EE16B
Designing Information Devices and Systems II

Lecture 8A
Observability and Observers

Outputs

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

Can't always measure state directly or all states…

Define output:

$$\vec{y}(t) = C\vec{x}(t)$$

p x n matrix for p outputs

Observability

A system is "observable" if, by watching

y(0),y(1),y(2),… we can determine the full state

Two stage approach:

1) Determine initial state *x(0)* from *y(0),y(1),….*

2) $\vec{x}(t) = A^t\vec{x}(0) + Bu(t)$

$$\vec{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^t \end{bmatrix}}_{\triangleq O_t} \vec{x}(0)$$

Observability

Q: What conditions on O_t, to determine x(0) uniquely?

$$\vec{y} = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^t \end{bmatrix}}_{\triangleq O_t} \vec{x}(0)$$

A: O_t must have n independent rows

strictly O_{n-1} has full rank

null-space is {0}

Observability $\Leftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has rank = n

Example

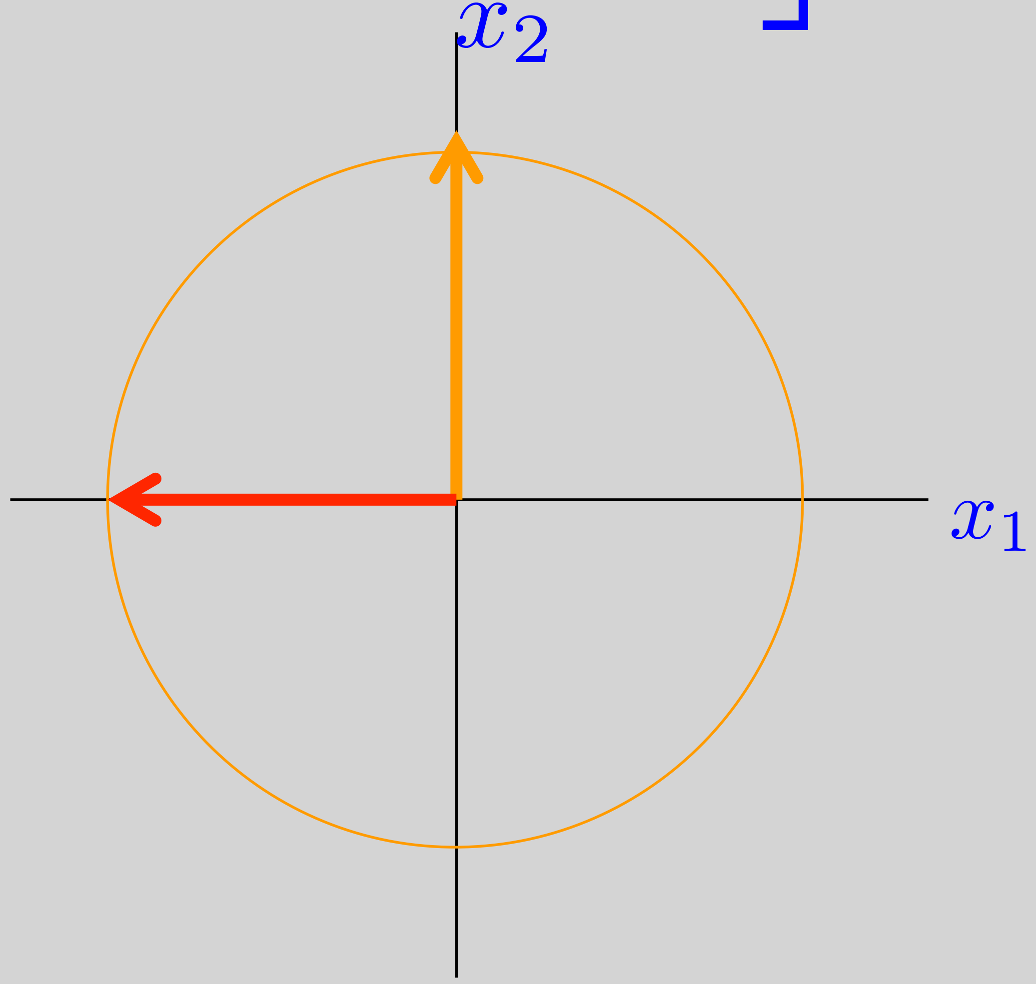
$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{A} \vec{x}(t)$$

A rotation matrix

$$\underbrace{y(t) = x_1(t)}_{C = [1 \;\; 0]}$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cos\theta & -\sin\theta \end{bmatrix} \Rightarrow \text{rank} = 2 \quad \text{if} \quad \theta \neq k\pi$$

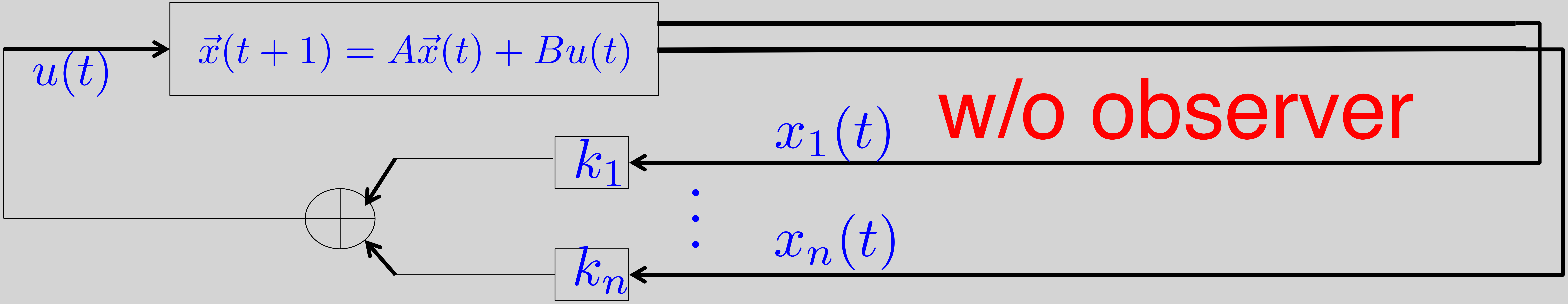
$$\theta = \frac{\pi}{2}$$



$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}$$

State Feedback Control

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) \qquad \vec{y}(t) = C\vec{x}(t)$$



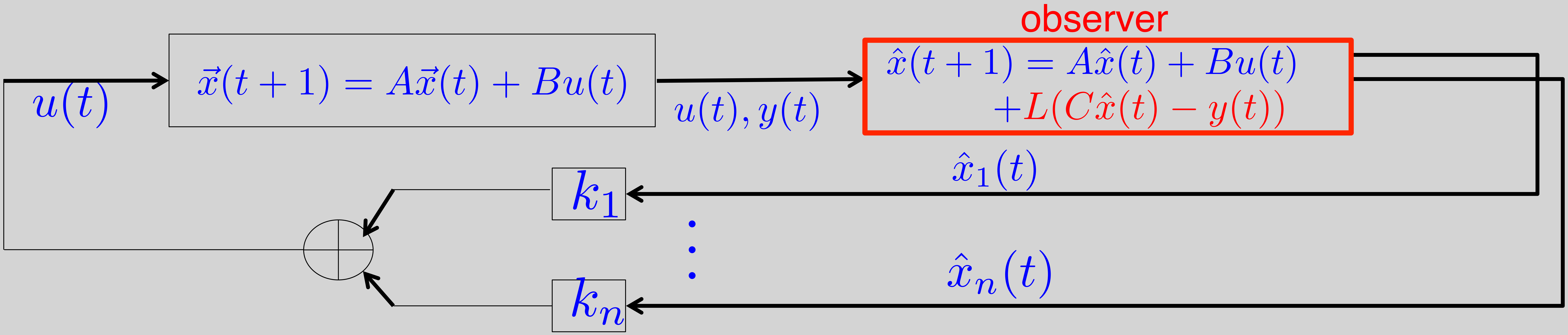
A Common Observer Algorithm

Start with initial guess $\hat{x}(0)$

Update estimate each time using:

n x p

$$\hat{x}(t+1) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{Copy of system model}} + \underbrace{L(C\hat{x}(t) - y(t))}_{\text{correction}}$$



Choosing L for Observer

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L(C\hat{x}(t) - \underbrace{y(t)}_{C\vec{x}(t)})$$

$$\vec{e}(t) \triangleq \hat{x}(t) - \vec{x}(t)$$

$$\vec{e}(t+1) = \hat{x}(t+1) - \vec{x}(t+1)$$

$$= A\underbrace{(\hat{x}(t) - \vec{x}(t))}_{\vec{e}(t)} - LC\underbrace{(\hat{x}(t) - x(t))}_{\vec{e}(t)}$$

$$\vec{e}(t+1) = (A + LC)\vec{e}(t)$$

$\vec{e}(t) \rightarrow 0$ If (A + LC) has eigenvalues inside unit circle

Choosing L for Observer

Claim: if (A,C) observable, then we can arbitrarily assign eigenvalues of A+LC

$$A + LC \xrightarrow{\text{transpose}} A^T + C^T L^T$$

$$A^T \triangleq \tilde{A}, \quad C^T \triangleq \tilde{B}, \quad L^T \triangleq \tilde{K}$$

$$\tilde{A} + \tilde{B}\tilde{K}$$

$$L = \tilde{K}^T \longleftarrow$$

If $(\tilde{A}, \tilde{B})$ controllable, we can design $\tilde{K}$ to assign eigenvalues of $\tilde{A} + \tilde{B}\tilde{K}$ (same eigenvalues as A+LC)

Given (A,C) observable, can we claim $\tilde{A} = A^T, \tilde{B} = C^T$ Controllable?

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ has rank = n

$$\begin{matrix} [C^T & A^TC^T & \cdots & (A^T)^{n-1}C^T] \\ \| & \| & & \| \\ [\tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B}] \end{matrix}$$ has rank = n

satisfies controllability!

Back to Example

$$\vec{x}(t+1) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{x}(t) \qquad y(t) = [1 \;\; 0]\vec{x}(t)$$

$$\hat{x}(t+1) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \hat{x}(t) + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \left([1 \;\; 0]\hat{x}(t) - y(t)\right)$$

eigenvalues of A+LC must be inside unit circle

$$\theta = \frac{\pi}{2} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1 \;\; 0] = \begin{bmatrix} l_1 & -1 \\ 1 + l_2 & 0 \end{bmatrix}$$

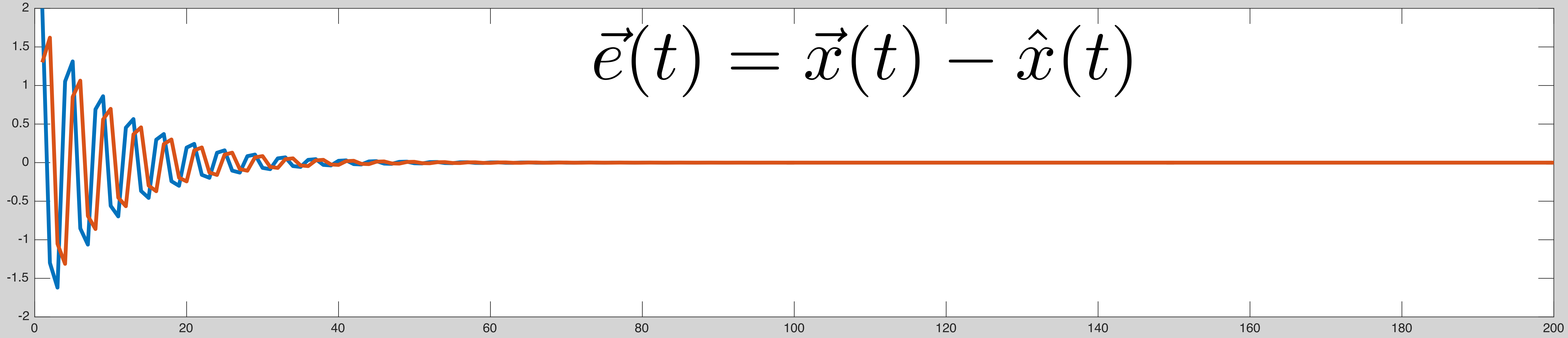
$$\lambda^2 - l_1\lambda + (l_2 + 1) = 0 \qquad \lambda_{1,2} = \pm 0.9i \qquad \Rightarrow \lambda^2 + 0.81 = 0$$

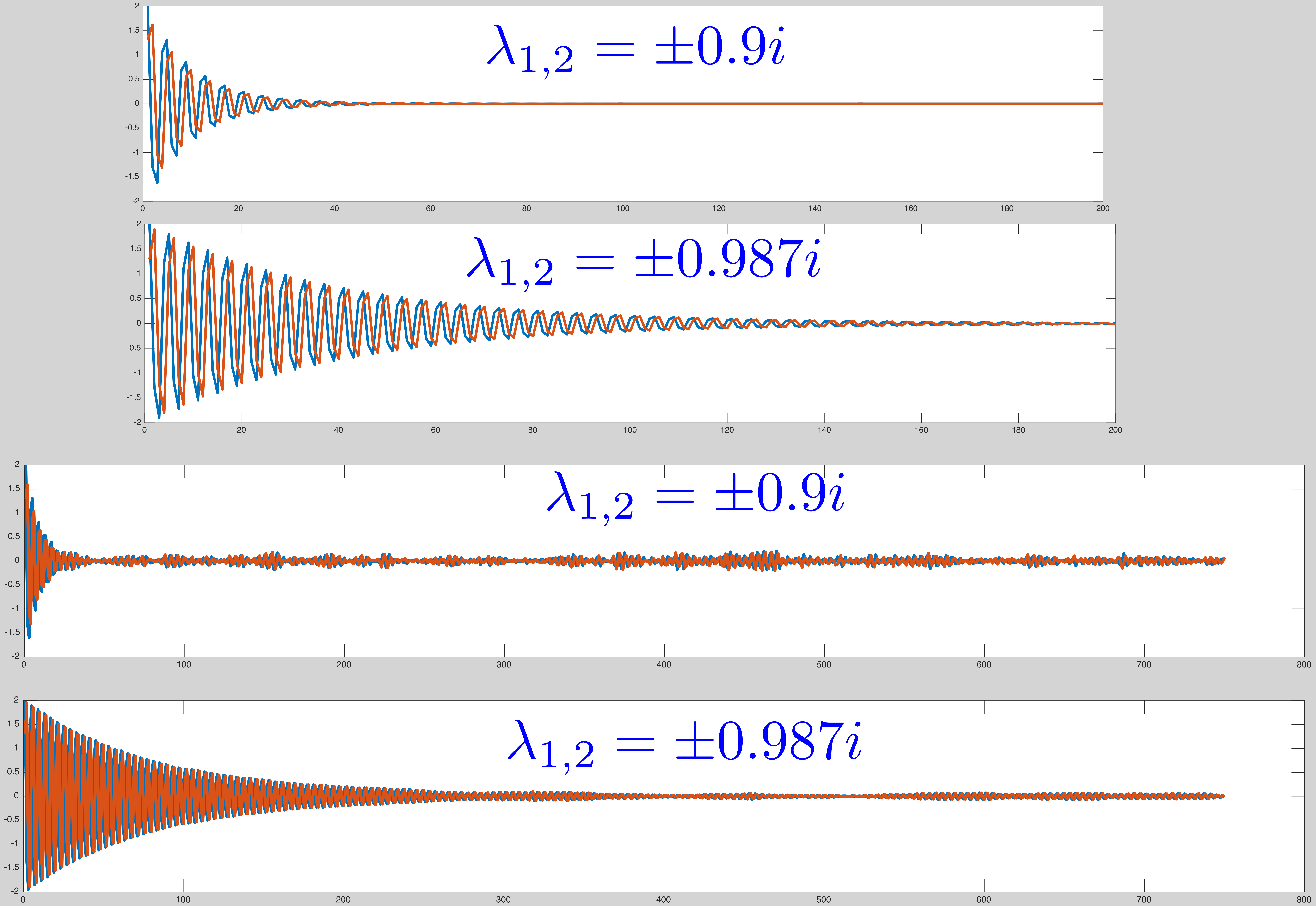
$$= 0 \qquad = 0.81$$

Example

$$\hat{x}(t+1) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \hat{x}(t) + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(t) - y(t) \right)$$

$$l_1 = 0 \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \hat{x}(0) = \begin{bmatrix} -1 \\ -1.3 \end{bmatrix}$$
$$l_2 = -0.19$$





$\lambda_{1,2} = \pm 0.9i$
$\lambda_{1,2} = \pm 0.987i$
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Kalman Filter

- We have not assumed noise and errors in our system model and inputs

$$\hat{x}(t+1) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{Copy of system model}} + \underbrace{L(C\hat{x}(t) - y(t))}_{\text{correction}}$$

A more elaborate form of the observer where the matrix L is also updated at each time, is known as the Kalman Filter and is the industry standard in navigation. The Kalman Filter takes into account the statistical properties of the noise that corrupts measurements and minimizes the mean square error between x(t) and xˆ(t)

Figure 3: **Rudolf Kalman (1930-2016)** introduced the Kalman Filter as well as many of the state space concepts we studied, such as controllability and observability. He was awarded the National Medal of Science in 2009.

Control Recap

- Controllability:

$$\vec{x}(n) - A^n\vec{x}(0) = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{bmatrix}$$

If Rn is full rank then we can move to any target value

Same rank test for continuous time

- Open loop control:

Can use the above equation to design an input sequence – and apply it blindly. Accuracy of result will depend on accuracy of model.

Control Recap – State Feedback

$$u(t) = K\vec{x}(t)$$

Closed-loop system: $\Rightarrow \quad \vec{x}(t+1) = (A + BK)\vec{x}(t)$

Must choose K s.t. A+BK has eigenvalues inside the unit circle (or left half-plane for coninuous time)

If controllable, can assign eigenvalues for A+BK arbitrarily

If not, some eigenvalues of A can not be changed!

(could be OK, if stable, bad news if not)

Control Recap - Observers

Not all state variable are measured, but we get "outputs"

$$\vec{y}(t) = C\vec{x}(t)$$

To estimate the state we estimate an initial guess and update:

$$\hat{x}(t+1) = \underbrace{A\hat{x}(t) + Bu(t)}_{\text{Copy of system model}} + \underbrace{L(C\hat{x}(t) - y(t))}_{\text{correction}}$$

Design L, such that A+LC has eigenvalues inside unit circle

$$\vec{e}(t+1) = (A + LC)\vec{e}(t)$$

Can assign arbitrary eigenvalues if system is observable

Control Recap

Observability:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \vec{x}(0)$$

O_{n-1} must have n independent rows (full rank) to determine x(0) uniquely from output

Duality: Observability of (C,A) is the same as controllability of (A^T,C^T)

Guidance, Navigation & Control (GNC) is aerospace engineering

Open loop

Observers or “kalman” filters

feedback