EE221A Linear System Theory
Problem Set 7

Professor C. Tomlin
Department of Electrical Engineering and Computer Sciences, UC Berkeley
Fall 2007
Issued 11/15; Due 11/27

Problem 1. Show that

$$\left\{ \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix} \right\}$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c^T \in \mathbb{R}^n$ is completely controllable iff (i) $\{A, b\}$ is completely controllable and (ii) the matrix

$$\begin{bmatrix} A & b \\ c & 0 \end{bmatrix}$$

is full rank.

Problem 2: Grammians under Similarity Transforms.

Consider the controllability and observability grammians W_c, W_o of a linear time-invariant system (A, B, C) over the time period $[0, \Delta]$. Determine what happens to them under similarity transformations of the state space. That is, determine the controllability and observability grammians of (TAT^{-1}, TB, CT^{-1}). Prove that the eigenvalues of the product $W_c W_o$ are constant under similarity transformations.

Problem 3: RL-RC circuit example. Consider the network shown in Figure 1 with voltage $v(t)$ as input and the current $i(t)$ as output. Give conditions under which: **(a)** the network is controllable; **(b)** the network is observable; and **(c)** the network has *constant resistance*, meaning that $\frac{v(s)}{i(s)}$ is a constant.

Problem 4. Consider a single input system

$$\dot{x} = Ax + bu \qquad (1)$$

with $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. State necessary and sufficient conditions for complete controllability. Now generalize to the multiple input case.

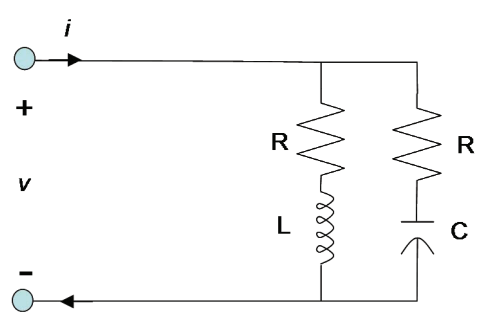

Figure 1: Showing a simple RL-RC circuit.

Problem 5: Echo Canceller. This problem addresses the design of a chip that is used to cancel echo on your telephone line. The echo $y(t) \in \mathbb{R}$ is represented as the linear combinations of delayed versions of your spoken (message signal) $m(t)$ as follows:

$$y(t) = \sum_{i=1}^{N} a_i m(t-i)$$

where t is a discrete time variable, representing the sampling rate of the voice signal (around 125 μ seconds). The coefficients $a_i \in \mathbb{R}$ model the characteristics of the line. These are assumed unknown when you pick up the telephone at $t = 0$, but you have estimates of them denoted $\hat{a_i}(t)$. The aim of the echo canceller is to update the estimates using the measurement of the echo $y(t)$, and the prediction error

$$e(t) := y(t) - \sum_{i=1}^{N} \hat{a}_i m(t-i)$$

Note that it will take N seconds after you pick up the phone to get all the $m(t-i)$, thus the echo canceller is initialized with $m(-1) = m(-2) = \cdots = m(-N+1) = 0$.

(a) For this problem, set the echo canceller up as an observability problem, with the vector $a \in \mathbb{R}^N$ representing the unknown (but constant) state vector to be estimated, no input, and $y(t)$ as the output function.

(b) Find $\hat{a}(1)$ so that it is the vector closest to $\hat{a}(0)$ in norm, that gives the correct value of $y(1)$.

(c) Try to make this recursive, so that you can determine $\hat{a}(t+1)$ from $\hat{a}(t), y(t)$.

Problem 6: Pole placement. Consider the dynamic system:

$$\frac{d^4\theta}{dt^4} + \alpha_1 \frac{d^3\theta}{dt^3} + \alpha_2 \frac{d^2\theta}{dt^2} + \alpha_3 \frac{d\theta}{dt} + \alpha_4 \theta = u$$

where u represents an input force, α_i are real scalars. Assuming that $\frac{d^3\theta}{dt^3}$, $\frac{d^2\theta}{dt^2}$, $\frac{d\theta}{dt}$, and θ can all be measured, design a state variable feedback control scheme which places the closed-loop eigenvalues at $s_1 = -1$, $s_2 = -1$, $s_3 = -1 + j1$, $s_4 = -1 - j1$.