Spring 2003: EE223 Homework 54
Solutions due Apr 7
Collaboration among students is encouraged

Problem 5.1 *Complete the derivation of the random coding error exponent for the discrete-time scalar AWGN channel that was started in class and by using the relaxed approximation for the indicator function for an error taken to the point where we had*

$$P_e \leq M^\rho \int_{\vec{y}} [\int_{\vec{x}} \left(f_{\vec{Y}|\vec{X}}(\vec{y}|\vec{x}) \right)^{\frac{1}{1+\rho}} f_{\vec{X}}(\vec{x}) dx]^{1+\rho} dy$$

and then we used the independence of the random entries in the code as well as the whiteness of the additive noise to get to:

$$P_e \leq M^\rho \left(\int_y [\int_x \left(f_{Y|X}(y|x) \right)^{\frac{1}{1+\rho}} f_X(x) dx]^{1+\rho} \right)^N dy$$

where N was the length of the code and M^ρ was just $(e^{\rho R \ln 2})^N$.

a. *Use the above to get $P_e \leq e^{-E_r(\rho,R)N}$. You will have to complete the square, etc. and will get a function of ρ that is an affine function of the rate R.*

b. *Calculate the intercepts of the affine function E_r with the R axis as a function of rho. What is the limit as $\rho \to 0$?*

c. *Notice that ρ was a parameter that we chose for the bound and does not correspond to anything about the code. So plot $E_r(R)$ for various values of ρ. What shape do you get for the curve?*

d. *Now suppose that we consider a channel where we can use 2-d vector valued inputs that we draw i.i.d. joint circularly symmetric Gaussian so that the power constraint is unchanged and the additive white noise has independent values with variance 1 along each dimension of the x. What happens at the same rate R to the error exponent?*

   (Hint: You can just think of this as cutting the average power constraint in half but then using twice as many symbols for the same size codebook M. Very little recalculation is required.)

   This corresponds to the physical case of doubling the allowed bandwidth but keeping the average power constraint the same while trying to signal over white Gaussian noise. How does the R axis intercept change?

e. *Take the limit of a very large bandwidth but the same power constraint. Compare with the exponent that we derived for orthogonal signaling.*

Problem 5.2 *Discrete cases.*

The argument we gave in class and that you have completed above was for the Gaussian case with continuous alphabet inputs to the noisy channel. Please repeat this argument from the beginning for the case of the Binary Symmetric Channel with crossover probability. Go through part c. above. Use fair coin tosses to generate the random codebook.

Now try it for the Binary Erasure Channel where the probability of an erasure is ϵ*. What changes do you need to make to the argument to get it to work?*