllable and non-controllable systems
- Today:
 - Open loop and state feedback control
 - Controller canonical form

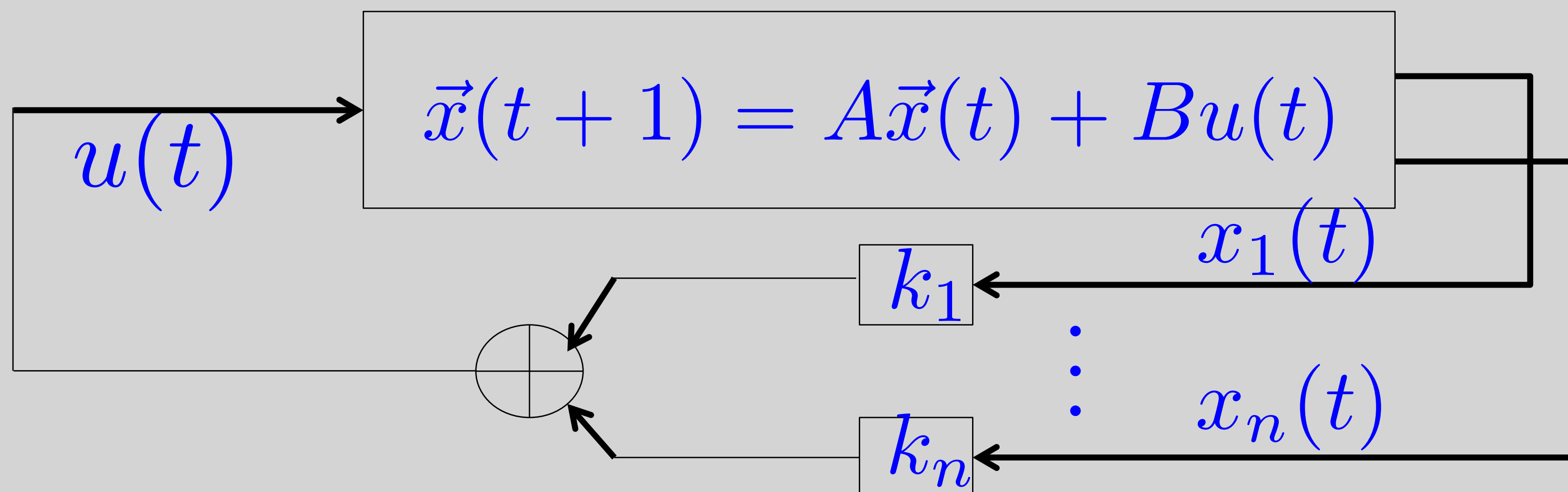
State Feedback Control

Discrete-time: $\vec{x}(t+1) = A\vec{x}(t) + Bu(t) \quad u \in \mathbb{R}$

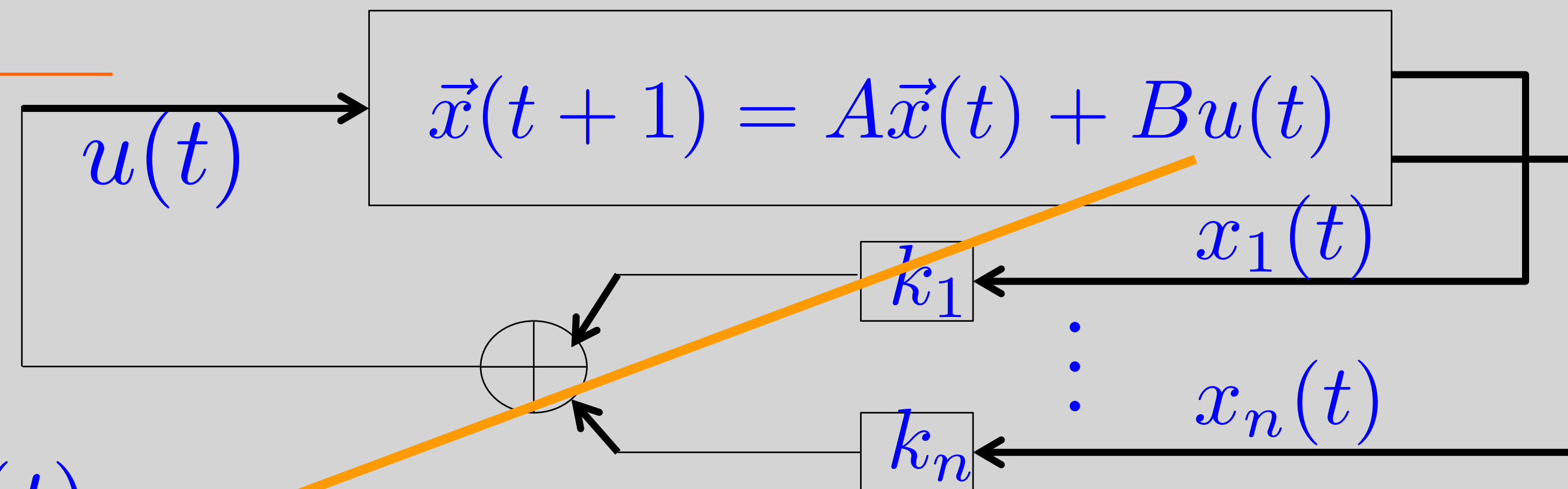
Goal: bring $\vec{x}(t)$ back to equilibrium $\vec{x} = 0$ from any initial condition $\vec{x}(0)$

“control policy” \ “control law”

$$u(t) = k_1 x_1(t) + k_2 x_2(t) + \cdots + k_n x_n(t)$$



State Feedback Control



$$u(t) = [k_1, \dots, k_n] \vec{x}(t) = K \vec{x}(t)$$

$$\Rightarrow \vec{x}(t+1) = (A + BK) \vec{x}(t)$$

If $(A+BK)$ satisfies the stability condition then,

$$\vec{x}(t) \rightarrow 0 \text{ from any initial condition!}$$

If the system is controllable, then we can also shape the eigenvalues arbitrarily...(later!)

Open Loop Control

$$\vec{x}(t) - A^t \vec{x}(0) = \begin{bmatrix} A^{t-1}B & A^{t-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$$

$\vec{x}_{\text{target}} = 0$

If the system is controllable, $u(t)$ exists to take the system from any initial state to a target state

$$\xrightarrow{u(t)} \boxed{\vec{x}(t+1) = A\vec{x}(t) + Bu(t)}$$

“Open loop control”

Q) What issues could occur in practical systems?

When would you use open or closed loop?

A) System is not robust to uncertainty or perturbations
open: to get from point A to B, closed to stay at a point

Example 1

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_A \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B u(t)$$

$$\lambda^2 - a_2\lambda - a_1$$

$$R_2 = [AB \quad B] = \begin{bmatrix} 1 & 0 \\ a_2 & 1 \end{bmatrix} \quad \text{Rank}=2 \Rightarrow \text{controllable!}$$


$$A + BK = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix}$$

$$\lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1)$$

Example 1 cont

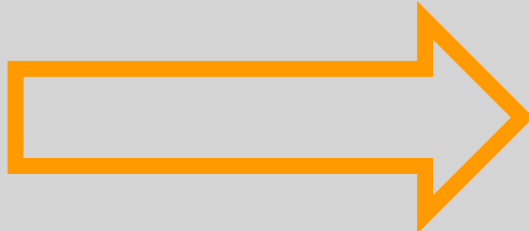
Suppose we want eigen-values at λ_1, λ_2

$$|\lambda I - (A + BK)| = \lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1)$$

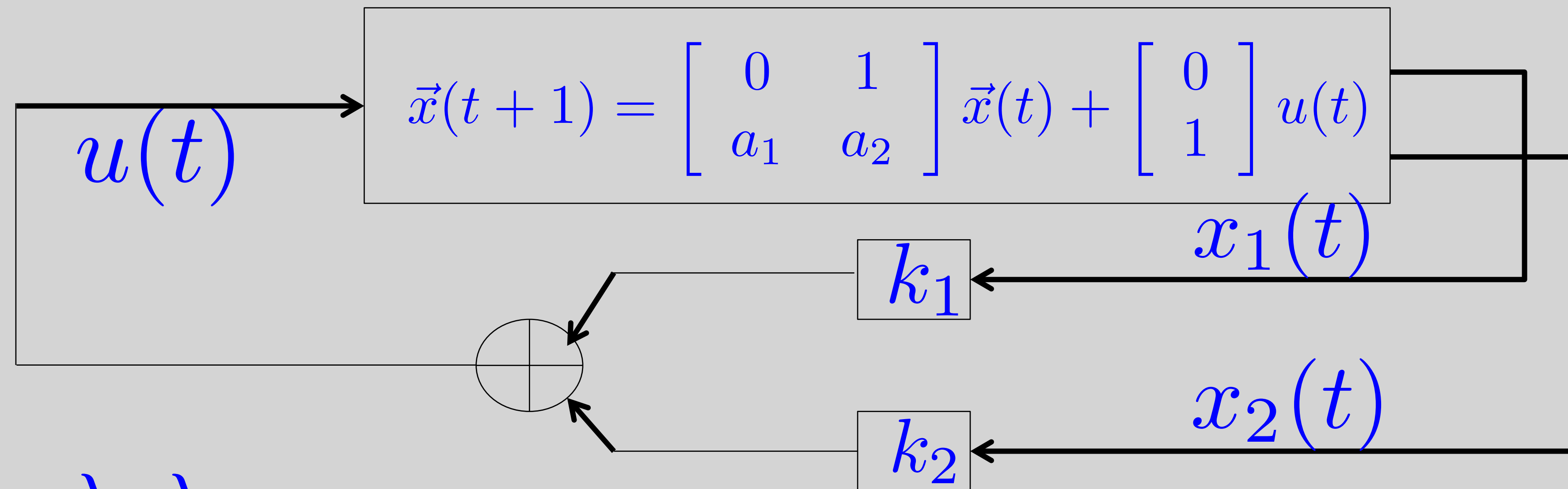
$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$


$$a_2 + k_2 = \lambda_1 + \lambda_2$$

$$a_1 + k_1 = -\lambda_1\lambda_2$$


$$\begin{aligned} k_1 &= -\lambda_1\lambda_2 - a_1 \\ k_2 &= \lambda_1 + \lambda_2 - a_2 \end{aligned}$$

Example 1: Summary



$$k_1 = -\lambda_1 \lambda_2 - a_1$$

$$k_2 = \lambda_1 + \lambda_2 - a_2$$

Eigen values of the state-feedback system will be at my chosen λ_1, λ_2 !

Example 2

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_A \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_B u(t)$$

$$A + BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} 1+k_1 & 1+k_2 \\ 0 & 2 \end{bmatrix}$$

$$R_2 = [AB \ B] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \leftarrow \begin{array}{l} \lambda_1 = k_1 + 1 \\ \lambda_2 = 2 \end{array}$$

rank = 1, uncontrollable

$$x_2(t+1) = 2x_2(t)$$

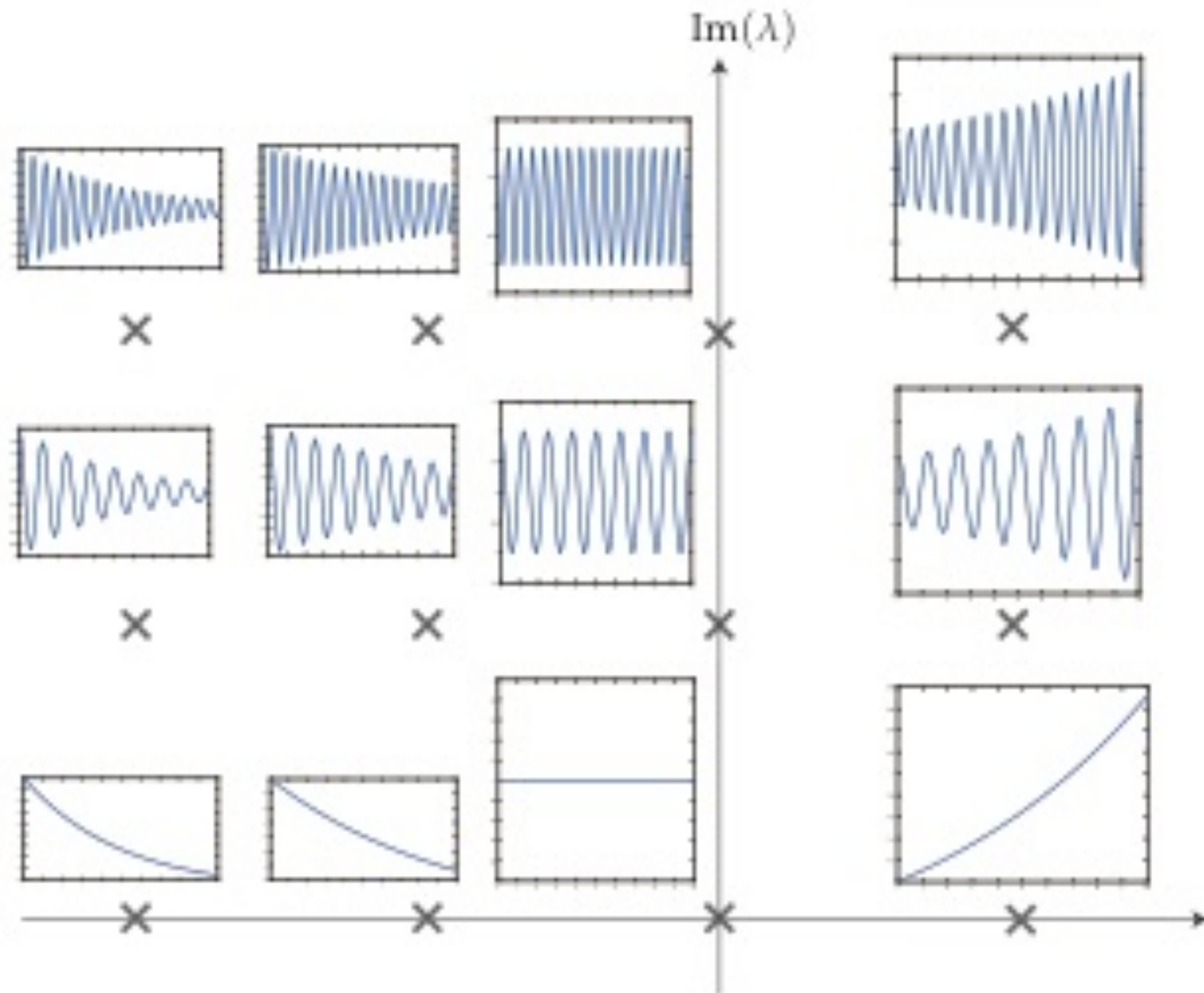
Continuous Time

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$$

$$u(t) = K\vec{x}(t)$$

$$\frac{d}{dt}\vec{x}(t) = (A + BK)\vec{x}(t)$$

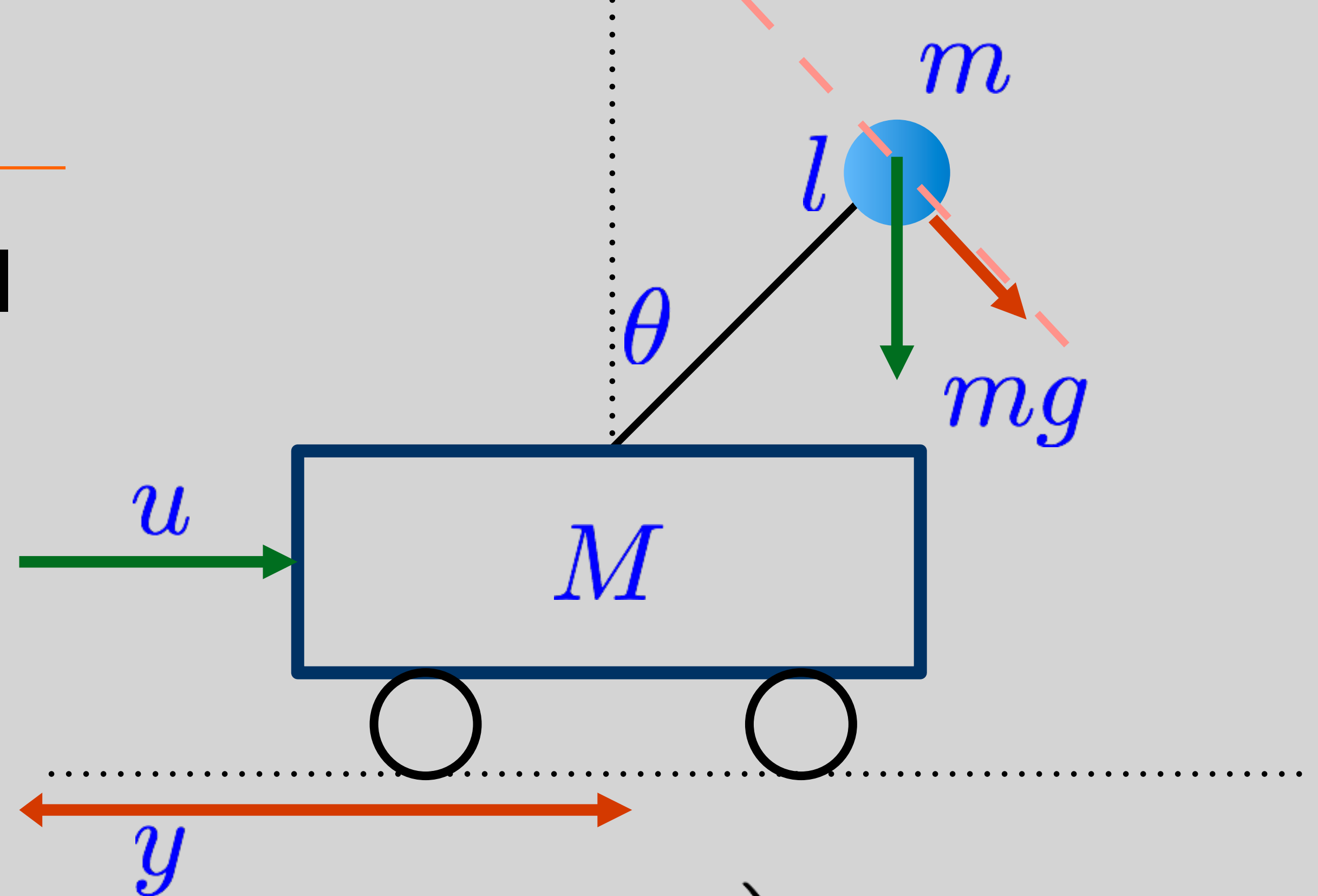
Choose K s.t. , $\text{Re } \lambda_i(A+BK) < 0$, $i=1,2,3\dots n$



Re

Example 3: Pole on a Cart

Design state-feedback control



$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2 \theta} \left(\frac{u}{m} + \underbrace{\dot{\theta}^2 l \sin \theta}_{\approx 0} - \underbrace{g \sin \theta}_{\approx \theta} \underbrace{\cos \theta}_{\approx 1} \right)$$

$$\ddot{\theta} = \frac{1}{l \left(\frac{M}{m} + \sin^2 \theta \right)} \left(-\frac{u}{m} \cos \theta - \dot{\theta}^2 l \sin \theta \cos \theta + \frac{M + m}{m} g \sin \theta \right)$$

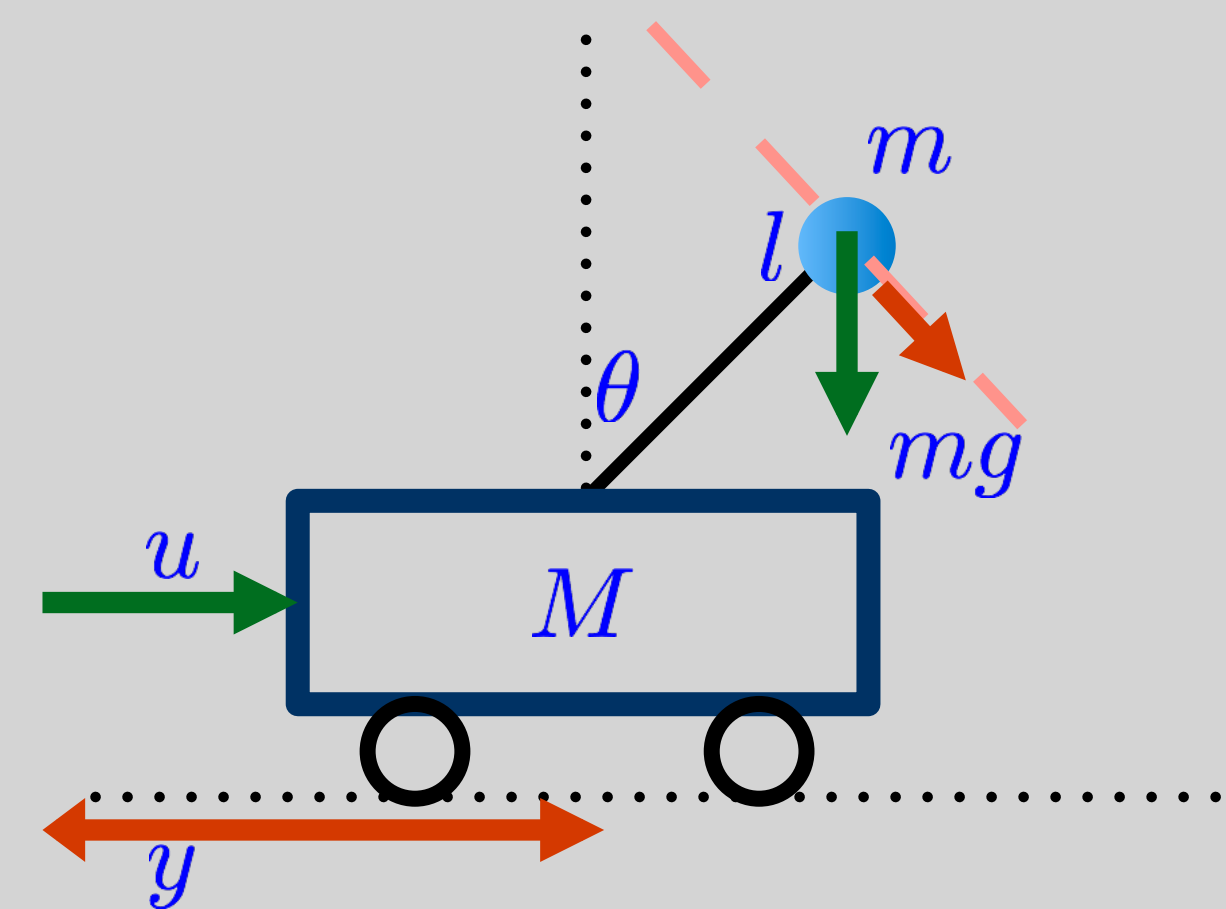
Example 3: Pole on a Cart

Linearization about $\theta = 0 \quad \dot{\theta} = 0$

$$\ddot{y} = -\frac{m}{M}g\theta + \frac{1}{M}u$$
$$\ddot{\theta} = -\frac{m+M}{Ml}g\theta - \frac{1}{Ml}u$$

State space model:

$$\frac{d}{dt} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{Ml}g & 0 & 0 \\ -\frac{m}{M}g & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ \frac{1}{M} \end{bmatrix} u(t)$$



Controller

$$M = 1$$

$$m = 0.1$$

$$l = 1$$

$$g = 10$$

$$u(t) = k_1 \theta(t) + k_2 \dot{\theta}(t) + k_3 \ddot{\theta}(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 11 - k_1 & -k_2 & -k_3 \\ -1 + k_1 & k_2 & k_3 \end{bmatrix}$$

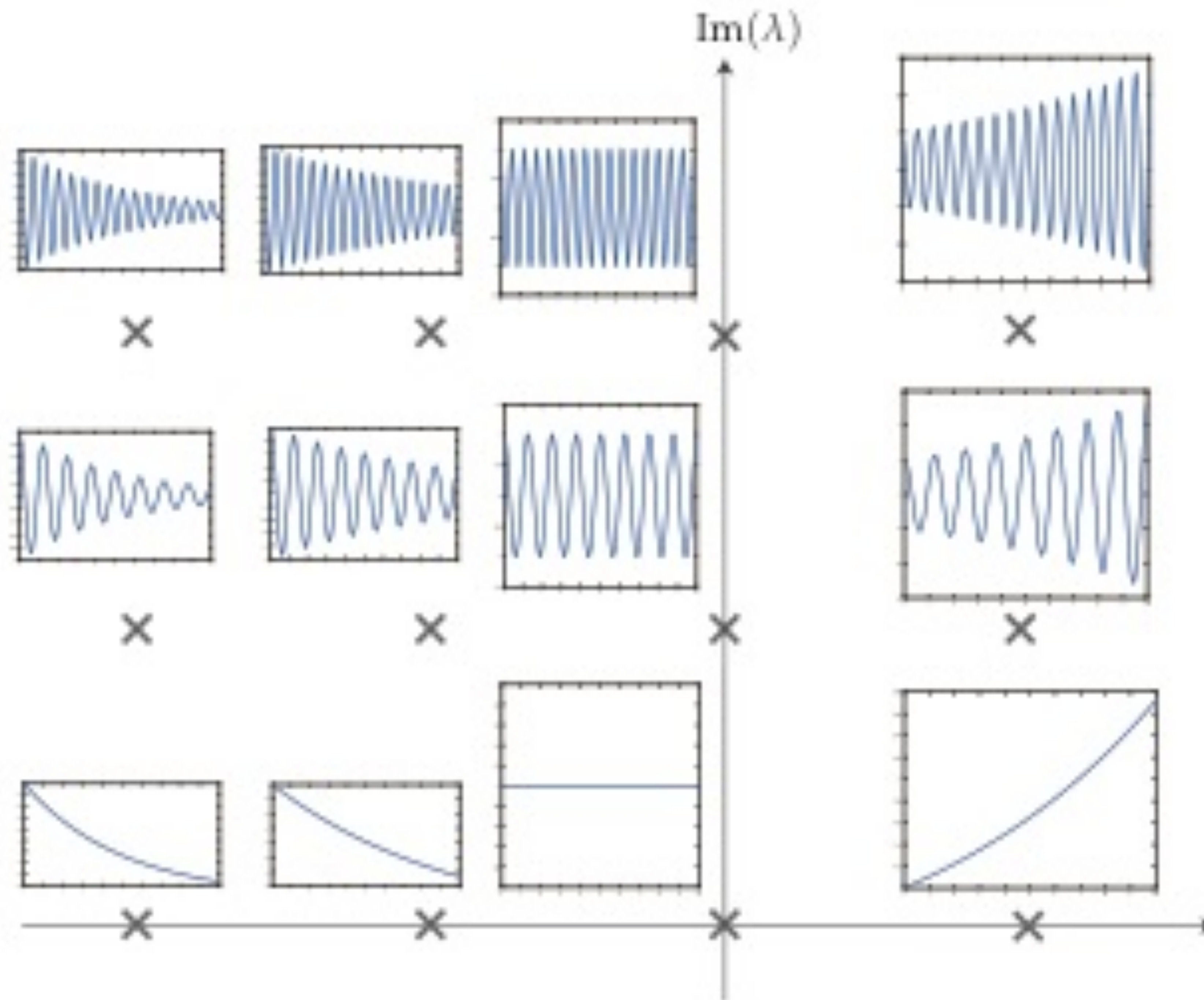
Characteristic polynomial:

$$\lambda^3 + (k_2 - k_3)\lambda^2 + (k_1 - 11)\lambda + 10k_3 = 0$$

Desired:

$$\lambda_1, \lambda_2, \lambda_3$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \quad \text{Match coeff.}$$



Re

Controller

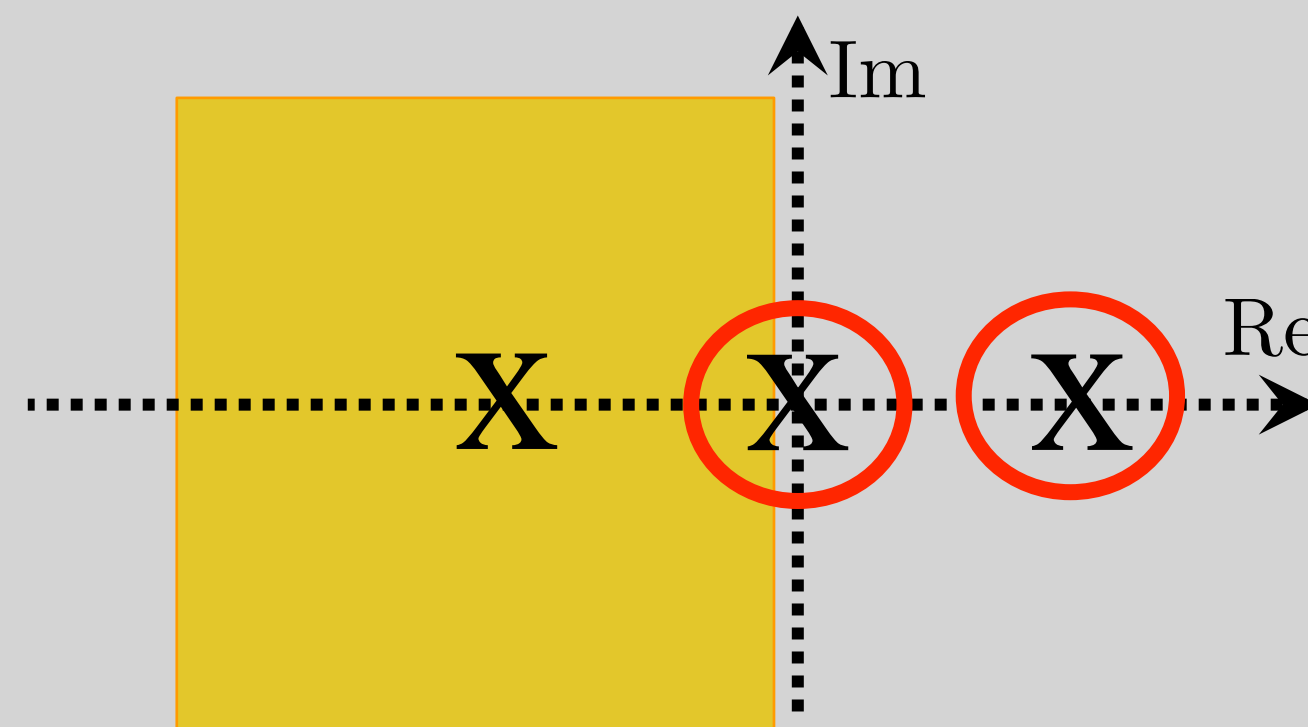
What is open loop? (no feedback control, $k_i=0$):

$$\lambda^3 + (k_2 - k_3)\lambda^2 + (k_1 - 11)\lambda + 10k_3 = 0$$

$$\lambda^3 - 11\lambda = 0$$

$$\lambda(\lambda^2 - 11) = 0$$

Ask yourself what if you can control just one, or two state variables?



Controller
moves bad
eigen-values
left!

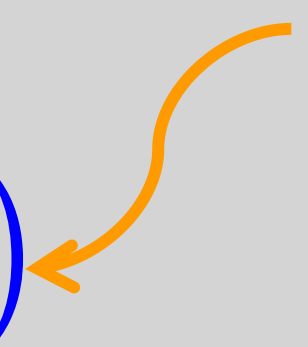
Controller Canonical Form

- I argued that with controllability one can assign eigenvalues of $A+BK$ arbitrarily
- Show
 - There's a special form for A and B that allows for arbitrary eigen-value assignment
 - Every controllable system can be brought to this form

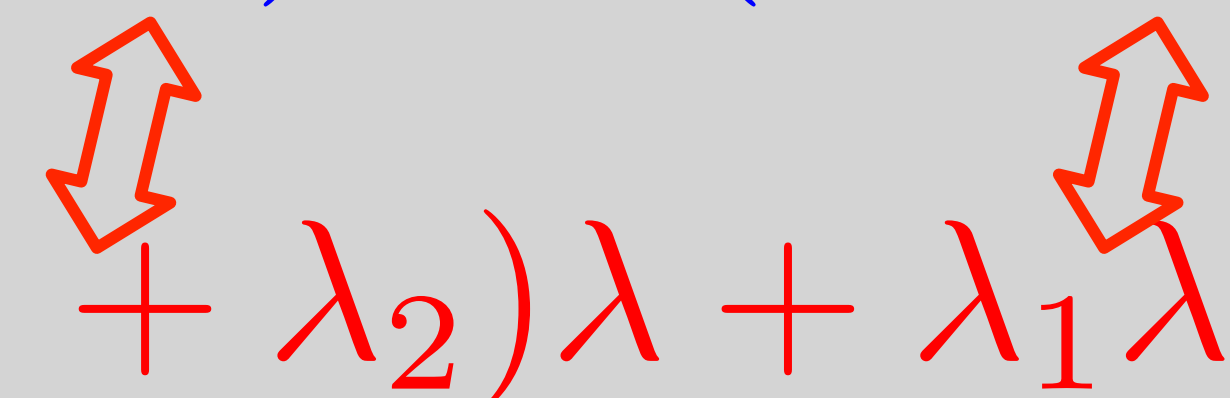
Back to Example 1

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_A \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B u(t)$$

$$A + BK = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \ k_2] = \begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix}$$

$$\lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1)$$


Match coefficients to:

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$


Canonical Form

For an n^{th} order system:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ & & & & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$A + BK =$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \\ & & & & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}}_A + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_B [k_1 \ k_2 \ \cdots \ k_n] = \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ a_1 + k_1 & a_2 + k_2 & \cdots & & \end{bmatrix}$$

$$|\lambda I - A| = \lambda^n - a_n \lambda^{n-1} - a_{n-1} \lambda^{n-2} \cdots - a_2 \lambda - a_1$$

$$|\lambda I - (A + BK)| = \lambda^n - (a_n + k_n) \lambda^{n-1} - \cdots - (a_2 + k_2) \lambda - (a_1 + k_1)$$

Match coefficients for solution:

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = \lambda^n - (\star) \lambda^{n-1} - \cdots - (\star\star) \lambda - (\star\star\star)$$

Quiz

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Q: Choose k_1, k_2, k_3 s.t. eigen values of $A+BK$ are:

$$\lambda_1 = \lambda_2 = \lambda_3 = 0$$

Quiz

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Q: Choose k_1, k_2, k_3 s.t. eigen values of $A+BK$ are:

$$\lambda_1 = \lambda_2 = \lambda_3 = 0$$

A: Characteristic polynomial of A is

$$\lambda^3 - 3\lambda^2 - 2\lambda - 1$$

Characteristic polynomial of $A+BK$ is

$$\lambda^3 - (3 + k_3)\lambda^2 - (2 + k_2)\lambda - (1 + k_1)$$

$\quad \quad \quad = -3 \quad \quad \quad = -2 \quad \quad \quad = -1$

Controllability in Canonical Form

Claim: If $\vec{x}(t+1) = \tilde{A}\vec{x}(t) + \tilde{B}u(t)$ is controllable then we can find an invertible T s.t.

$$\begin{aligned} T\tilde{A}T^{-1} &= A \\ T\tilde{B} &= B \end{aligned} \quad \text{Canonical form}$$

$$\vec{z}(t) = T\vec{x}(t) \qquad \vec{z}(t+1) = A\vec{z}(t) + Bu(t)$$

We can design k s.t. $A+BK$ has eigenvalues wherever we want:

$$u(t) = K \underbrace{\vec{z}(t)}_{=T\vec{x}(t)} = \underbrace{KT}_{=\tilde{K}} \vec{x}(t)$$

$$T(\tilde{A} + \tilde{B}\tilde{K})T^{-1} = A + BK$$

- How do we know T exists s.t. : $T\tilde{A}T^{-1} = A$
 $T\tilde{B} = B$

- When (\tilde{A}, \tilde{B}) are controllable?

$$\tilde{R}_n = \begin{bmatrix} \tilde{A}^{n-1}\tilde{B} & \tilde{A}^{n-2}\tilde{B} & \dots & \tilde{A}\tilde{B} & \tilde{B} \end{bmatrix} \text{ Invertible!}$$

$$R_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & AB & B \end{bmatrix} = T \begin{bmatrix} \underbrace{\tilde{A}^{n-1}\tilde{B} \quad \tilde{A}^{n-2}\tilde{B} \quad \dots \quad \tilde{A}\tilde{B} \quad \tilde{B}}_{\tilde{R}_n} \end{bmatrix}$$

Diagrammatic annotations for the matrix equality:

- Red double-headed arrow between $A^{n-1}B$ and $T\tilde{A}^{n-1}\tilde{B}$.
- Red double-headed arrow between AB and $\underbrace{T\tilde{A}T^{-1}(T\tilde{B})}_{T\tilde{A}\tilde{B}}$.
- Red double-headed arrow between B and $T\tilde{B}$.

$$R_n = T\tilde{R}_n \Rightarrow T = R\tilde{R}^{-1}$$

You don't have to go to canonical form to solve the problem! It's just easier.

Summary

- Discussed State-feedback Control
- Discussed open-loop control
- When the system is controllable, can assign eigenvalues arbitrarily
- Discussed Controller Canonical Form

- Next time:
 - Inputs, Observables, and Observability – when we can't see everything needed to control!