EE16B
Designing Information Devices and Systems II

Lecture 9A
Computing the SVD

SVD

SVD decomposes a rank r matrix $A \in \mathrm{R}^{m \times n}$ into a sum of r rank-1 matrices:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \Rightarrow ||\vec{u}_i|| = 1 \quad \vec{u}_i \perp \vec{u}_j$

2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \Rightarrow ||\vec{v}_i|| = 1 \quad \vec{v}_i \perp \vec{v}_j$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

Matrix Form of SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

$$U_1 = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix} \quad S = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \quad V_1 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix}$$

$$m \times r \qquad\qquad r \times r \qquad\qquad n \times r$$

$$A = U_1 S V_1^T$$

$$U_1^T U_1 = I_{r \times r}$$

$$V_1^T V_1 = I_{r \times r}$$

$$S \succ 0 \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

Full Matrix Form of SVD

$$U_1 = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_r \end{bmatrix}}_{m \times r} \qquad S = \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}}_{r \times r} \qquad V_1 = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_r \end{bmatrix}}_{n \times r}$$

$$U = \underbrace{\begin{bmatrix} U_1 & U_2 \end{bmatrix}}_{m \times m} \qquad \Sigma = \underbrace{\left[\begin{array}{c|c} S & 0 \\ \hline 0 & 0 \end{array}\right]}_{m \times n} \qquad V = \underbrace{\begin{bmatrix} V_1 & V_2 \end{bmatrix}}_{n \times n}$$

$$A = U\Sigma V^T$$

$$U^T U = I_{m\times m}$$
$$V^T V = I_{n\times n}$$
$$\Sigma \succeq 0$$

Computing the SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ 2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

- What's the singular value of A:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Computing the SVD

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$$

1) $\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ 2) $\vec{v}_i^T \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

3) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

- What's the singular value of A:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \underbrace{\sqrt{2}}_{\sigma_1} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}_{\vec{u}_1} \underbrace{[1 \quad 0]}_{\vec{v}_1^T}$$

General Procedure for SVD

$A \in \mathrm{R}^{m \times n}$

1) Procedures based on A^TA (…and AA^T …later!)

A^TA has only real eigenvalues, r of them are positive and the rest are zero
A^TA has orthonormal eigenvectors (to be proven next time)

Step1: Find eigenvalues of A^TA and order them from biggest to smallest $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \cdots 0$

Step2: Find orthonormal vectors: $\vec{v}_1, \cdots, \vec{v}_r : \; A\vec{v}_i = \lambda \vec{v}_i$

$$A = a \quad \Rightarrow \quad A^TA = a^2 \Rightarrow \lambda = a^2$$
$$A = a \quad \Rightarrow \quad \sigma = |a|$$

Step3: Set $\sigma_i = \sqrt{\lambda_i}$, and $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$

Example

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = 4 \qquad \lambda_2 = 1$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\sigma_1 = 2 \qquad \sigma_2 = 1$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$A = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Computing the SVD with $A^T A$

$$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i$$

- Proof concept: let $A^T A\vec{v}_i = \lambda_i \vec{v}_i \Rightarrow A^T A V_1 = \Lambda V_1$

$$\sigma_i^2 = \lambda_i \qquad S^2 = \Lambda$$

Show that $A\vec{v}_i = \sigma_i \vec{u}_i$, where

$$\vec{u}_i^T \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \Rightarrow U_1^T U_1 = I_{r\times r}$$

Show that $A = U_1 S V_1^T$

Proof U_1 is orthonormal

- Let,

$$A\vec{v}_i = \hat{\sigma}_i \vec{u}_i \qquad i = 1, \cdots, r$$

$$(A\vec{v}_j)^T A\vec{v}_i = (A\vec{v}_j)^T \hat{\sigma}_i \vec{u}_i$$

$$(A\vec{v}_j)^T A\vec{v}_i = \hat{\sigma}_j \vec{u}_j^T \hat{\sigma}_i \vec{u}_i$$

$$\vec{v}_j^T \underbrace{A^T A\vec{v}_i}_{\sigma_i^2 \vec{v}_i} = \hat{\sigma}_j \hat{\sigma}_i \vec{u}_j^T \vec{u}_i$$

$$\sigma_i^2 \vec{v}_j^T \vec{v}_i = \hat{\sigma}_j \hat{\sigma}_i \vec{u}_j^T \vec{u}_i$$

Orthonormal!

$$\hat{\sigma}_j \hat{\sigma}_i \vec{u}_j^T \vec{u}_i = \begin{cases} \sigma_i^2 & i = j \\ 0 & i \neq j \end{cases}$$

Proof $A=U_1SV_1^T$

$$A\vec{v}_i = \sigma_i \vec{u}_i \qquad i = 1, \cdots, r$$

$$\Rightarrow AV_1 = U_1 S$$

$$AV_1V_1^T = U_1SV_1^T$$

form we want!

- Need to show:

$$AV_1V_1^T = A$$

- We know:

$$A\underbrace{[V_1 \quad V_2][V_1 \quad V_2]^T}_{VV^T = I_{n\times n}} = A$$

$$AV_1V_1^T + AV_2V_2^T = A$$

Show =0

Proof $A=U_1SV_1^T$

$$AV_1V_1^T = U_1SV_1^T$$

form we want!

$$AV_1V_1^T + AV_2V_2^T = A$$

Show =0

- We know:

$$A^TAV_2 = 0$$

$$V_2^TA^TAV_2 = 0$$

$$(AV_2)^TAV_2 = 0$$

$$(A\vec{v}_i)^TA\vec{v}_i = 0 \qquad i = r+1, r+2, \cdots, n$$

$$\Rightarrow ||A\vec{v}_i||^2 = 0$$

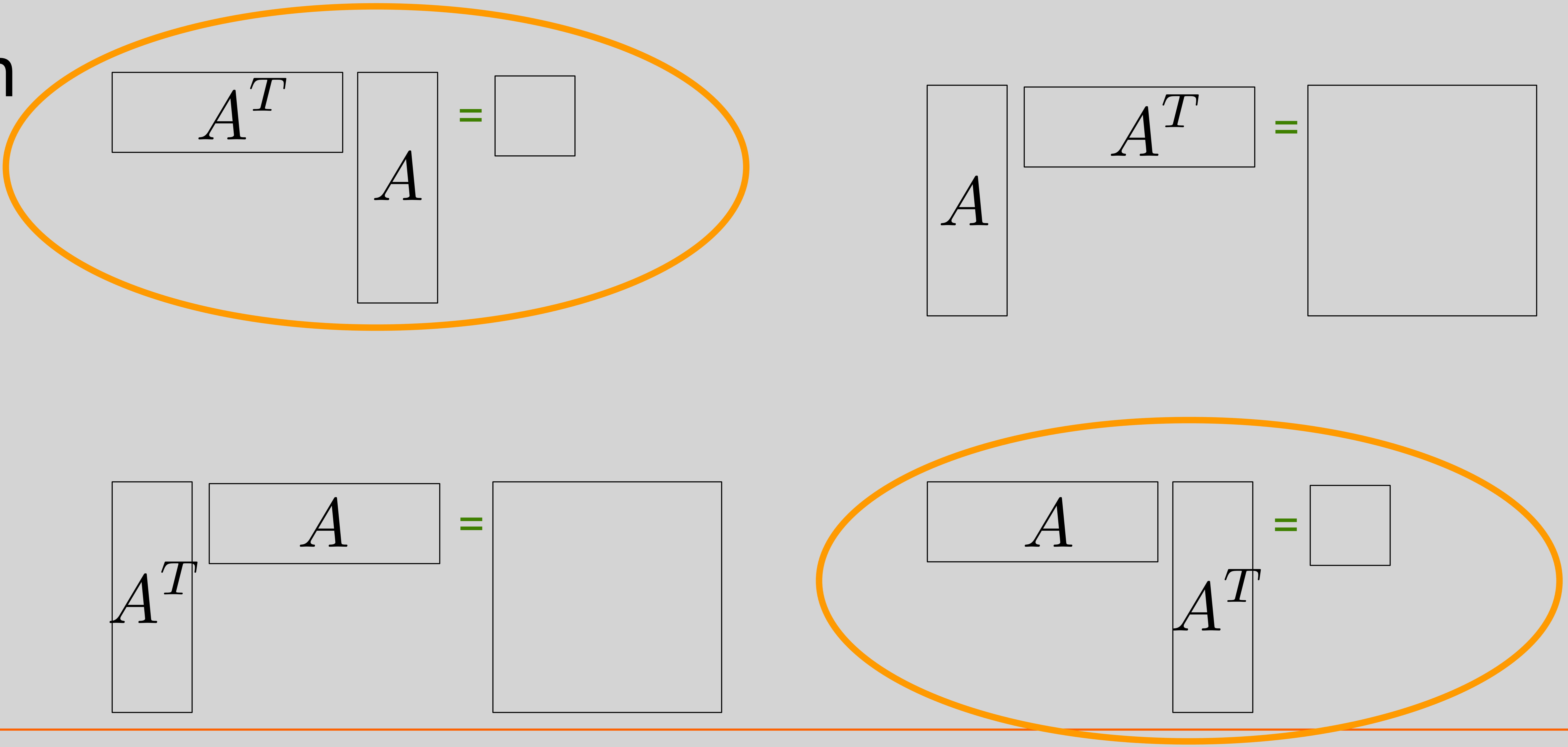
$$\Rightarrow A\vec{v}_i = 0 \;\Rightarrow AV_2 = 0$$

Alternate Procedure using AA^T

$A^T A$ $\quad$ AA^T

$n \times n$ $\quad$ $m \times m$

- If, m > n
- If m < n



Alternate Procedure using AA^T

Step 1: Find eigenvalues of AA^T and order s.t.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \cdots 0$$

Step 2: Find orthonormal eigenvectors of AA^T:

$$AA^T \vec{u}_i = \lambda_i \vec{u}_i \qquad i = 1, \cdots, r$$

Step 3: Set,

$$\sigma_i = \sqrt{\lambda_i} \qquad \vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i$$

Example

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \qquad r = 2$$

$$A^T A = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} \qquad AA^T = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

$$\lambda_1 = 32 \qquad \lambda_2 = 18$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i \qquad \vec{v}_1 = \frac{1}{4\sqrt{2}} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Signs of u_1,v_1 (u_2,v_2) can be flipped!

Uniqueness of the SVD

Find SVD of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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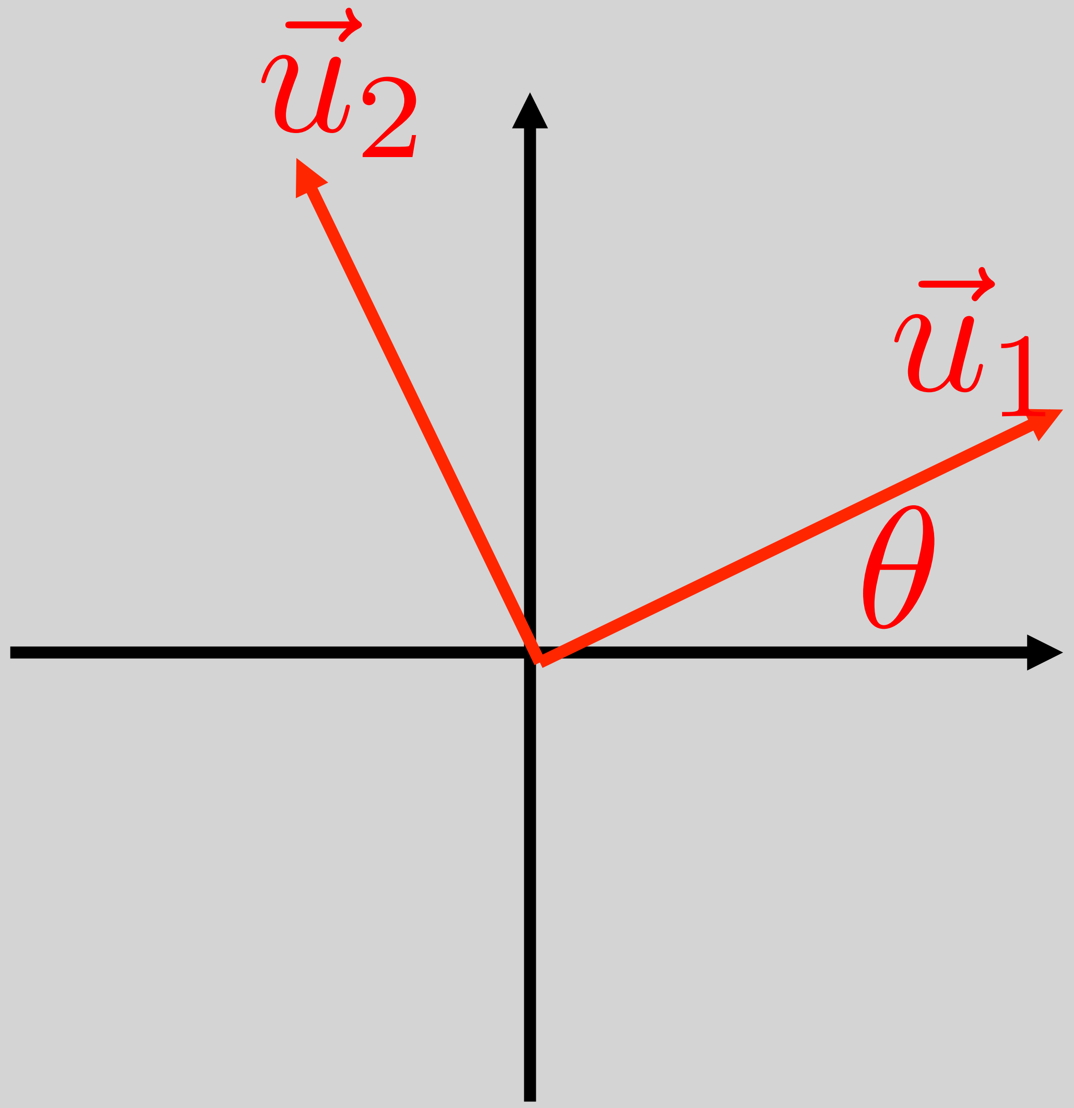
Find SVD of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \lambda_1 = \lambda_2 = 1 \qquad \Rightarrow \sigma_1 = \sigma_2 = 1$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\vec{u}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Uniqueness of the SVD

Find SVD of A

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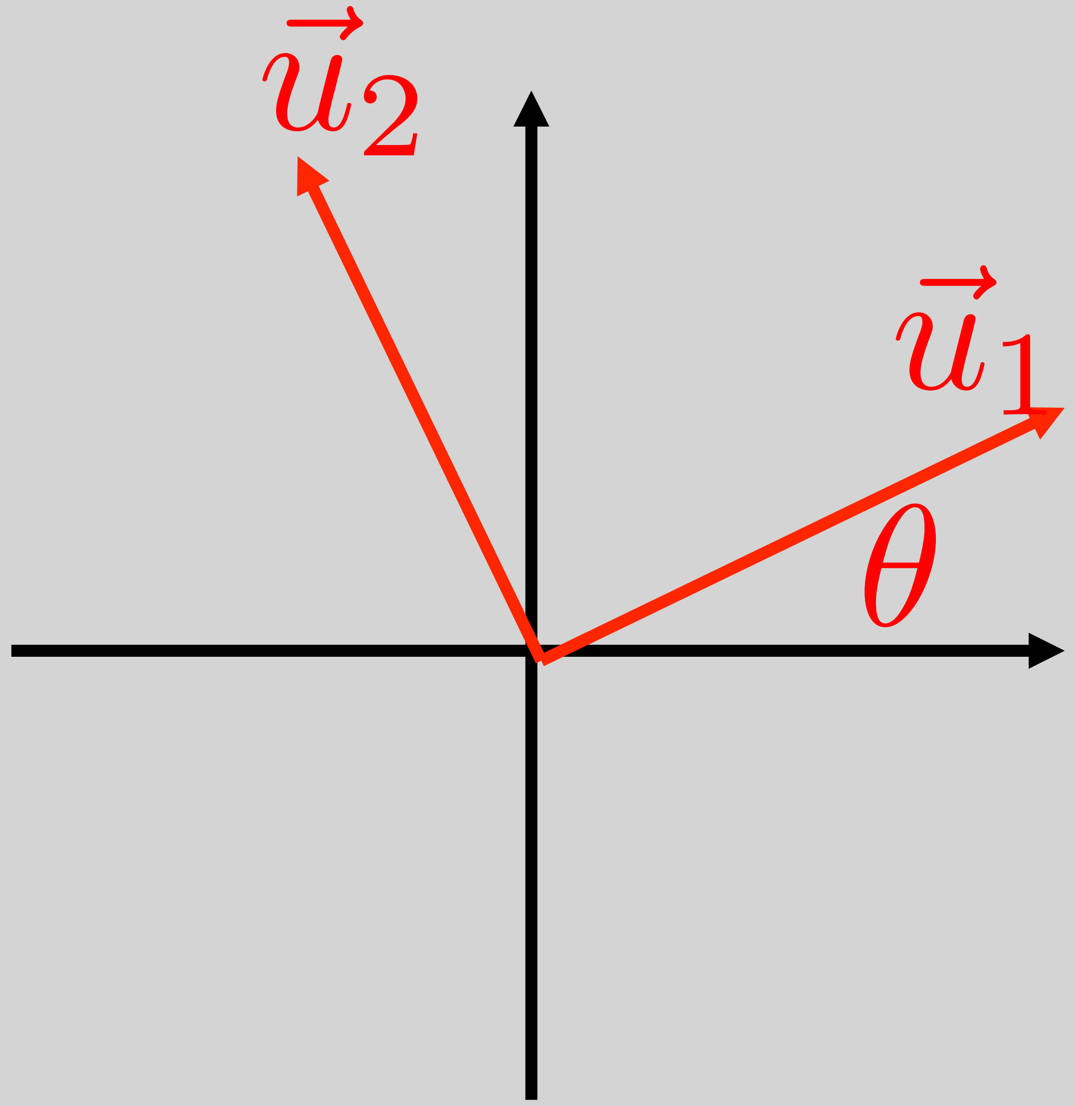
$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \lambda_1 = \lambda_2 = 1 \qquad \Rightarrow \sigma_1 = \sigma_2 = 1$$



$$\vec{u}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\vec{v}_1 = \frac{1}{\sigma_1} A^T \vec{u}_1 = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -\sin\theta \\ -\cos\theta \end{bmatrix}$$

Uniqueness of the SVD

Find SVD of A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \sigma_1 = \sigma_2 = 1$$

$$\vec{u}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \; \vec{u}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \quad \vec{v}_1 = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \vec{v}_2 = \begin{bmatrix} -\sin\theta \\ -\cos\theta \end{bmatrix}$$

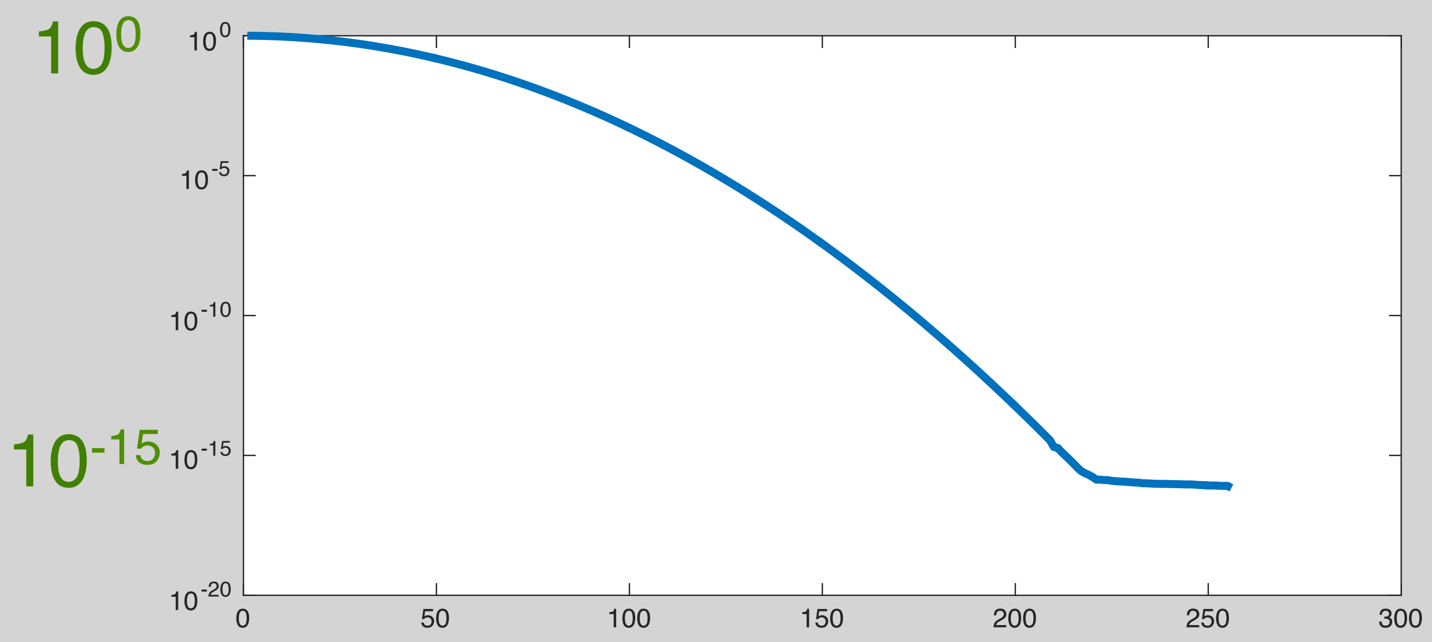
$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$$

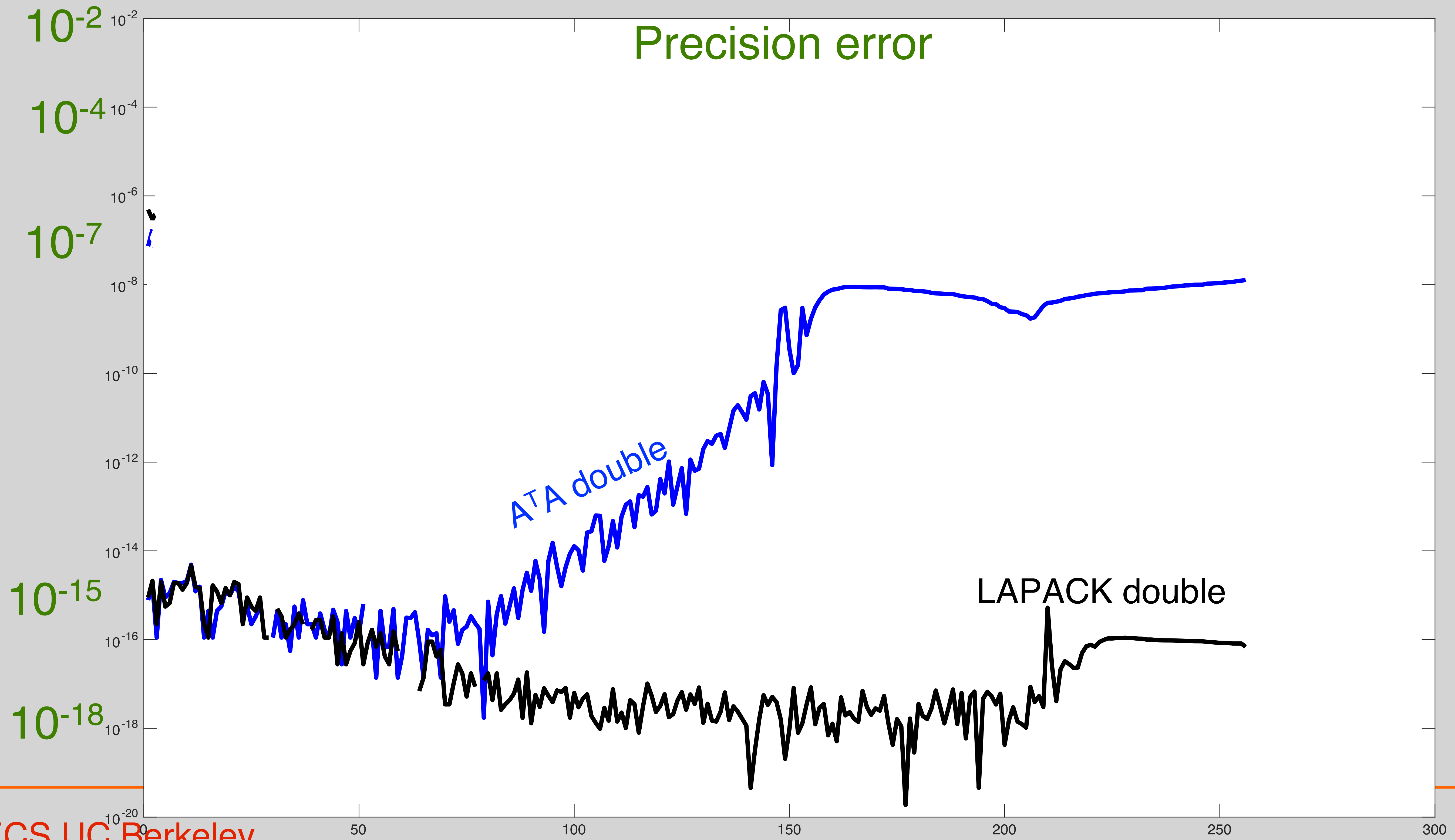
$$= \begin{bmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & -\sin^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Accuracy with Finite Precision

Consider matrix $A \in R^{512x256}$ with the following singular values:





Accuracy with Finite Precision

Consider matrix $A \in R^{512x256}$ with the following singular values: