EE221A Linear System Theory
Problem Set 4

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Dear 221A folks: Some problems like the ones below may be included on the midterm on October 16. Thus, you may hand in your homework solutions to me on Monday October 15 and receive solutions to use in your midterm preparation.

Problem 1: A linear time-invariant system.

Consider a single-input, single-output, time invariant linear state equation

$$\dot{x}(t) = Ax(t) + bu(t), x(0) = x_0 \quad (1)$$
$$y(t) = cx(t) \quad (2)$$

If the nominal input is a non-zero constant, $u(t) = \overline{u}$, under what conditions does there exist a constant nominal solution $\overline{x}(t) = x_0$, for some x_0?

Under what conditions is the corresponding nominal output zero?

Under what conditions do there exist constant nominal solutions that satisfy $\overline{y} = \overline{u}$ for all $\overline{u}$?

Problem 2. Preservation of Eigenvalues under Similarity Transform.

Consider a matrix $A \in \mathbb{R}^{n\times n}$, and a non-singular matrix $P \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of $\overline{A} = PAP^{-1}$ are the same as those of A.

Remark: This important fact in linear algebra is the basis for the similarity transform that a redefinition of the state (to a new set of state variables in which the equations above may have simpler representation) does not affect the eigenvalues of the A matrix, and thus the stability of the system. We will use this similarity transform in our analysis of linear systems.

Problem 3: Sampled Data System

You are given a linear, time-invariant system

$$\dot{x} = Ax + Bu \quad (3)$$

which is sampled every T seconds. Denote $x(kT)$ by x_k. Further, the input u is held constant between kT and $(k+1)T$, that is, $u(t) = u_k$ for $t \in [kT, (k+1)T]$. Derive the state equation for the sampled data system, that is, give a formula for x_{k+1} in terms of x_k and u_k.

Problem 4: Linear Quadratic Optimization.

Consider an object of mass $m = 1$ moving along the x-axis in response to a force input $u(t)$. The object's dynamics can be described simply as $\ddot{x} = u(t)$. Suppose you would like to design an input $u(t)$ which will move the object from any initial position and velocity, to come to rest at the position $x = 4$. Using the linear quadratic regulator discussed in class, formulate an appropriate quadratic cost functional, and solve the problem in MATLAB, showing simulations of your results for different weightings on the state and input.

Problem 5: Linear Quadratic Optimization.

Consider the control system

$$\begin{aligned} \dot{x}_1 &= ax_2 + u \\ \dot{x}_2 &= bx_2 \end{aligned}$$

with cost $J = \int_0^T (x_1^2 + hx_2^2 + u^2)dt$ with $h > 0$ and no terminal cost.

(a) Show that the optimal control for this system is given by $u(t) = -p_1(t)x_1(t) - p_2(t)x_2(t)$ where $p_1(t)$ and $p_2(t)$ are obtained by solving two coupled differential equations over $[0, T]$. Determine (but don't solve) these differential equations.

(b) Change h to $2h$. Is there any change in the optimal control? Discuss, referring to the system equations, why this makes sense.