EE221A Linear System Theory
Final Exam

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Fall 2006
12/14/06, 8.00-11.00am

Your answers must be supported by analysis, proof, or counterexample.

There are 13 questions: Please make sure your exam paper has all 13 questions. This exam paper has 16 pages.

Approximate points for each question are indicated. The exam is out of 66 points total.

Read all problems and do the easy ones first.

You are allowed to use 1 8.5×11 *crib sheet (both sides).*

Problem 1: Achieving a desired $x(T)$ (3 points).

Given the system

$$\dot{x} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u; \qquad x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

True or false: there exists $u[0,T]$ such that $x(T) = [10, -10, 0]^T$. If true, give a sketch of the proof, if false, explain.

Problem 2: A-Invariance.

Consider a MIMO completely controllable, completely observable LTI system $R = (A, B, C, D)$ with $A \in \mathbb{R}^{n \times n}$, with controllability matrix $\mathcal{Q}$ and observability matrix $\mathcal{O}$.

(a) (3 points) Prove that $R(Q)$ is the smallest A-invariant subspace containing $R(B)$.

(b) (1 point) Let $\mathcal{N}$ be an A-invariant subspace in the nullspace of C. What can you say about $\mathcal{N}$?

Problem 3: Controllability (6 points).

For an LTI system $R = (A, B, C, D)$ with $A \in \mathbb{R}^{n \times n}$, directly prove the standard controllability result that $rank[sI - A|B] = n, \forall s \in \mathbb{C}$ implies that $rank[B\ AB\ \cdots\ A^{n-1}B] = n$.

Problem 4: Orthogonal Projection (6 points).

You are given a linear operator $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ both finite dimensional spaces with $\mathcal{A}$ injective. Prove that the operator $\mathcal{P} = \mathcal{A}(\mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^* : \mathcal{Y} \to \mathcal{Y}$ is an orthogonal projection operator from $\mathcal{Y}$ onto $R(\mathcal{A})$, meaning, prove that:

(i) $\mathcal{P}^2 = \mathcal{P}$; and

(ii) for all $x \in R(\mathcal{A})$, $\mathcal{P}x = x$, and for all $x \in N(\mathcal{A}^*), \mathcal{P}x = 0$.

Explain why $\mathcal{A}^*\mathcal{A}$ is invertible. For the case that $\mathcal{A}^*$ is surjective, find the orthogonal projection operator from $\mathcal{X}$ onto $R(\mathcal{A}^*)$.

Problem 5: Characteristic and minimal polynomials (3 points).

Let the characteristic polynomial of A be

$$\chi_A(s) = (s-1)^5 \tag{2}$$

and its minimal polynomial be $\Psi_A(s) = (s-1)^3$. True or False: there exists a nonsingular matrix P such that

$$PAP^{-1} = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \tag{3}$$

Explain your answer.

Problem 6: Sampled data systems.

Consider a *minimal* linear, time invariant sampled data system of the form

$$\dot{x} = Ax + Bu \qquad y = Cx$$

with the states, outputs at sample times kT defined to be $x_k := x(kT), y_k := y(kT)$. Further, assume that the inputs are held constant between sample times, that is, $u(t) \equiv u_k, t \in [kT, (k+1)T)$.

(a) (2 points) Find the sampled data system

$$x_{k+1} = Fx_k + Gu_k \qquad y_k = Hx_k$$

i.e. express F, G, H in terms of A, B, C.

(b) (4 points) Is the pair (F, G) completely controllable? Is the pair (F, H) completely observable? If you answer "no" to either of these questions, give conditions on A under which complete controllability and observability are preserved under sampling.

Problem 7: Observability.

You are told that $A(t) \in \mathbb{R}^{n \times n}, C(t) \in \mathbb{R}^{n_o \times n}$ are *uniformly* completely observable, meaning there exist $\Delta, \alpha > 0$ such that the observability grammian

$$W_o(t_0, t_0 + \Delta) = \int_{t_0}^{t_0+\Delta} \Phi^*(\tau, t_0) C^*(\tau) C(\tau) \Phi(\tau, t_o) d\tau > \alpha I$$

(recall that $>$ for matrices means "more positive definite"). Now, for a given *exponentially stable*, linear time varying system

$$\dot{x} = A(t)x$$

define

$$P(t) := W_o(t, \infty)$$

(a) (3 points) Is $P(t) \in \mathbb{R}^{n \times n}$ well defined (meaning does the infinite integral exist)? Is it positive definite?

(b) (3 points) Find the formula for $\dot{P}$.

Hint: Recall the definition of exponential stability, for all $x(t_0) \in \mathbb{R}^n$, there exists $M, \lambda > 0$ such that for all t

$$||x(t)|| \leq M \exp^{-\lambda(t-t_0)} ||x(t_0)||$$

Problem 8. Lyapunov stability (4 points).

Suppose that there exist positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$ and some $\lambda > 0$ such that

$$A^T P + PA - 2\lambda P = -Q$$

What can you say about the eigenvalues of A?

Problem 9: Stability (4 points).

You are given a SISO transfer function

$$\frac{(s+1)(s+3)}{s^2(s+2)^2(s+4)}$$

and are told that it has a minimal realization A, b, c with $A \in \mathbb{R}^{5\times5}, b \in \mathbb{R}^5, c \in \mathbb{R}^{1\times5}$. Is the undriven linear system

$$\dot{x} = Ax$$

stable? Is it asymptotically stable? Prove your conclusions.

Problem 10: Integral control.

Consider the following simple plant input-output transfer function:

$$G(s) = \frac{1}{s+1}$$

(a) (1 point) Derive the state space representation of this plant in controllable canonical form.

(b) (2 points) Consider this plant in state feedback with state feedback gain F and reference input R: derive an expression for the plant input $u(t)$ to ensure that $Y(t)$ tracks constant reference inputs $R(t) = R$ with zero steady state error.

(c) (2 points) Now, suppose that the actual plant is subject to modeling errors

$$G(s) = \frac{1}{s+1+\epsilon}$$

where ϵ is an unknown, but fixed, deviation from the known model. Discuss the effect of this modeling error on the steady state error computed in part (b) above.

(d) (3 points) Define the output error as $e = R - Y$, and define a new state x_2 to be the integral of this error:

$$x_2 = \int_0^t e d\tau \tag{4}$$

Consider the feedback diagram shown in Figure 1.

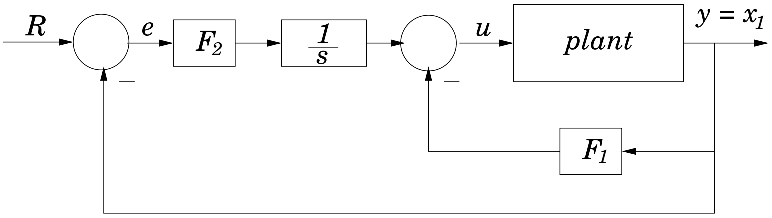

Figure 1: Integral control.

Write out the augmented state space equations for this new closed loop system, using the plant model from part (a) and this new error state.

Now, with a new stabilizing state feedback control law $F = [F_1 \; F_2]$, with reference input R for this augmented system, discuss the effect of this "integral control" on the steady state error between output and reference when modeling errors are present.

(Extra space for Problem 10)

Problem 11: Controllability and observability.

Consider the LTI system $\dot{x} = Ax + Bu$.

(a) (2 points) Suppose that this open loop system is controllable. Show that the closed loop system resulting from state feedback $u = Fx + v$ is controllable (from new input v).

(b) (3 points) Now, in addition, assume that $y = Cx$ and that the open loop system is observable. Suppose that $\psi(y)$ is a known, possibly nonlinear, function of y. Show, by designing an appropriate observer and analyzing the convergence of the state estimate error, that the system resulting from *output injection*: $\dot{x} = Ax + \psi(y) + Bu$, $y = Cx$, is observable. (Hint: consider the block diagram of the observer)

Problem 12: Observer design.

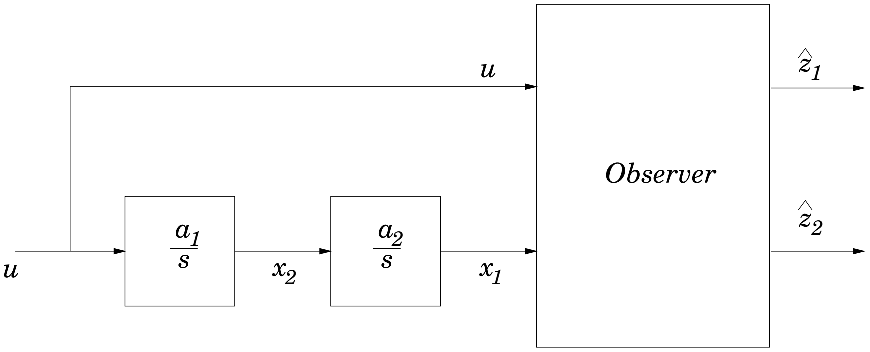

Figure 2: Simple model of a DC Servo system, for Problem 4.

Figure 2 shows a block diagram representation of a simple model of a DC servo system: x_1 is a voltage signal proportional to the output angular velocity x_2.

(a) (3 points) Design a full order observer, with observer gain matrix T given by

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \tag{5}$$

for x_1 and x_2 so that the characteristic polynomial associated with the (exponentially stable) error dynamics is given by:

$$\Delta_e(s) = s^2 + 2\zeta_e \omega_e s + \omega_e^2 \tag{6}$$

("Design" means write the equations for the observer, with expressions for gains T_1 and T_2.)

(b) (2 points) Now, the observer is a system with inputs u and x_1, and outputs $\hat{z_1}$ and $\hat{z_2}$. Thus, there are four possible transfer functions between inputs and outputs – these may be included as elements in a 2×2 matrix. Evaluate the following *matrix of transfer functions* $M(s)$ between the inputs to the observer u and x_1, and its outputs $\hat{z_1}$ and $\hat{z_2}$:

$$M(s) = \begin{bmatrix} \hat{z}_1(s)/u(s) & \hat{z}_1(s)/x_1(s) \\ \hat{z}_2(s)/u(s) & \hat{z}_2(s)/x_1(s) \end{bmatrix} \tag{7}$$

as a function of gains T_1 and T_2, as well as system parameters a_1 and a_2.

(c) (2 points) Now determine $M(s)$ as $T_2 \to \infty$. Discuss the meaning of the result.

(Extra space for Problem 12)

Problem 13: Stability (4 points).

True or False: If a linear time-varying system is asymptotically stable, it is also exponentially stable. If true, prove, if false, give a counterexample.