EE16B
Designing Information Devices and Systems II

Lecture 7B
Outputs and Observability

Announcements

- Survey just posted to piazza. If > 85% of students fill it out, everyone gets a free point on the next midterm
- Pay attention for discussion section cancellations posts on Piazza
- Talk of interest, EESE Colloquium
  - Wednesday October 18 , 306 Soda Hall
  - Larse Blackmore (SpaceX)

SpaceX's reusable rocket program aims to reduce the cost of space travel by making rockets that can land, refuel and refly, instead of being thrown away after every flight. **Autonomous precision landing of a rocket** is a unique problem, which has been **likened to balancing a rubber broomstick on your hand in a windstorm**. Rockets do not have wings (unlike airplanes) and they cannot rely on a high ballistic coefficient to fly in a straight line (unlike missiles). In the past two years, **SpaceX has successfully landed sixteen rockets**, some of which were on dry land at Cape Canaveral, and some of which were on a floating platform in the Atlantic. This talk will discuss the challenges involved, how these **challenges** were overcome, and **next steps** towards rapid reusability.

Intro

- Last time
  - State feedback control
  - Eigen value asignment
  - Controller Canonical Form

- Today:
  - Finish Controller canonical form
  - Example of cooperative, adaptive cruise control
  - Output and observability
  - MRI as a dynamical system

Review: Canonical Form

For an n^{th} order system:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \\ & & & & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$A + BK =$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \\ & & & & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ & & \ddots & \\ & & & 1 \\ a_1 + k_1 & a_2 + k_2 & \cdots & a_n + k_n \end{bmatrix}$$

$$|\lambda I - A| = \lambda^n - a_n \lambda^{n-1} - a_{n-1} \lambda^{n-2} \cdots - a_2 \lambda - a_1$$

$$|\lambda I - (A + BK)| = \lambda^n - (a_n + k_n)\lambda^{n-1} - \cdots - (a_2 + k_2)\lambda - (a_1 + k_1)$$

Match coefficients for solution:

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = \lambda^n - (\star)\lambda^{n-1} - \cdots - (\star\star)\lambda - (\star\star\star)$$

Controllability in Canonical Form

Claim: If $\vec{x}(t+1) = \tilde{A}\vec{x}(t) + \tilde{B}u(t)$ is controllable then we can find an invertible T s.t.

$$T\tilde{A}T^{-1} = A$$

$$T\tilde{B} = B$$

Canonical form

$$\vec{z}(t) = T\vec{x}(t) \qquad \vec{z}(t+1) = A\vec{z}(t) + Bu(t)$$

We can design k s.t. A+BK has eigenvalues wherever we want:

$$u(t) = K\underbrace{\vec{z}(t)}_{=T\vec{x}(t)} = \underbrace{KT}_{=\tilde{K}}\vec{x}(t)$$

$$T(\tilde{A} + \tilde{B}\tilde{K})T^{-1} = A + BK$$

- How do we know T exists s.t. : $$T\tilde{A}T^{-1} = A$$ $$T\tilde{B} = B$$

- If the system with $(\tilde{A}, \tilde{B})$ is controllable,

$$\tilde{R}_n = \begin{bmatrix} \tilde{A}^{n-1}\tilde{B} & \tilde{A}^{n-2}\tilde{B} & \cdots & \tilde{A}\tilde{B} & \tilde{B} \end{bmatrix}$$ Invertible!

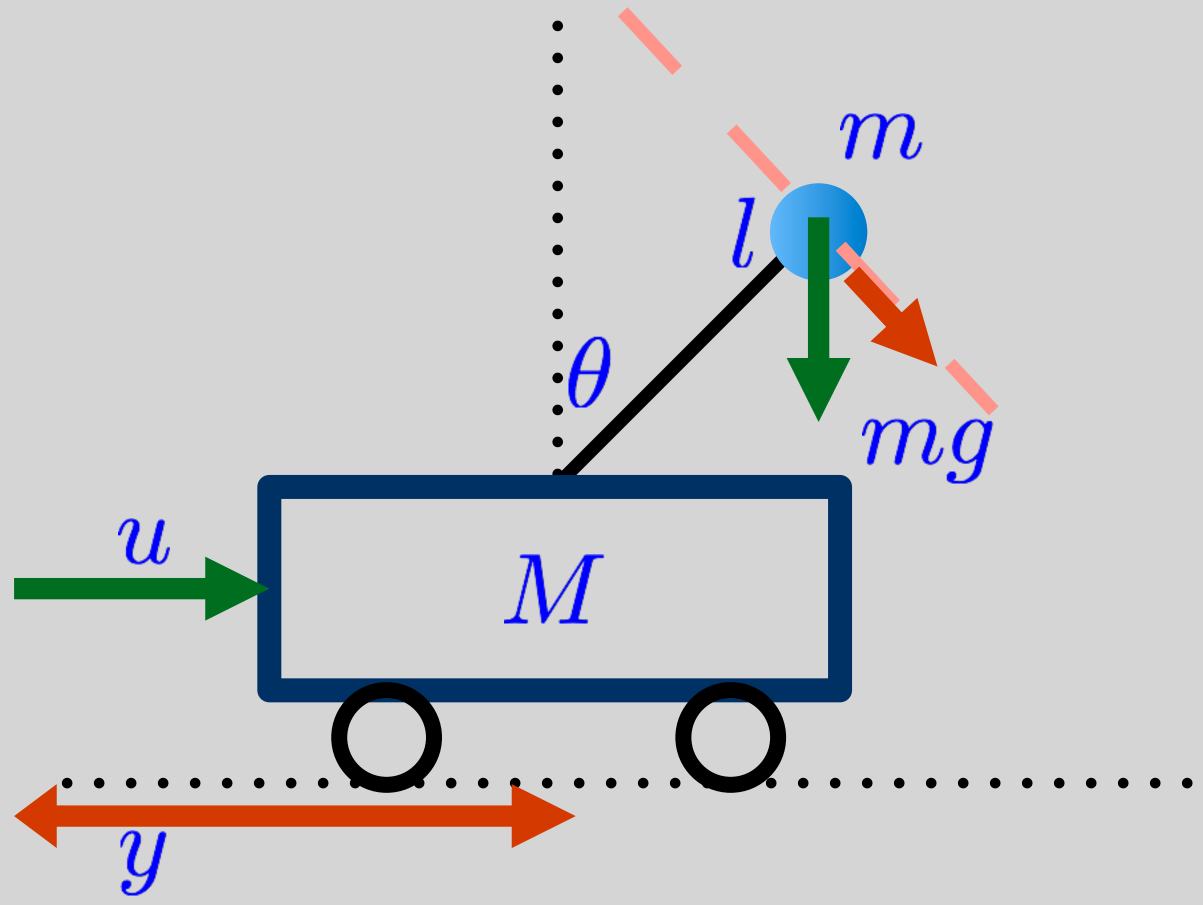
$$R_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix} = T \underbrace{\begin{bmatrix} \tilde{A}^{n-1}\tilde{B} & \tilde{A}^{n-2}\tilde{B} & \cdots & \tilde{A}\tilde{B} & \tilde{B} \end{bmatrix}}_{\tilde{R}_n}$$

$A^{n-1}B \leftrightarrow T\tilde{A}^{n-1}\tilde{B}$, $\quad AB \leftrightarrow \underbrace{T\tilde{A}T^{-1}(T\tilde{B})}_{T\tilde{A}\tilde{B}}$, $\quad B \leftrightarrow T\tilde{B}$

$$R_n = T\tilde{R}_n \qquad \Rightarrow \boxed{T = R_n\tilde{R}_n^{-1}}$$

You don’t have to go to canonical form to solve the problem! It’s just easier.

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 11 - k_1 & -k_2 & -k_3 \\ -1 + k_1 & k_2 & k_3 \end{bmatrix}$$



$$\lambda^3 + (k_2 - k_3)\lambda^2 + (k_1 - 11)\lambda + 10k_3 = 0$$

$$\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0$$

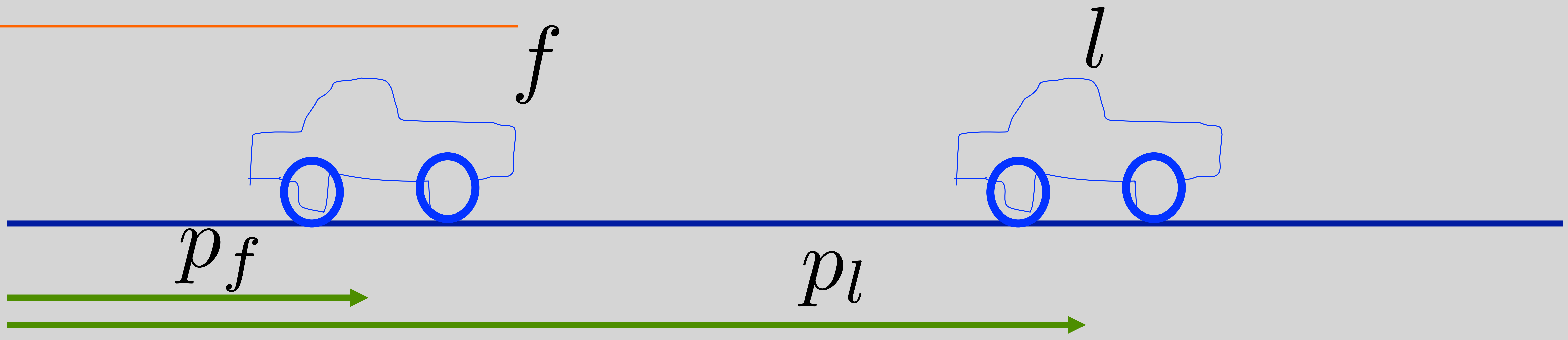
$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -(\lambda_1 + \lambda_2 + \lambda_3) \\ -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + 11 \\ \lambda_1\lambda_2\lambda_3 \end{bmatrix}$$

inv

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0.1 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} -(\lambda_1 + \lambda_2 + \lambda_3) \\ -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + 11 \\ -\lambda_1\lambda_2\lambda_3 \end{bmatrix}$$

Cooperative Adaptive Cruise control

Example:



$$\frac{d}{dt}p_l(t) = v_l(t) \qquad \frac{d}{dt}p_f(t) = v_f(t)$$

$$\frac{d}{dt}v_l(t) = u_l(t) \qquad \frac{d}{dt}v_f(t) = u_f(t)$$

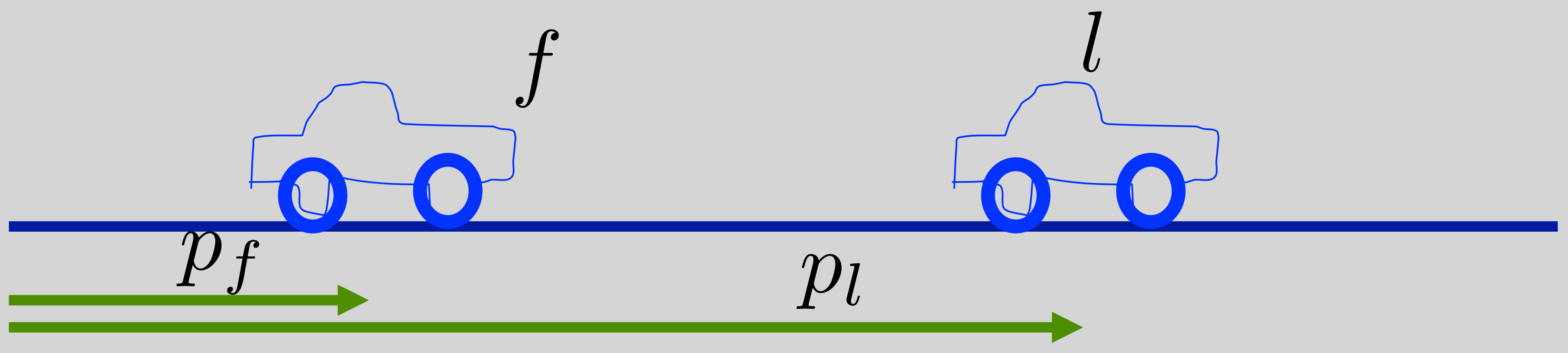
$$x_1(t) = p_l(t) - p_f(t) - \delta$$

$$x_2(t) = v_l(t) - v_f(t)$$

$$\frac{d}{dt}x_1(t) = v_l(t) - v_f(t)$$

$$\frac{d}{dt}x_2(t) = u_l(t) - u_f(t) \triangleq u(t)$$

Cooperative Adaptive Cruise control



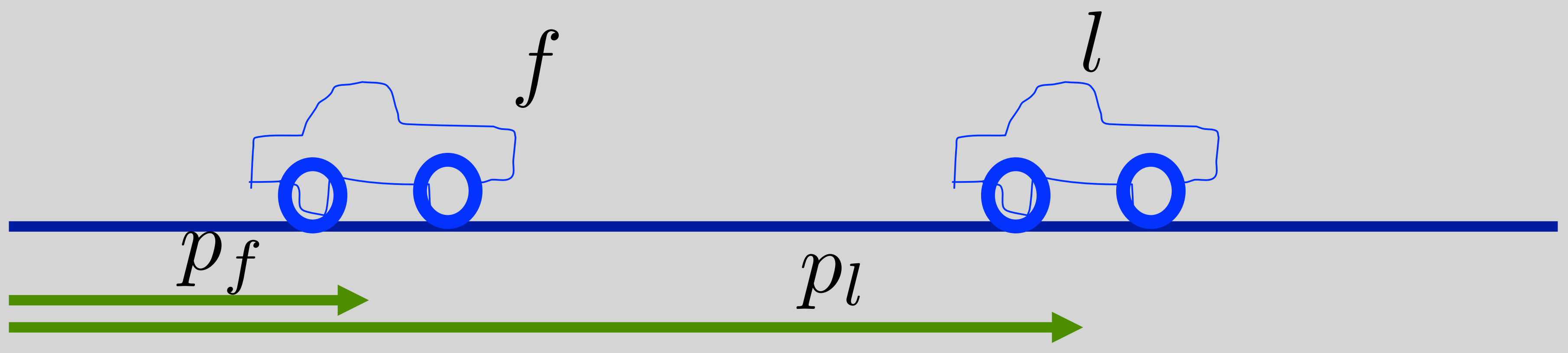
$$\frac{d}{dt}x_1(t) = v_l(t) - v_f(t)$$

$$\frac{d}{dt}x_2(t) = u_l(t) - u_f(t) \triangleq u(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow A + BK = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}$$

Q) What eigen-values will you want here?

Let's look at input more closely…



$$u(t) = k_1 x_1(t) + k_2 x_2(t)$$

$$\Rightarrow u(t) = k_1(p_l(t) - p_f(t) - \delta) + k_2(v_l(t) - v_f(t))$$

But leader chooses his own acceleration $u_l(t)$

$$u(t) = u_l(t) - u_f(t)$$

$$\begin{aligned} u_f(t) &= u_l(t) - u(t) \\ &= u_l(t) - k_1(p_l(t) - p_f(t) - \delta) - k_2(v_l(t) - v_f(t)) \end{aligned}$$

Q) What does the follower need to know to implement?

A) Cooperative (vehicle2vehicle comm.)
range sensor (for distance and velocity)

PLATOONING

Outputs

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

Can't always measure state directly or all states…

Define output:

$$\vec{y}(t) = C\vec{x}(t)$$

p x n matrix for p outputs

$$y = x_1 \quad \Rightarrow C = [1\ 0\ 0\ \cdots\ 0]$$

$$y = x_1 + x_2 \quad \Rightarrow C = [1\ 1\ 0\ \cdots\ 0]$$

$$\vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Observability

A system is "observable" if, by watching
y(0),y(1),y(2),...   we can determine the full state

Two stage approach:

1) Determine initial state *x(0)* from *y(0),y(1),....*

2) $\vec{x}(t) = A^t\vec{x}(0) + Bu(t)$

Ignore input: $u(t) = 0$

$$y(0) = C\vec{x}(0)$$
$$y(1) = C\vec{x}(1) = CA\vec{x}(0)$$
$$\vdots$$
$$y(t) = CA^t\vec{x}(0)$$

$$\vec{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^t \end{bmatrix} \vec{x}(0)$$

Observability

Q: What conditions on O_t, to determine x(0) uniquely?

$$\vec{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^t \end{bmatrix}}_{\triangleq O_t} \vec{x}(0)$$

A: O_t must have n independent rows

strictly O_{n-1} has full rank

null-space is {0}

$$\text{Observability} \Leftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \text{ has rank} = n$$

Observability

With input:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \vec{x}(0) + \begin{bmatrix} Bu(0) \\ Bu(1) \\ \vdots \\ Bu(n-1) \end{bmatrix}$$

$$\begin{bmatrix} y(0) - Bu(0) \\ y(1) - Bu(1) \\ \vdots \\ y(n-1) - Bu(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \vec{x}(0)$$

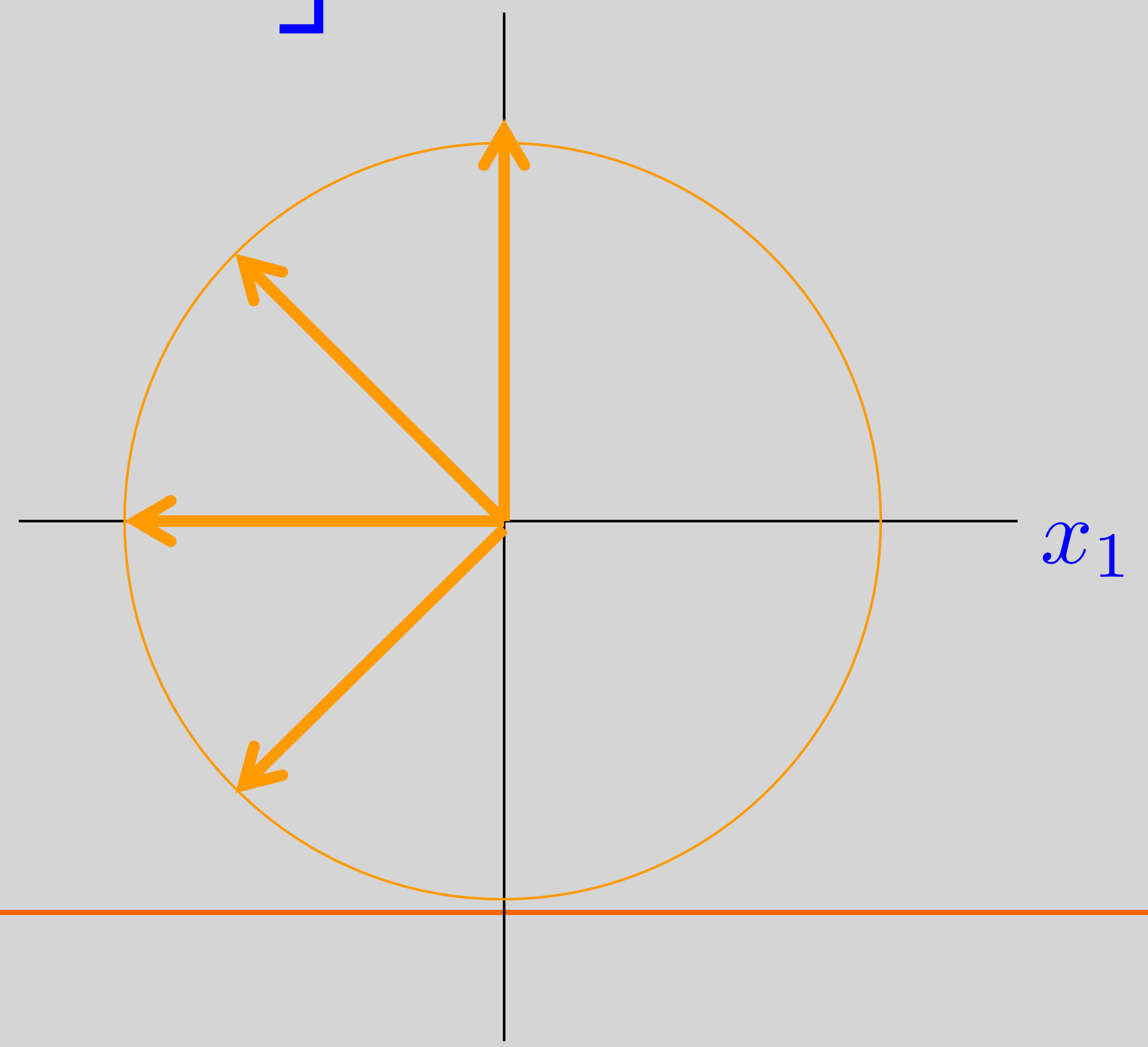
Example

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A}\vec{x}(t)$$

A rotation matrix

$$\underbrace{y(t) = x_1(t)}_{C = [1 \;\; 0]}$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos\theta & -\sin\theta \end{bmatrix} \Rightarrow \text{rank} = 2 \quad \text{if} \quad \theta \neq k\pi$$



Example

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A}\vec{x}(t)$$

A rotation matrix

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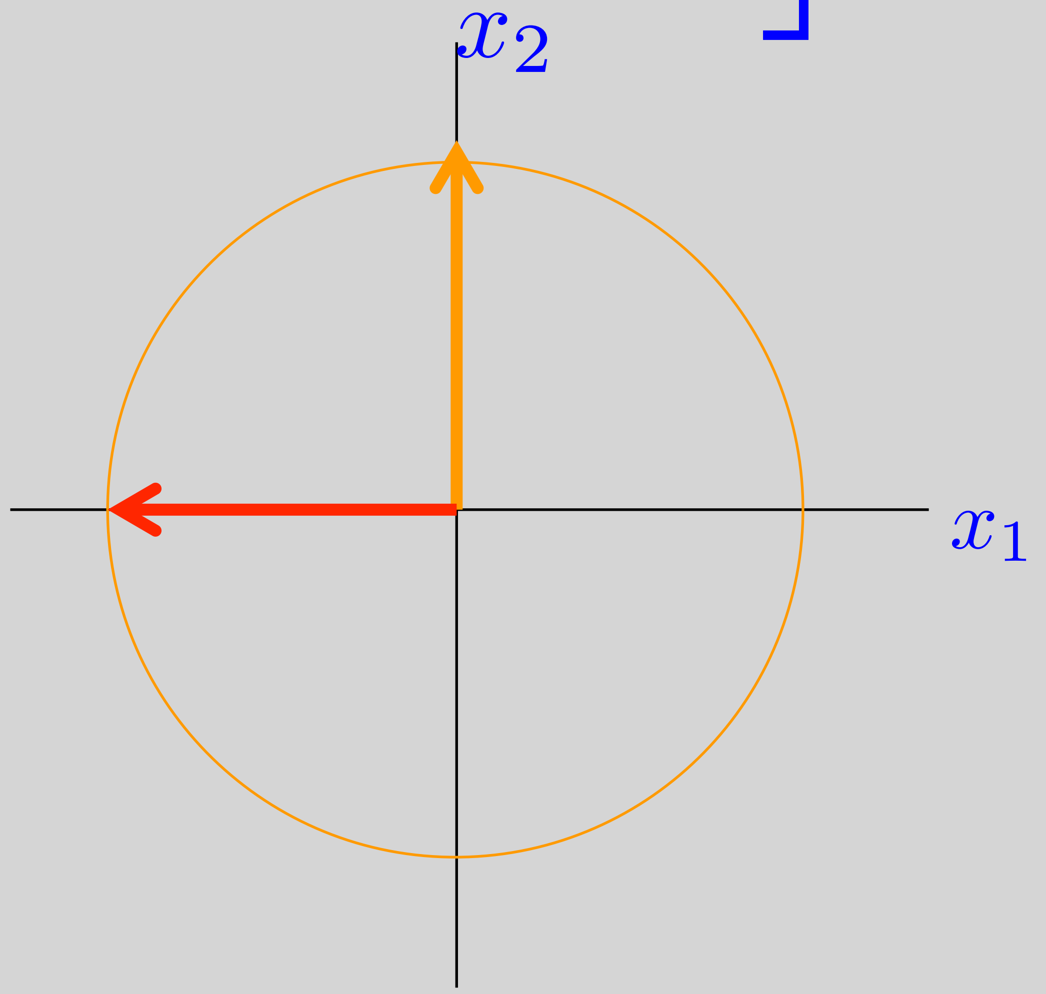
$\theta = \frac{\pi}{2}$

$x_1(0) = 0$

$x_1(1) = -1$

$x_2(1) = 0$

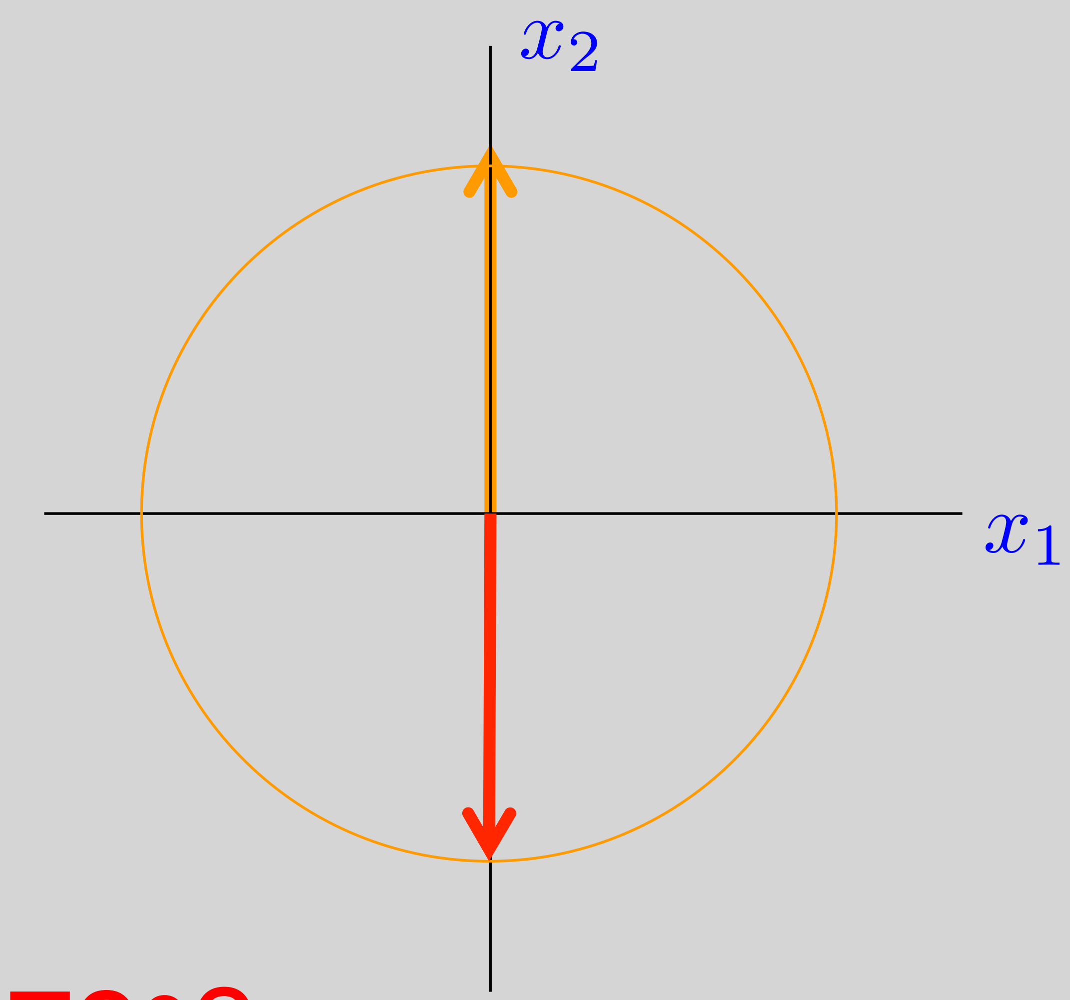


$\theta = \pi$

$x_1(0) = 0$

$x_1(1) = 0$

$x_2(1) = ?$



Q: What is Θ = 179°?

Observer

Would like to implement:

$$u(t) = k_1 x_1(t) + k_2 x_2(t)$$

but C = [1 0], hence $y(t) = x_1(t)$

If the system is observable, then can estimate $x_2(t)$!

Observer: algorithm that estimates the full state vector x(t) from measurements y(t)… more next time!

How Does MRI Work? (some today – more later!)

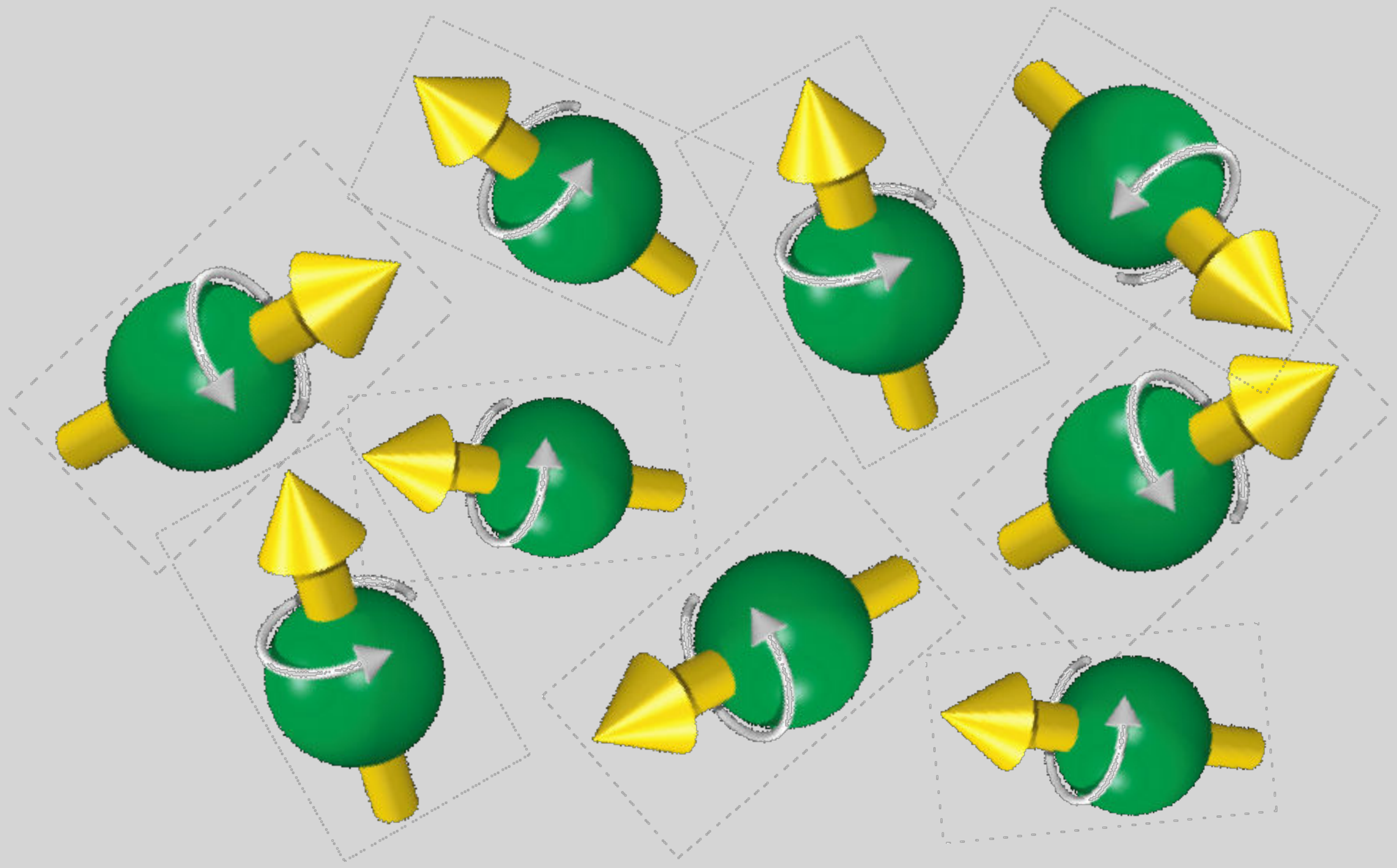
- Magnetic Polarization
  -- Very strong uniform magnet
- Excitation
  -- Very powerful RF transmitter
- Acquisition
  -- Location is encoded by gradient magnetic fields
  -- Very powerful audio amps

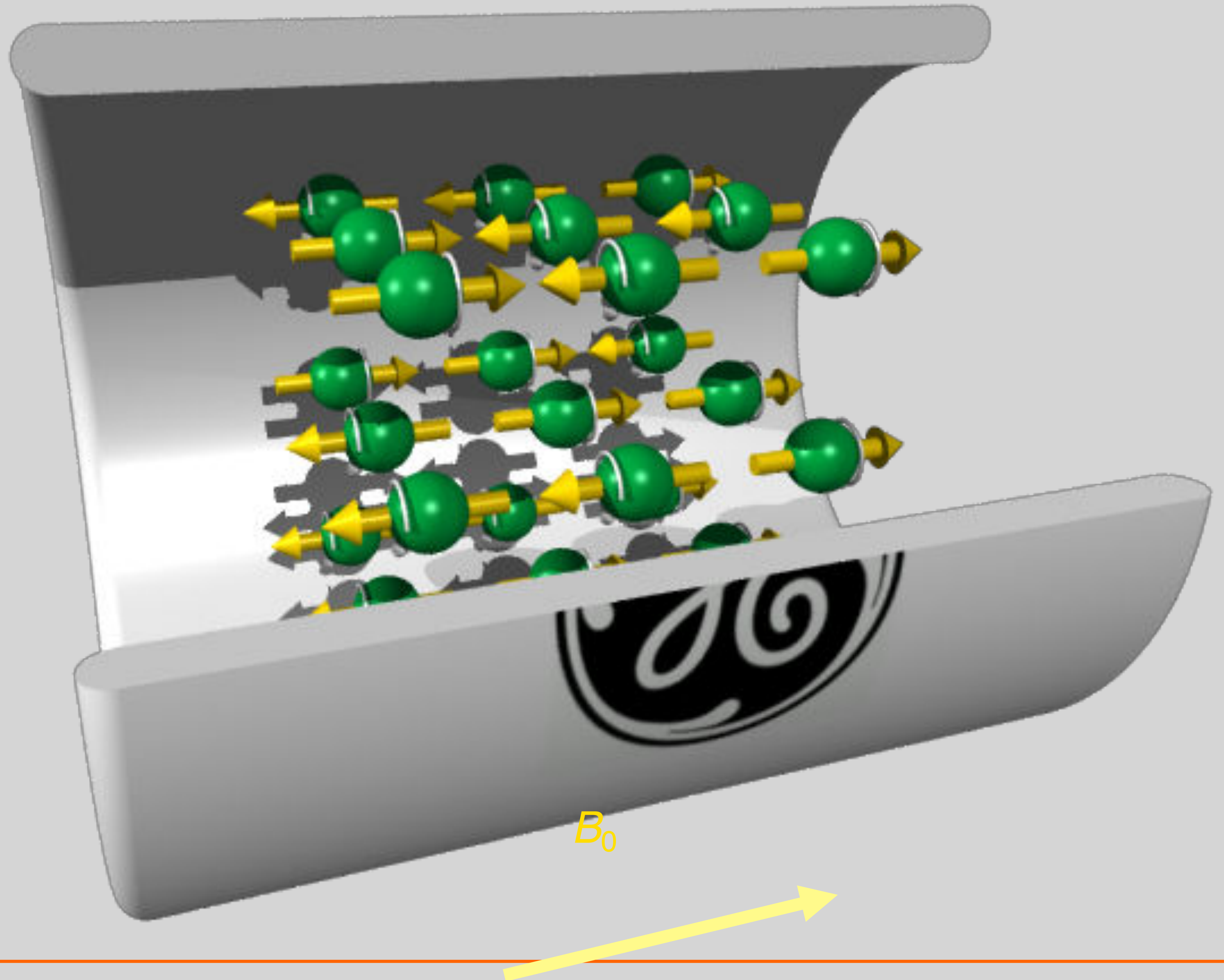
Polarization

- Protons have a magnetic moment
- Protons have spins
- Like rotating magnets

Polarization

- Body has a lot of protons
- In a strong magnetic field B0, spins align with B0 giving a net magnetization

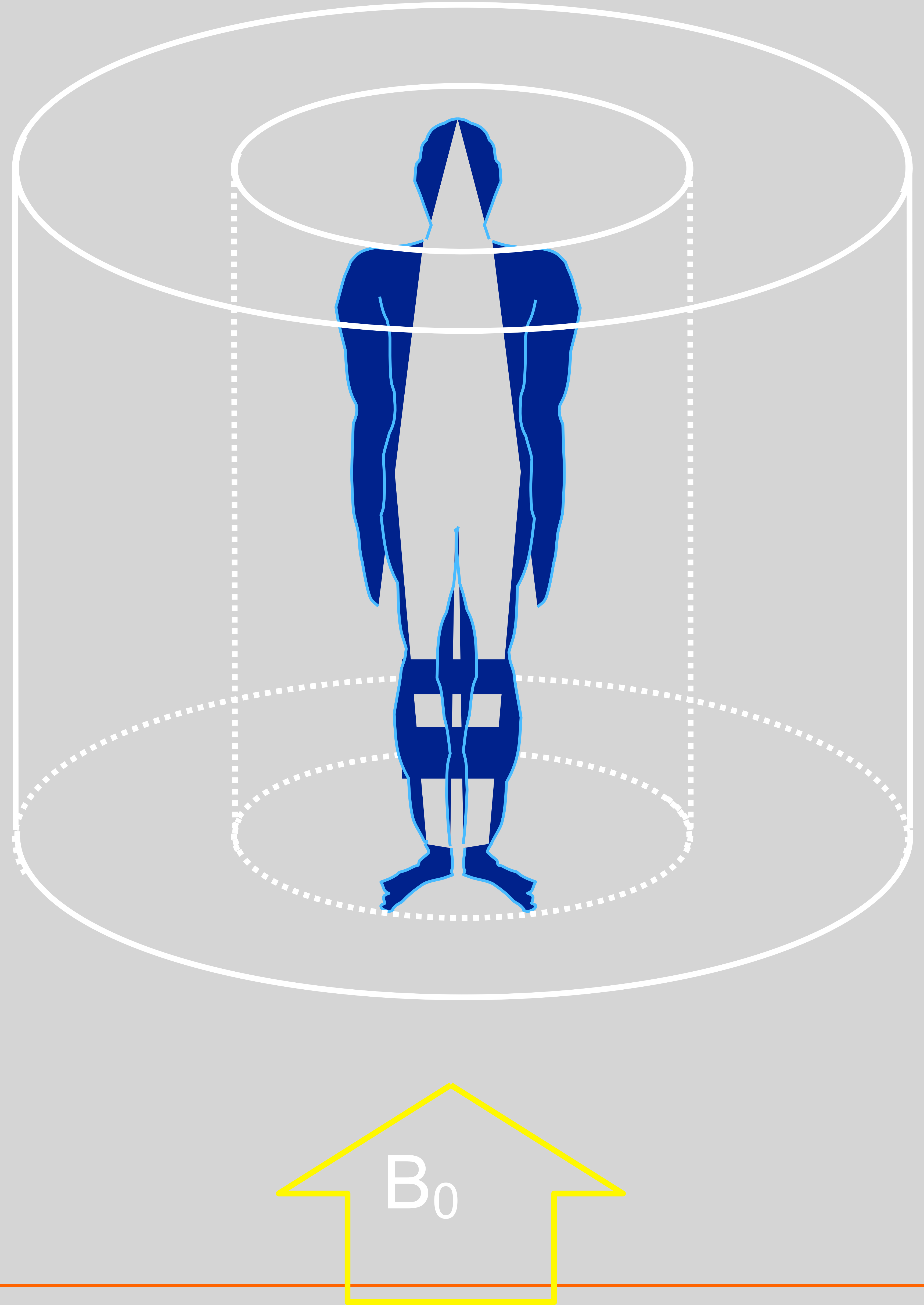




*Graphic rendering Bill Overall

Polarizing Magnet

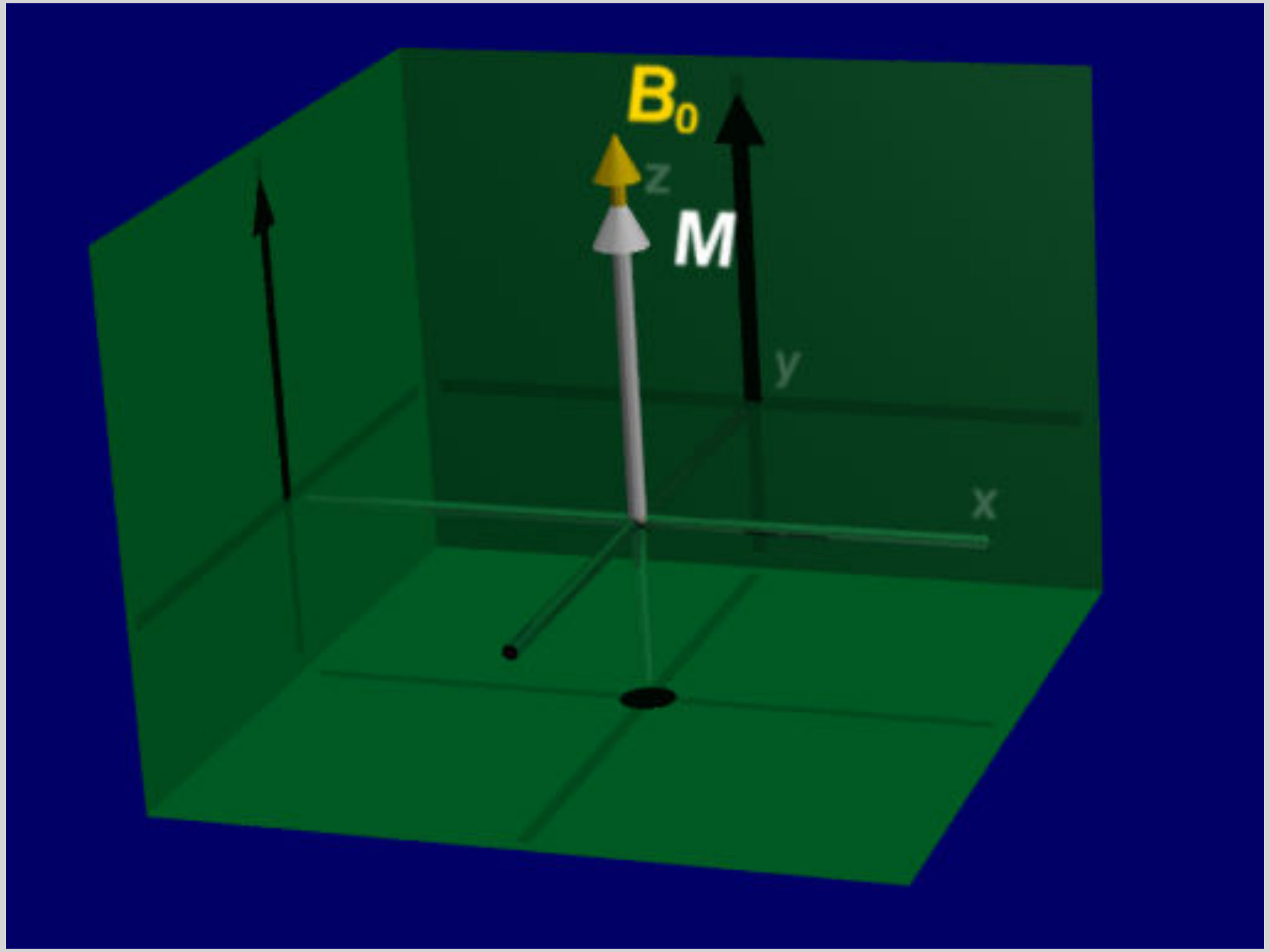
- 0.1 to 12 Tesla
- 0.5 to 3 T common
- 1 T is 10,000 Gauss
- Earth's field is 0.5G
- Typically a superconducting magnet

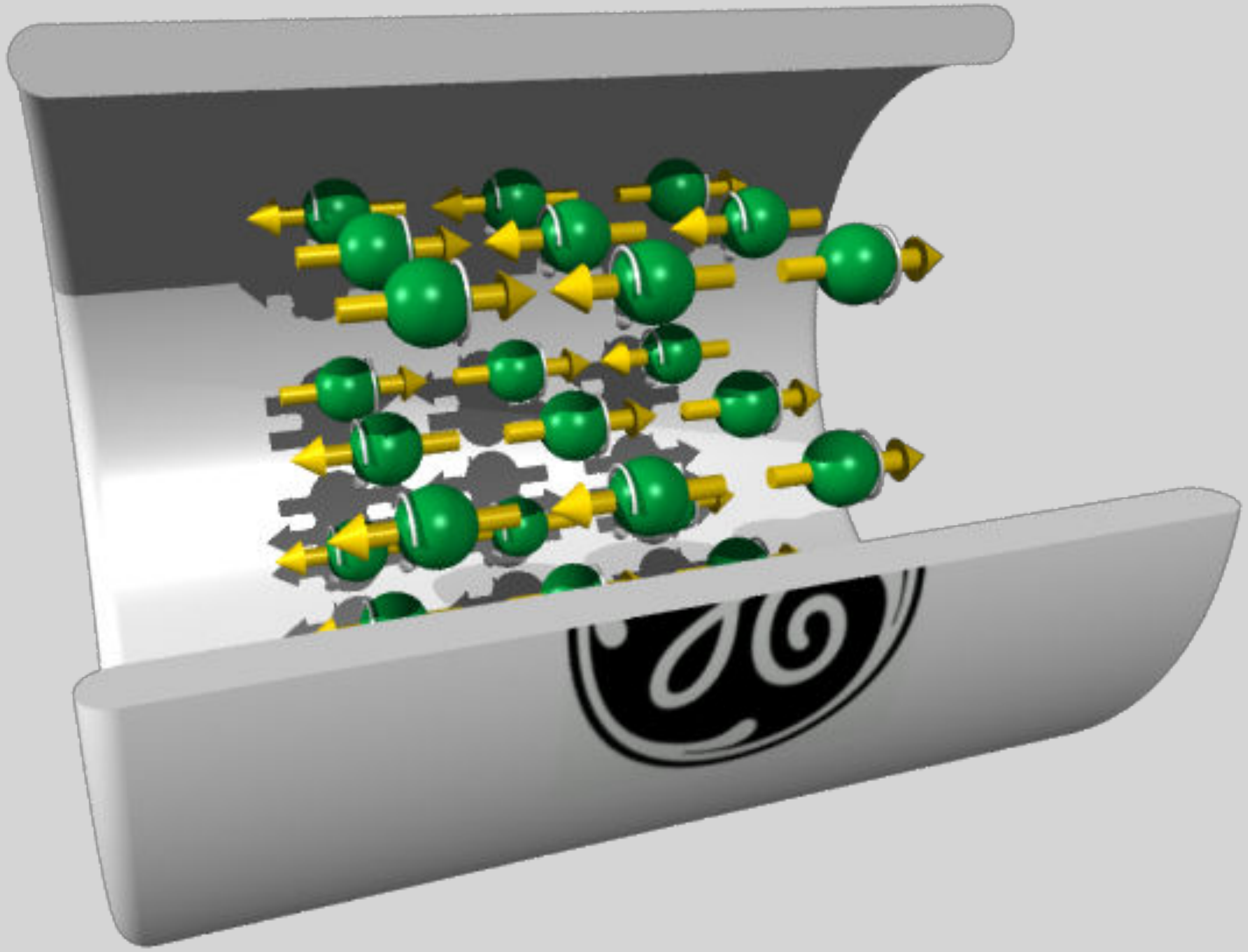


Typical MRI Scanner


A magnetic
object

Polarizaion

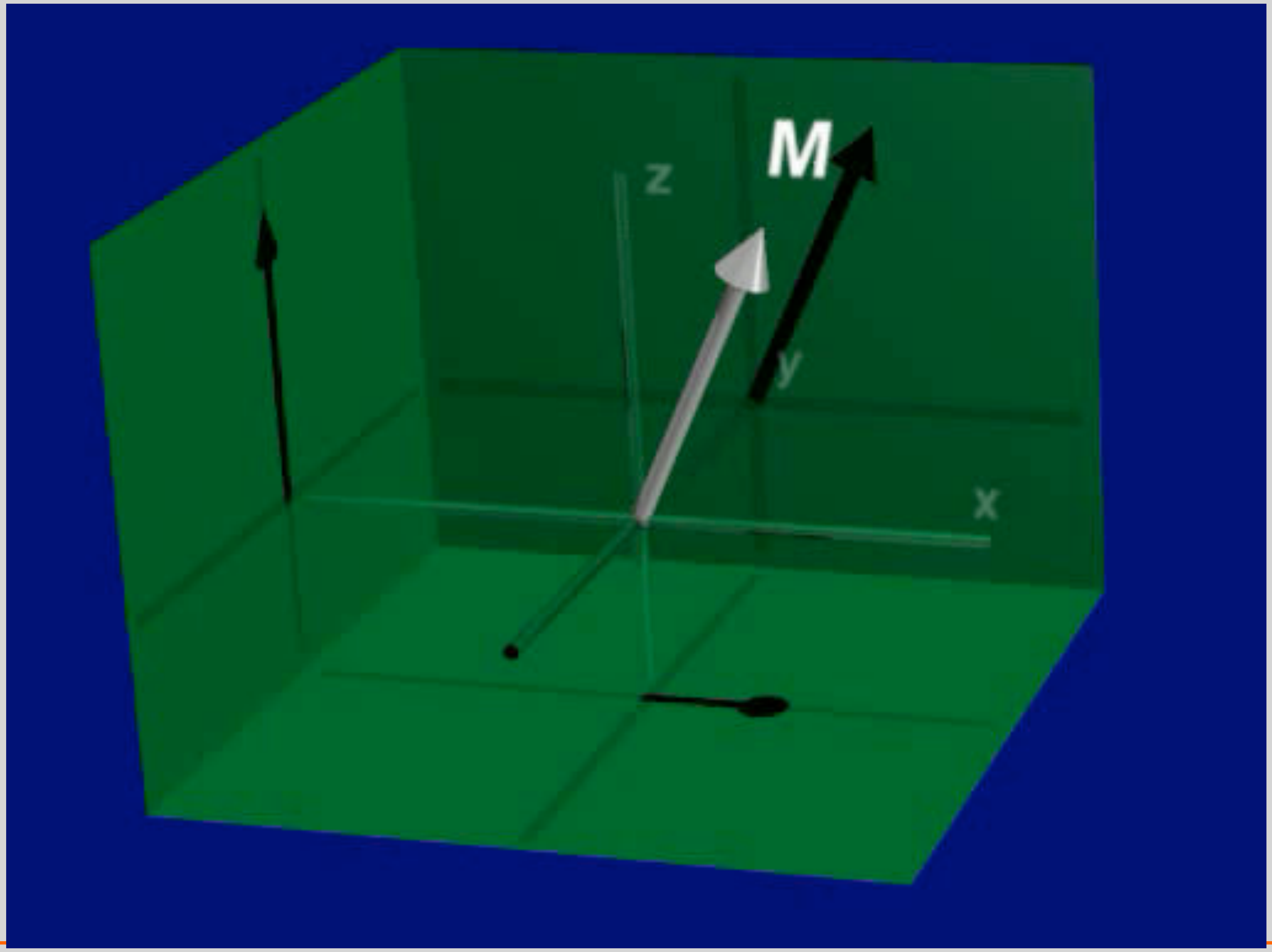




Free Precession

- Much like a spinning top
- Frequency proportional to the field
- f = 127MhZ @ 3T

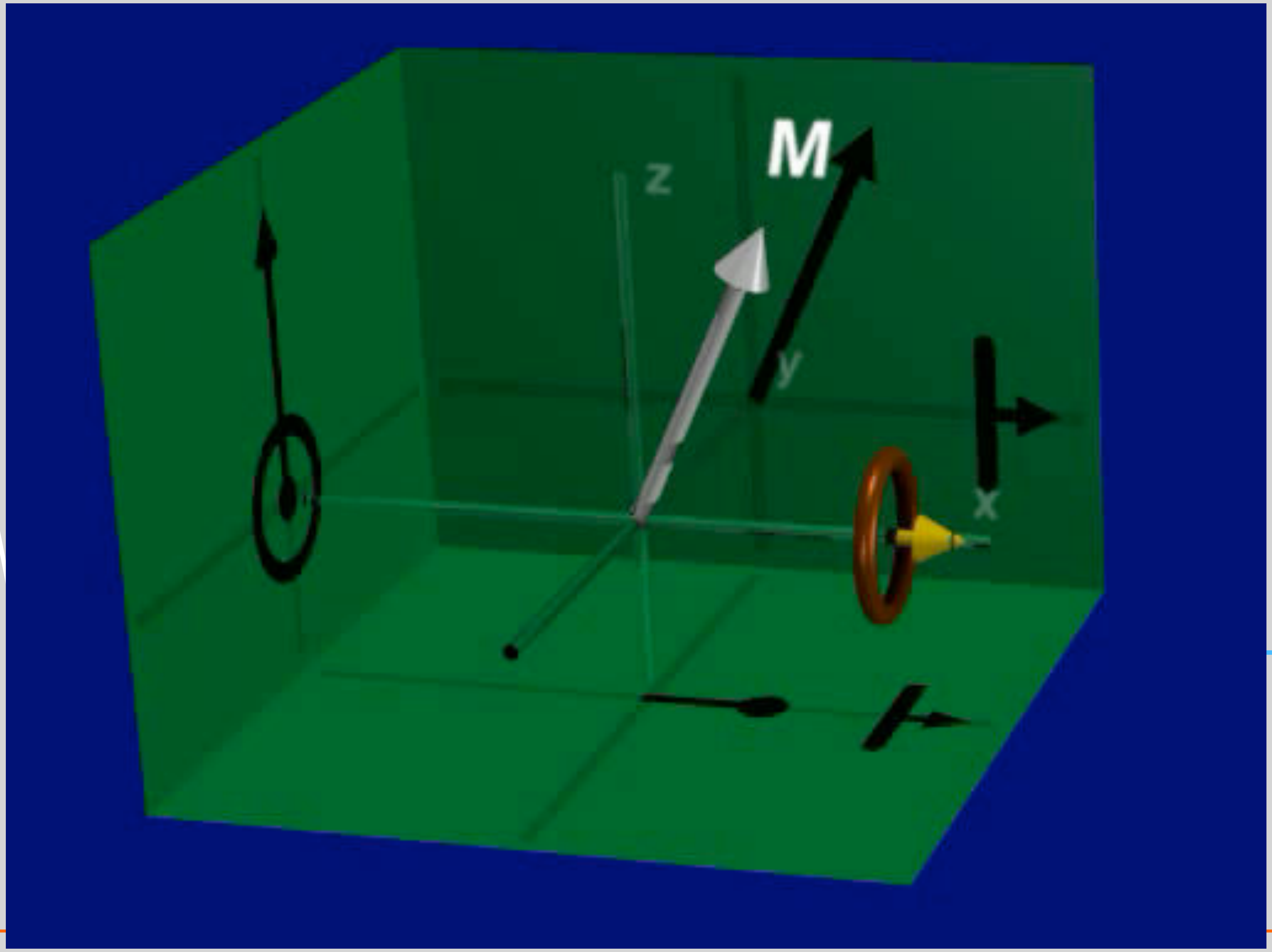
$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$



MIT physics demos

Free Precession

- Precession induces magnetic flux
- Flux induces voltage in a coil

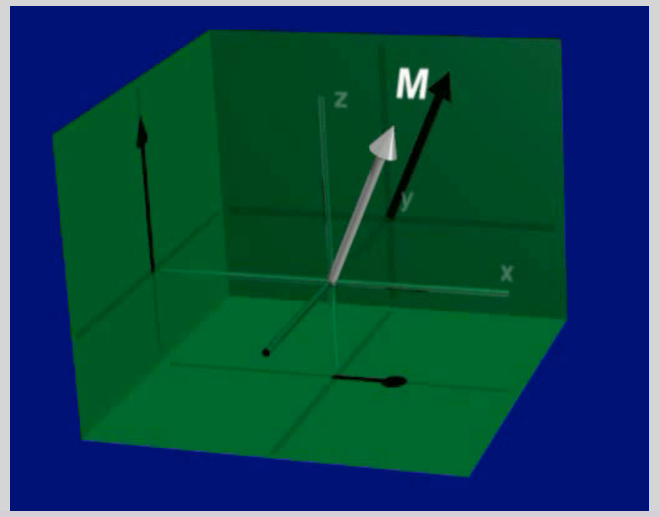


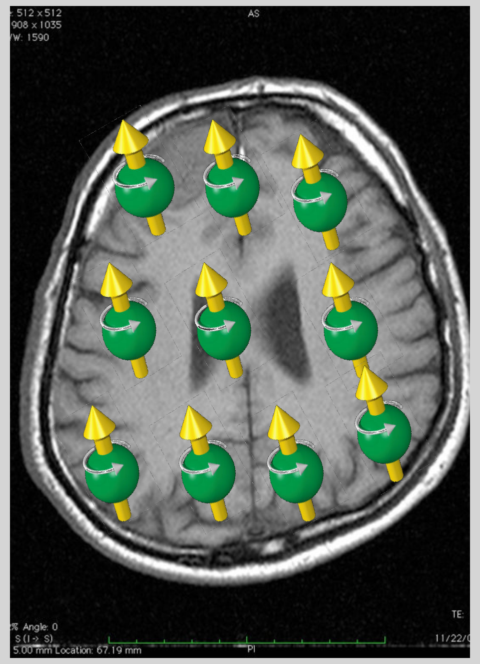
Signal


Audacity
Default Input Source
15.0
16.0
17.0
18.0
19.0
20.0
21.0
22.0
23.0
24.0
25.0
26.0
27.0
28.0
Audio Trac
tereo, 8192Hz
32-bit float
Mute
Solo
0.05
0.00
0.05
0.00
isk space remains for recording 761 hours and 21 minutes
oject rate: 8192
Cursor: 0:00.000000 min:sec [Snap-To Off]

Free precession

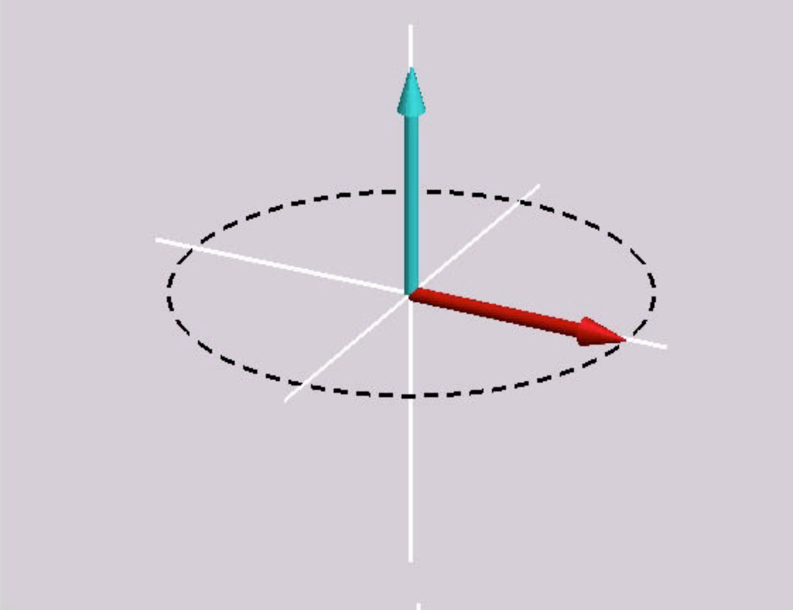
$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} 0 & \gamma B_0 & 0 \\ -\gamma B_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix}$$





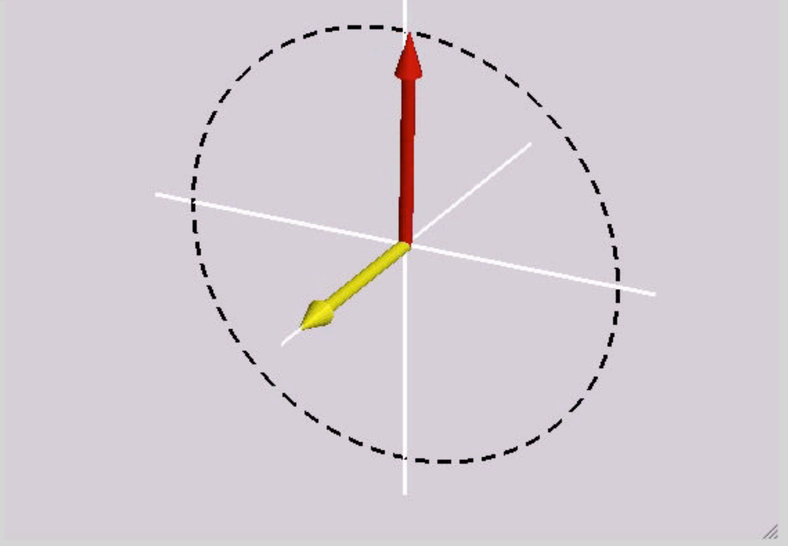
Gradient encoding

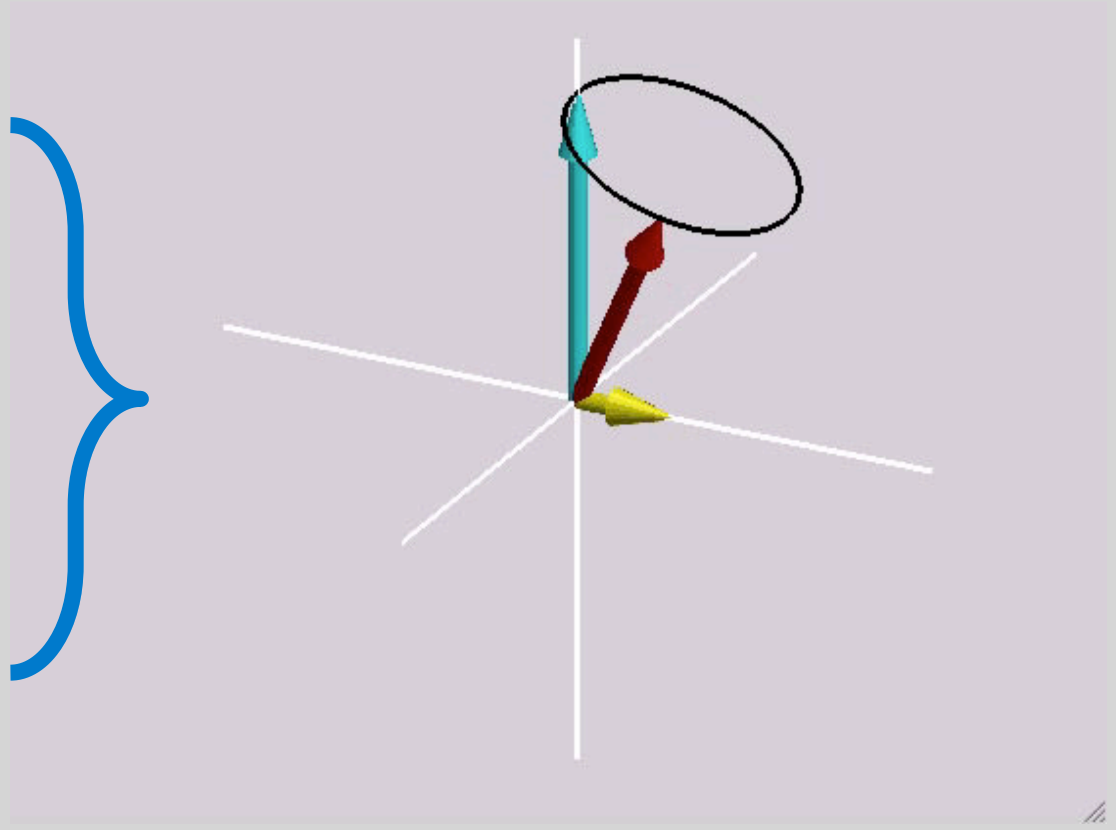
$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} 0 & \gamma \vec{G}(t)\cdot\vec{r} & 0 \\ -\gamma \vec{G}(t)\cdot\vec{r} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix}$$



RF excitation

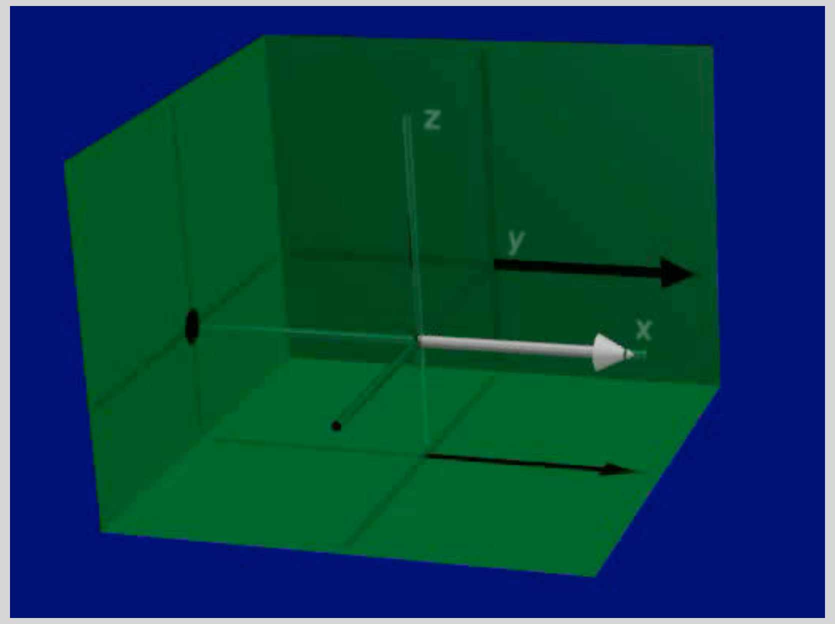
$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma B_{1x}(t) \\ 0 & -\gamma B_{1x}(t) & 0 \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix}$$





Relaxation

$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & 0 & 0 \\ 0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0(\vec{r})$$



The Bloch Equation

$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2}(\vec{r}) & \gamma \vec{G}(t)\cdot\vec{r} & -\gamma B_{1y}(t) \\ -\gamma \vec{G}(t)\cdot\vec{r} & -\frac{1}{T_2}(\vec{r}) & \gamma B_{1x}(t) \\ \gamma B_{1y}(t) & -\gamma B_{1x}(t) & -\frac{1}{T_1}(\vec{r}) \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0(\vec{r})$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Observe: $\vec{y}(t) = \int_{\vec{R}} C M(\vec{r},t) d\vec{r}$

Inputs: $G_x(t), G_y(t), G_z(t), B_{1x}(t), B_{1y}(t)$

Unknown: $M_0(\vec{r})$ The image!


MRI is all about contrast......
http://thundafunda.com

The Toilette Analogy (©2009 Al Macovski)

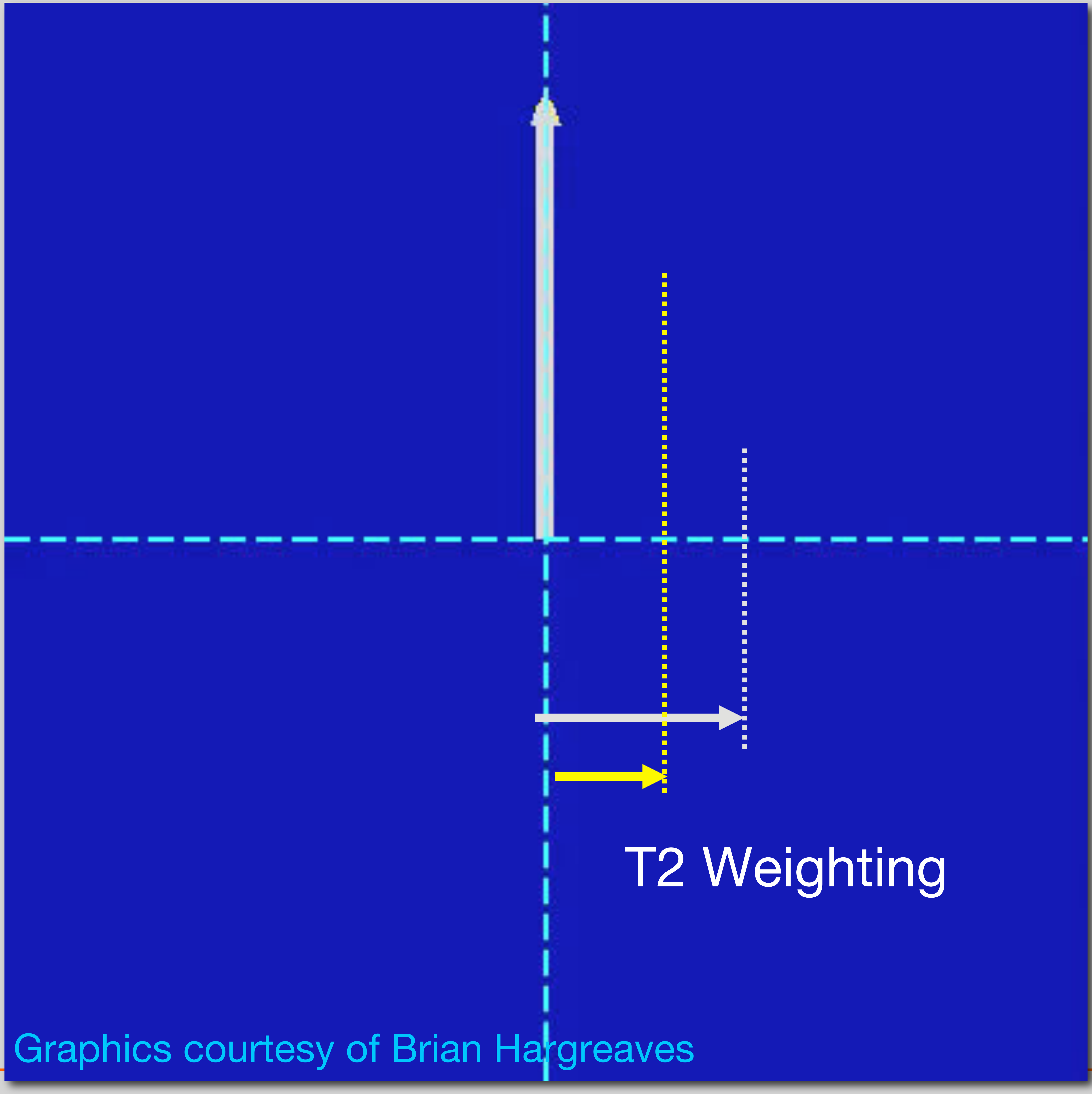
- Excitation = Flush

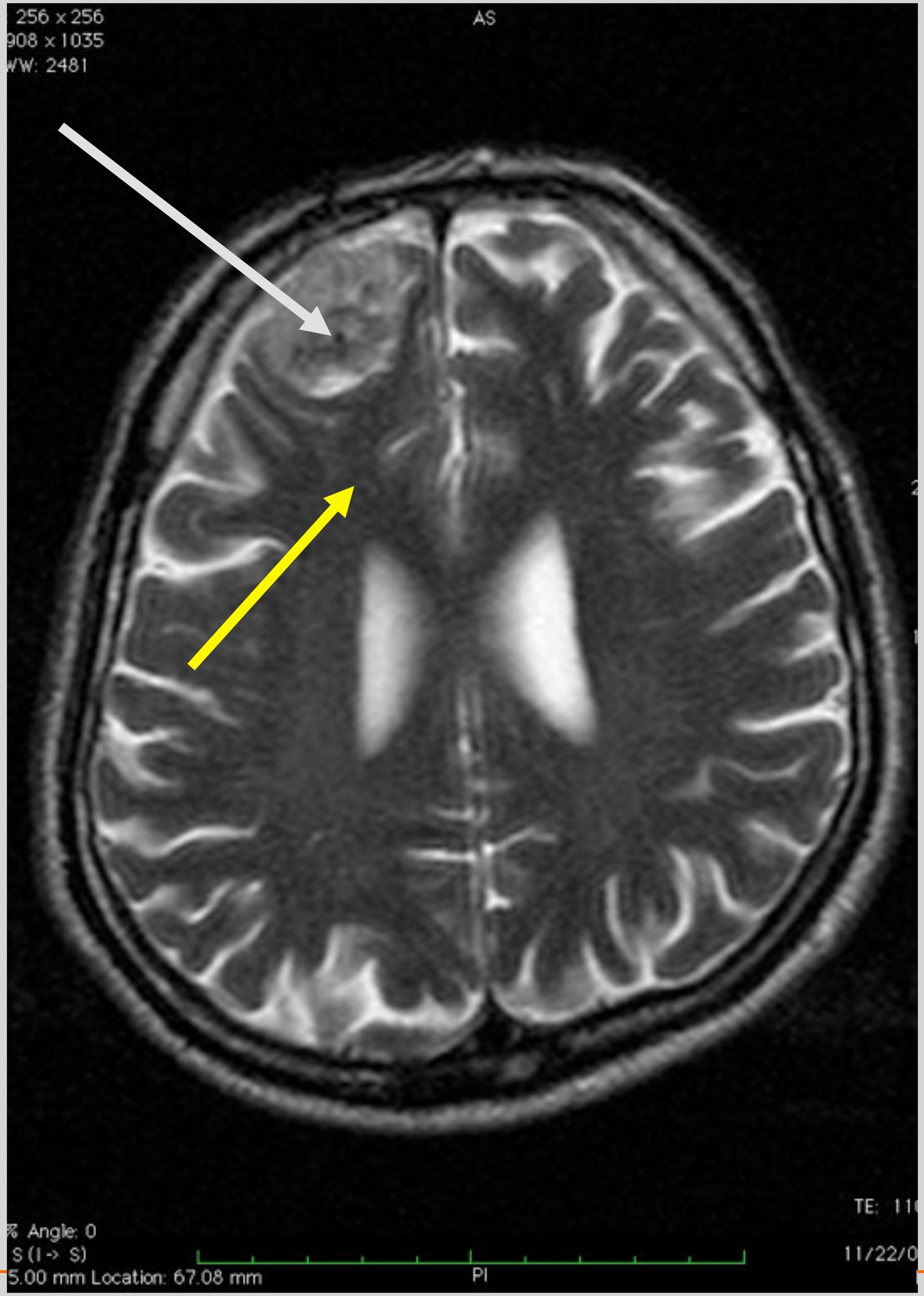
- Dynamics:
  - Water drains = signal decays, equivalent to T2
  - Tank refills = Magn. recovers, equivalent to T1

- Observed signal = water in the bowl
- Different toilettes = different tissues

T2-Weighting

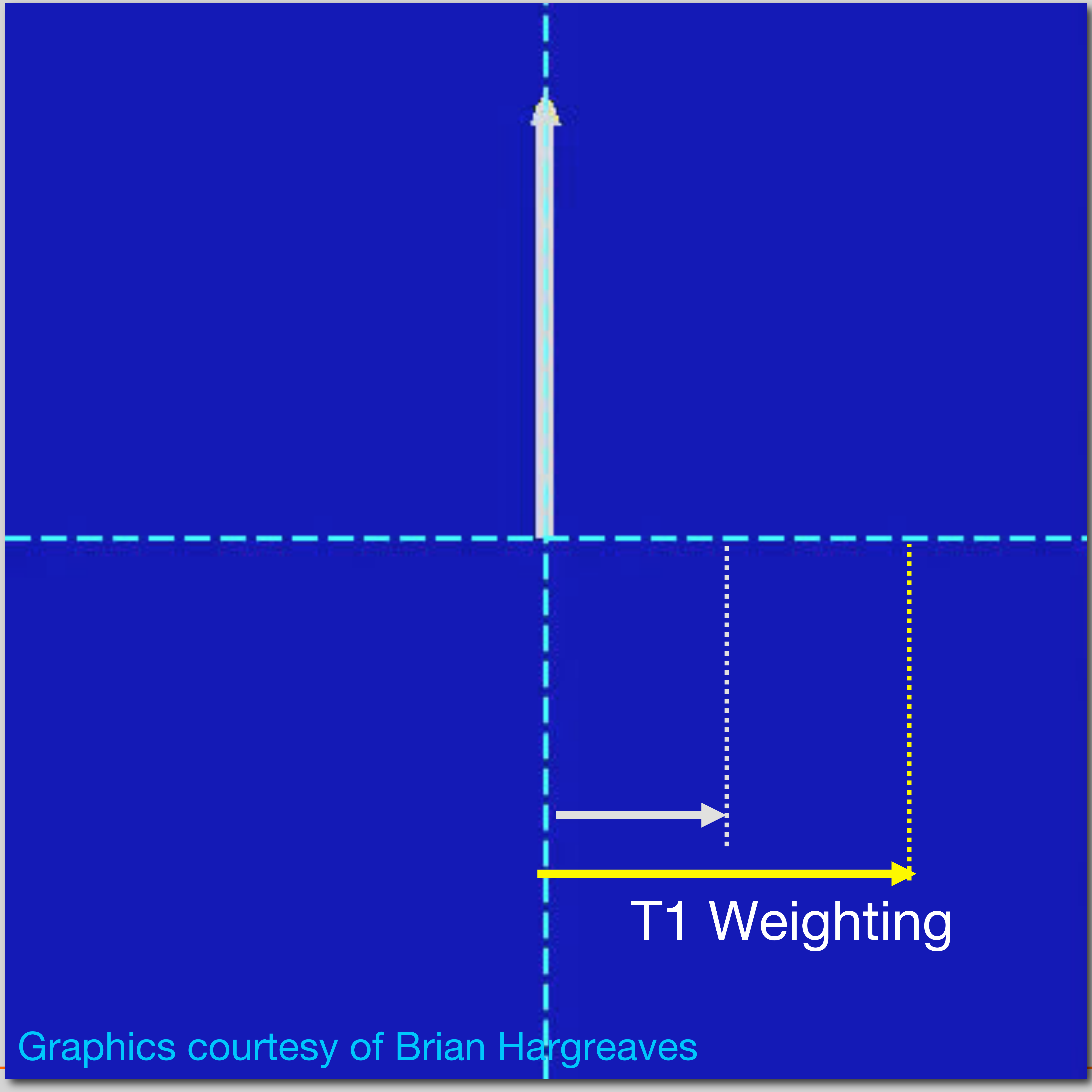
$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & 0 & 0 \\ 0 & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0(\vec{r})$$

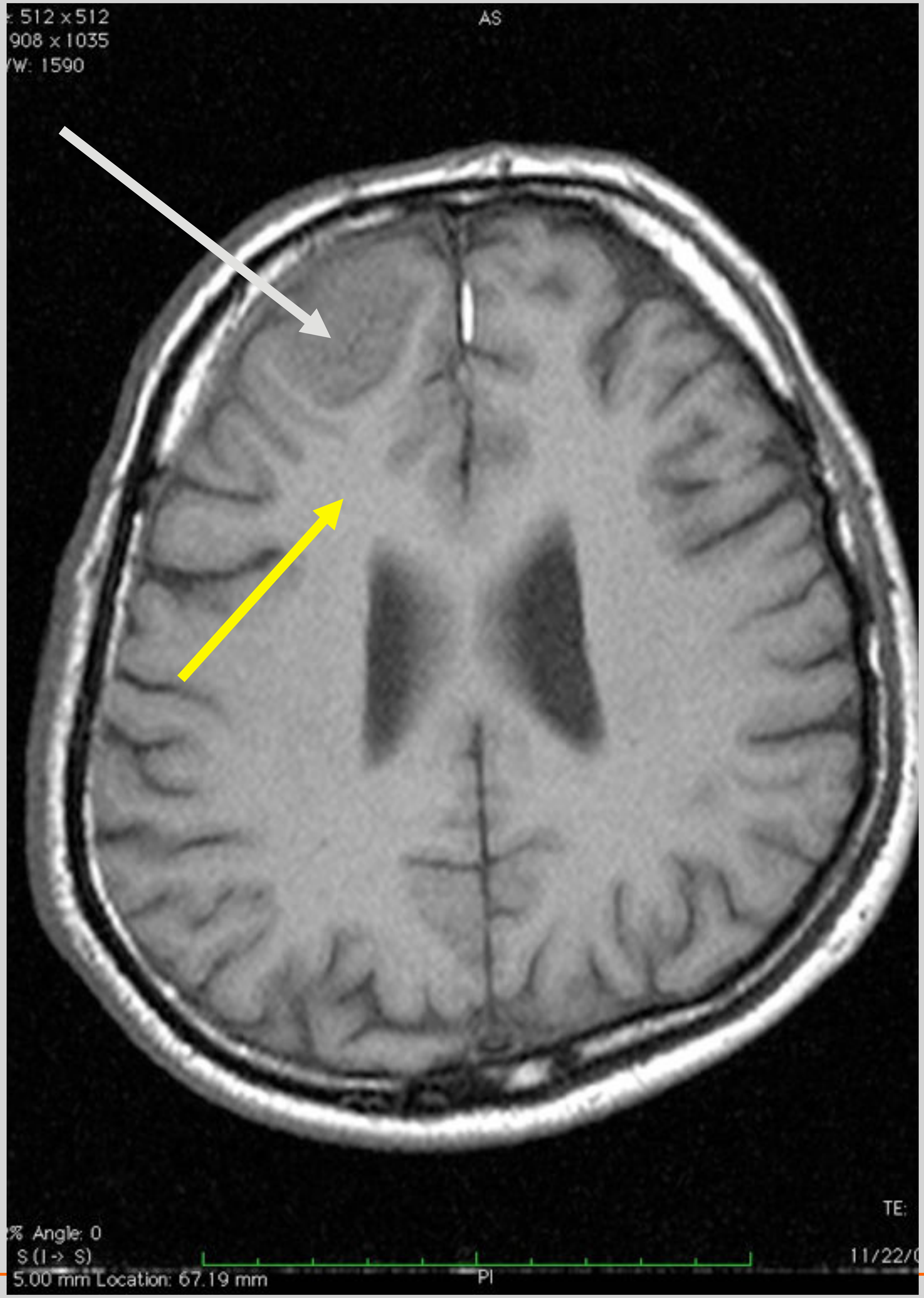




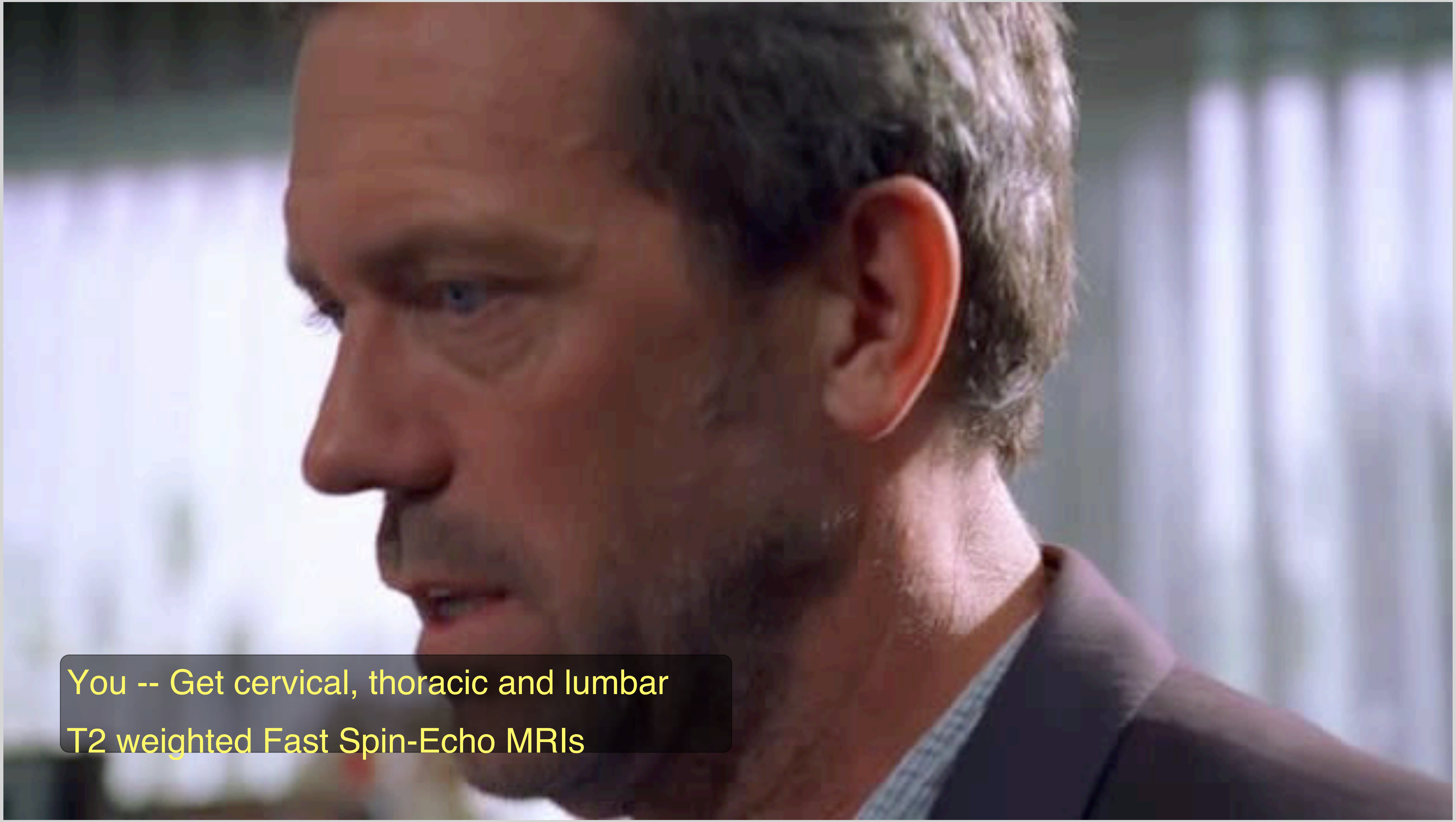
T1 Weighting

$$\begin{bmatrix}\dot{M}_x(\vec{r},t)\\ \dot{M}_y(\vec{r},t)\\ \dot{M}_z(\vec{r},t)\end{bmatrix} = \begin{bmatrix}-\frac{1}{T_2} & 0 & 0\\ 0 & -\frac{1}{T_2} & 0\\ 0 & 0 & -\frac{1}{T_1}\end{bmatrix}\begin{bmatrix}M_x(\vec{r},t)\\ M_y(\vec{r},t)\\ M_z(\vec{r},t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{T_1}\end{bmatrix} M_0(\vec{r})$$



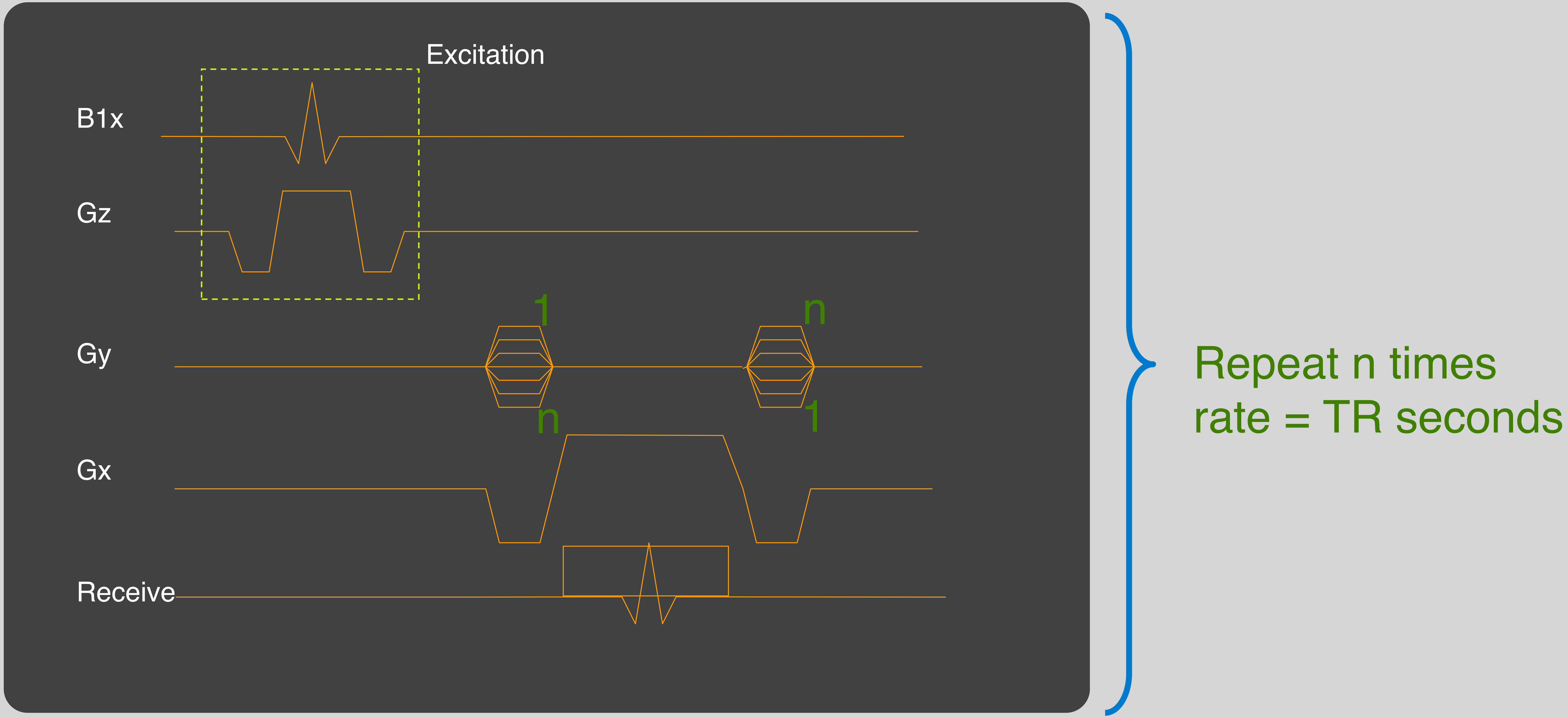


House Prefers T2 .....



MRI Pulse Sequence

$$\begin{bmatrix} \dot{M}_x(\vec{r},t) \\ \dot{M}_y(\vec{r},t) \\ \dot{M}_z(\vec{r},t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2(\vec{r})} & \gamma \vec{G}(t)\cdot\vec{r} & -\gamma B_{1y}(t) \\ -\gamma \vec{G}(t)\cdot\vec{r} & -\frac{1}{T_2(\vec{r})} & \gamma B_{1x}(t) \\ \gamma B_{1y}(t) & -\gamma B_{1x}(t) & -\frac{1}{T_1(\vec{r})} \end{bmatrix} \begin{bmatrix} M_x(\vec{r},t) \\ M_y(\vec{r},t) \\ M_z(\vec{r},t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix} M_0(\vec{r})$$

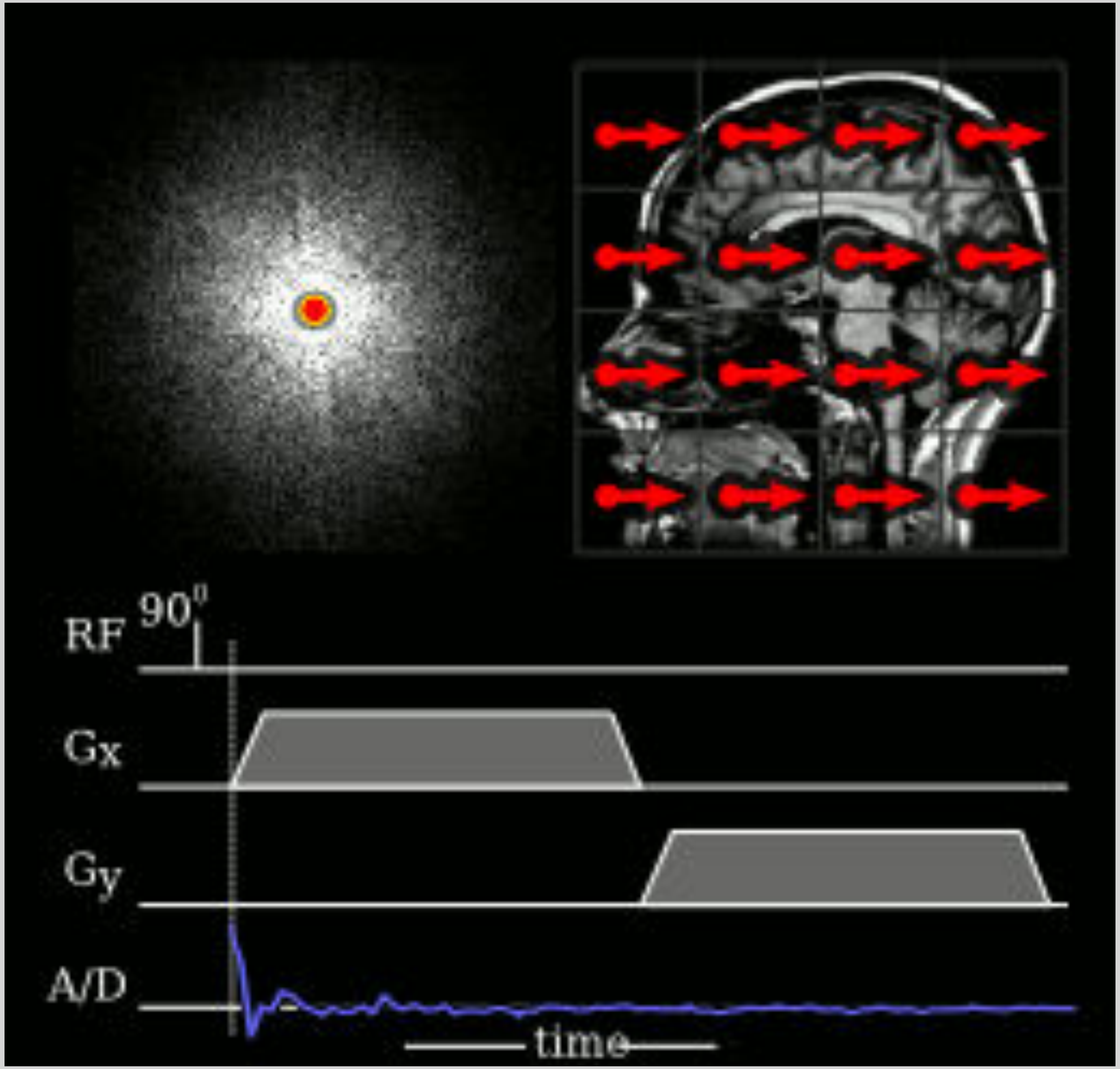


Acquisition

Observe:

$$\vec{y}(t) = \int_{\vec{R}} CM(\vec{r},t)d\vec{r}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Fourier Reconstruction

$$y(t) = \int_R M_x(r,t) + jM_y(r,t)dr$$

Data (showing just magnitude)