

Reconstruction of 2-D Signals from Partial Fourier Information

1/27/06

- motivations: electron microscopy.
optical astronomy
- crystallography. . . .

- 3 problems:
- ① FTM = magnitude of
Fourier Transform
 - ② Recon from Phase
Fourier Transform
 - ③ Recon from level
crossings

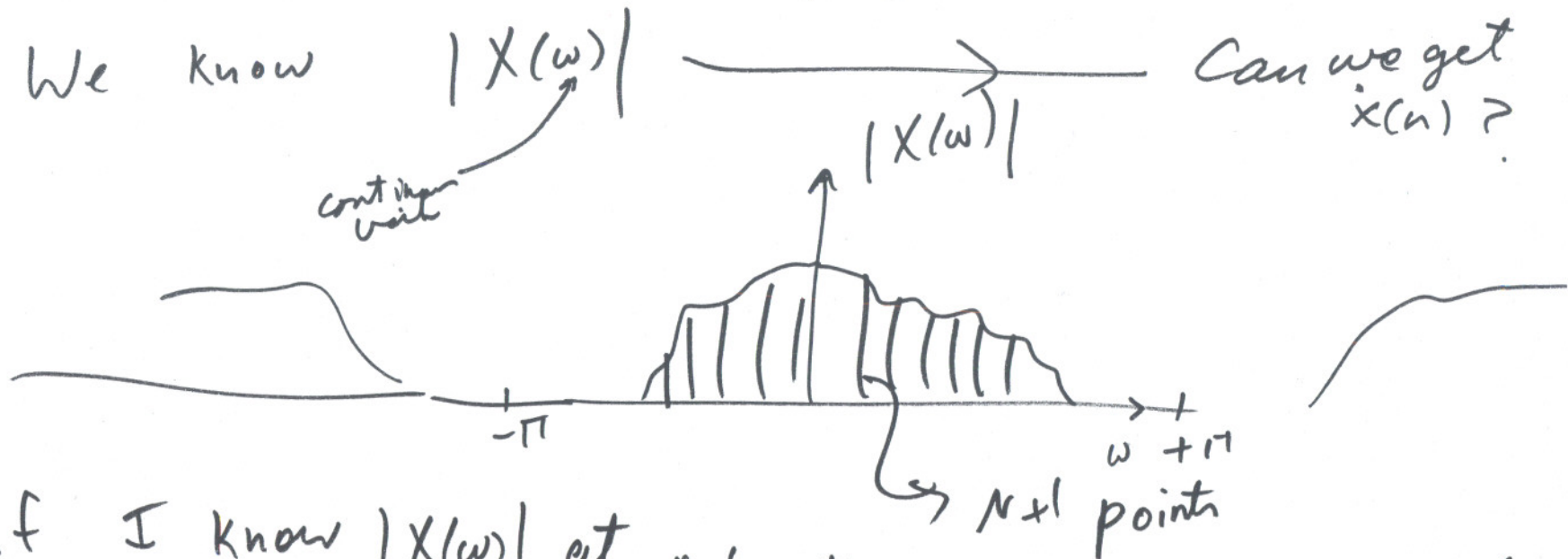
Recon from FTM

1-D Case:

- $X(n)$: discrete time 1-D.
 $N+1$ non zero points.
 $n=0, \dots, N$

$$\text{D.T.F.T}\{x(n)\} = X(\omega) = \sum_{n=0}^N x(n) e^{-j\omega n} \quad e^{-j\omega n} = y$$

IF We know $|X(\omega)|$ → Can we get $x(n)$?

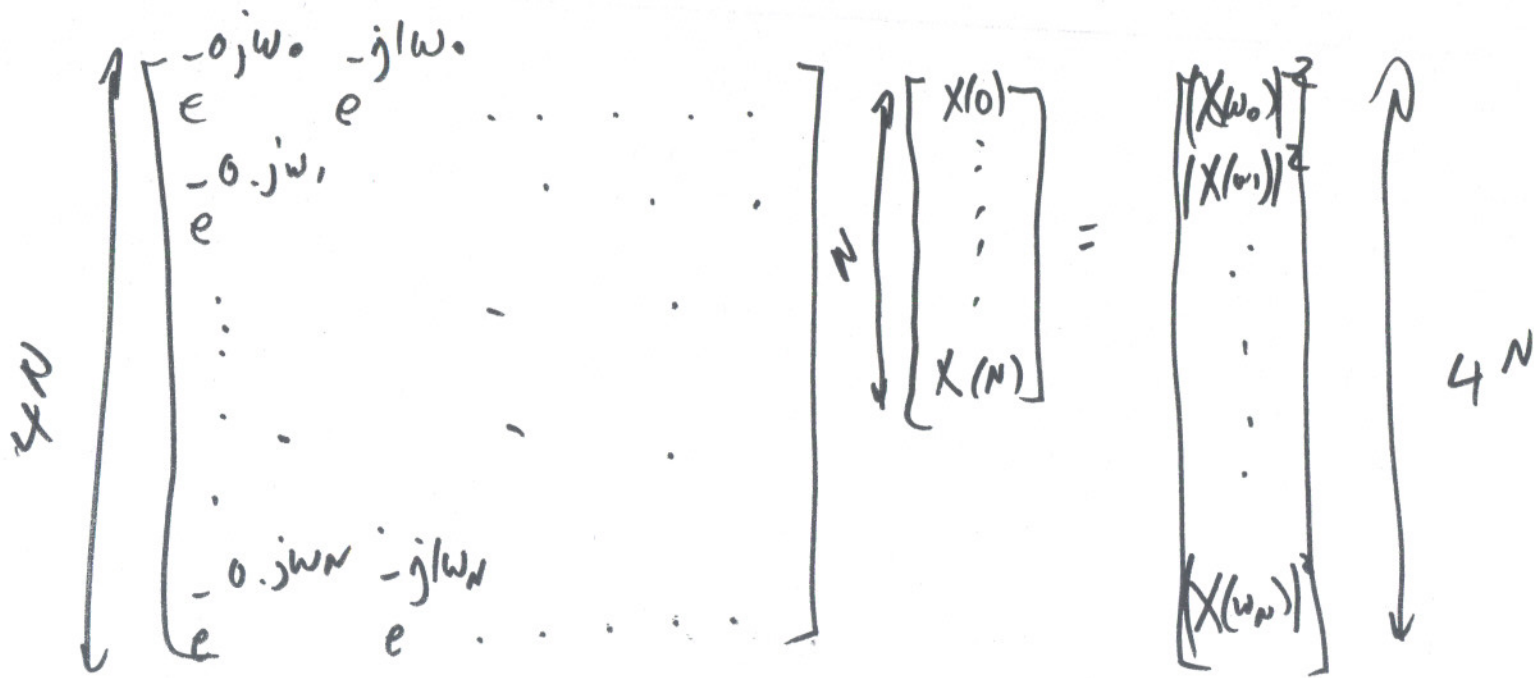


if I know $|X(\omega)|$ at $N+1$ pts \implies cannot reconstruct

$$\begin{bmatrix}
 e^{-j\omega_0} & e^{-j2\omega_0} & e^{-j3\omega_0} & \dots & e^{-jN\omega_0} \\
 e^{-j\omega_1} & e^{-j2\omega_1} & e^{-j3\omega_1} & \dots & e^{-jN\omega_1} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 e^{-j\omega_{N-1}} & e^{-j2\omega_{N-1}} & e^{-j3\omega_{N-1}} & \dots & e^{-jN\omega_{N-1}}
 \end{bmatrix}
 \begin{bmatrix}
 x(0) \\
 x(1) \\
 \vdots \\
 x(N)
 \end{bmatrix}
 =
 \begin{bmatrix}
 |X(\omega_0)| \\
 |X(\omega_1)| \\
 \vdots \\
 |X(\omega_{N-1})|
 \end{bmatrix}$$

$$\sum_{n=0}^N x(n) y^n = X(\omega)$$

Sampling a
 2-D poly of degree
 at arbitrary
 N
 points
 results
 in unique
 recon of
 polynomial



Q: Can we uniquely reconstruct if we oversampled F.D.?

A for ID signal \longrightarrow No 2^N ambiguity
 MD signal \longrightarrow yes

auto-correlation fn of $x(n)$.

$$r(n) = \sum_l x(l) x^*(l+n)$$

$$R(\omega) = \text{D.T.F.T} \{r(n)\} = \sum r(n) e^{-j\omega n} = |X(\omega)|^2$$

$$\Rightarrow F^{-1} \{ |X(\omega)|^2 \} = r(n)$$

Q: Can we obtain $x(n)$ from $r(n)$?

$$\text{z.T.} \{r(n)\} = R(z) = X(z) X^* \left(\frac{1}{z^*} \right) = \sum_{n=-N}^{+N} r(n) z^{-n}$$

Associated polynomial to $R(z) \triangleq P_r(z) = \sum_{n=0}^{2N} r[N-n] z^n$

Since $P_r(z)$ are self symmetric \Rightarrow

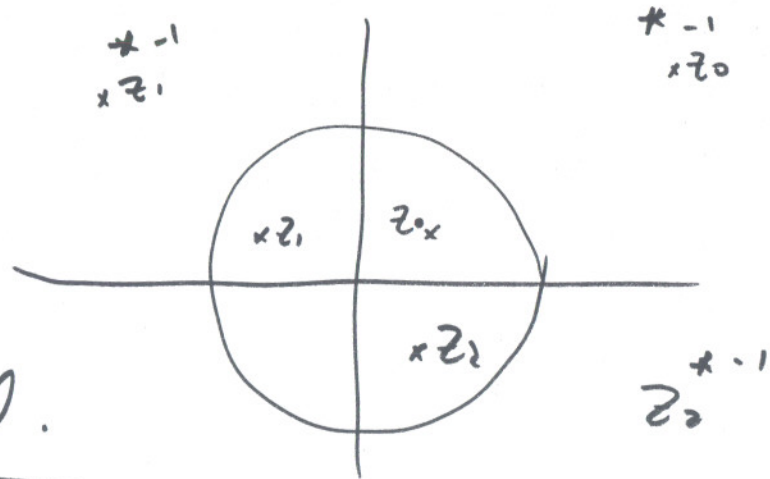
If z_0 is a zero of $P_r(z)$, so is z_0^*

symmetric

z_0^*

⇒ Can factor $P_r(z)$

$$P_r(z) = A \prod_{i=1}^N (z - z_i) (1 - \bar{z}_i^* z)$$



Def mirror of a polynomial.

Associated with any polynomial, there is a mirror polynomial consisting of coefficients in reverse order and conjugated.

$$\text{If } P(z) = \sum_{n=0}^N p_n z^n \xrightarrow{\text{mirror}} \tilde{P}(z) = \sum_{n=0}^N p_{N-n}^* z^n$$

- Assume $y(n) \rightarrow$ autocorrelation $r(n)$

$$P_r(z) = P_y(z) \overset{u}{P}_y(z)$$

2^N ways in order to generate $P_y(z)$.

$$P_y(z) = \sqrt{A} \prod_{i \in I} (z - z_i) \prod_{j \notin I} (z - z_j^*)$$

I any subset of $[1, \dots, N]$

$$(z_0, z_0^{*-1})$$

$$(z_1, z_1^{*-1})$$

⋮

$$(z_N, z_N^{*-1})$$

$$P(z)$$

$$P_r(z)$$

$$r(n)$$



$$X(n)$$

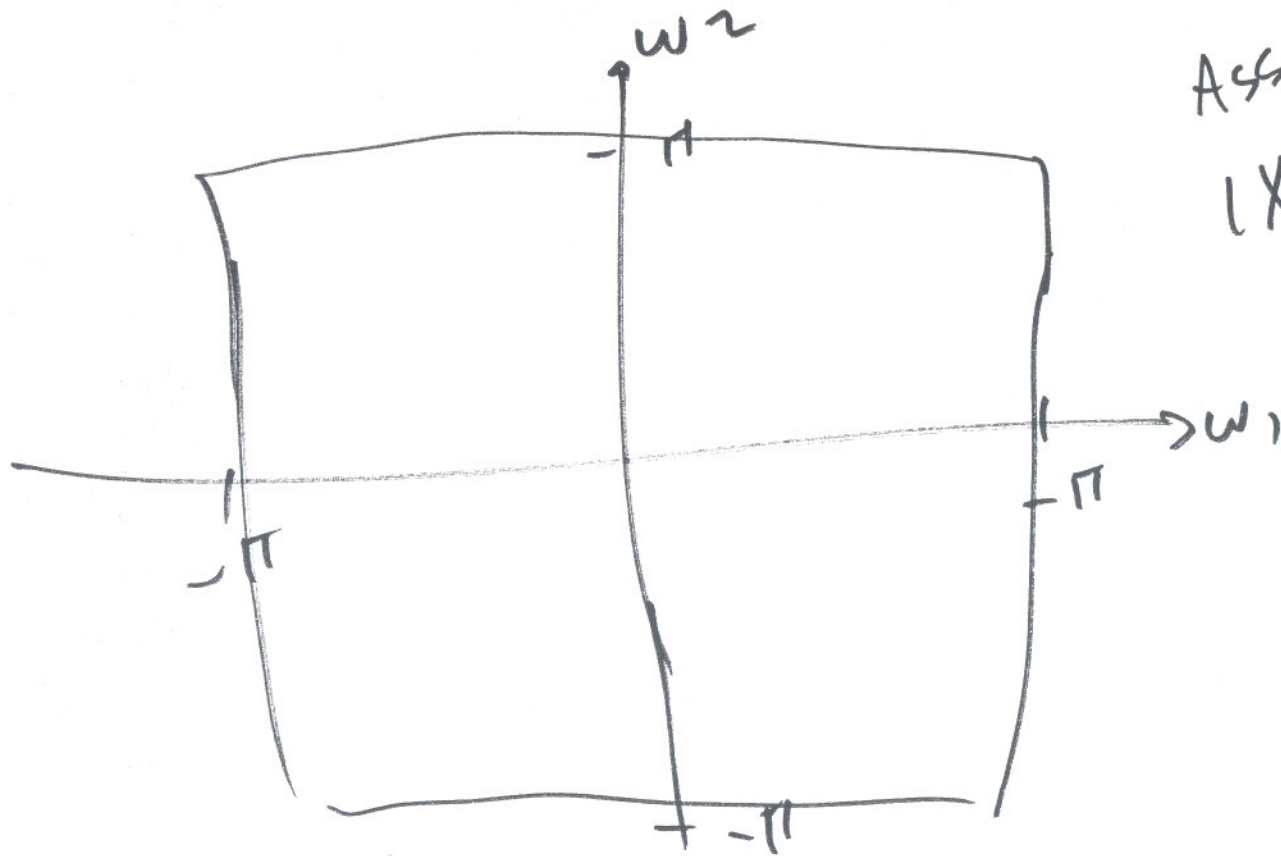
2^N

$$X(n)$$



same $r(n)$

2-D core FTM



Assume
 $|X(w_1, w_2)|^2$

$X(w_1, w_2) : N \times N$

\Rightarrow in 2-D, unlike 1-D, most 2D polynomials
are irreducible \Rightarrow non factorable.

1982, M. Hayes: If $x(n_1, n_2)$ has irreducible associated polynomials, then all other $y(n)$ which have same F.T.M are equivalent to

x . Equivalent means:

$$y(n_1, n_2) \sim x(n_1, n_2) \text{ if}$$
~~$$y(n_1, n_2) = e^{j\theta} x(n_1, n_2)$$~~

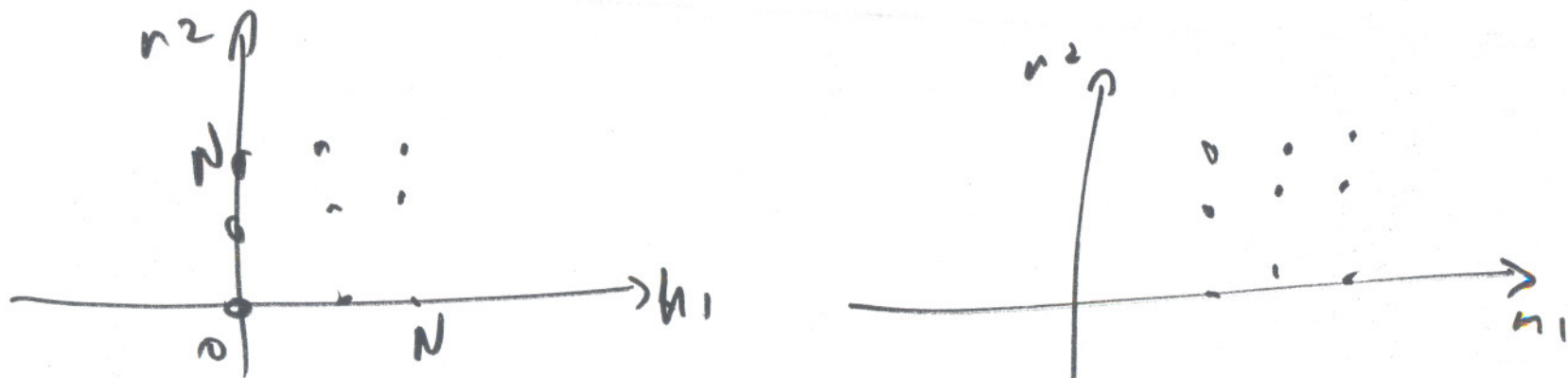
$$y(n_1, n_2) = e^{j\theta} x(k_1 \pm n_1, k_2 \pm n_2)$$

- observation ①: $x(n_1, n_2) \rightarrow |X(\omega_1, \omega_2)|^2$

$$e^{j\theta} x(n_1, n_2) \rightarrow |X(\omega_1, \omega_2)|^2$$

- observation ② $x(n_1, n_2) \rightarrow |X(\omega_1, \omega_2)|^2$

Shift by k_1, k_2 in space domain \Rightarrow same FTM



- Observation 3: Rotational Sector.

$$x(n_1, n_2) \longrightarrow |X(\omega_1, \omega_2)|^2$$

$$y(n_1, n_2) = x(N - n_1, N - n_2)$$

also has same FTM.

① Assume signal $x(n_1, n_2)$ is real

$$e^{j\theta}$$

$$, \theta = 0, \text{ or } \theta = \pi$$

- Assume signal is positive $\implies \theta = 0$

② - Assumption: extent of $x(n_1, n_2)$ is known.

③ Z.T of signal $x \implies$ irreducible.

Hayer \rightarrow Nail signal to factor of 2:

either $x(n_1, n_2)$ or $x(N-n_1, N-n_2)$

same FTM.

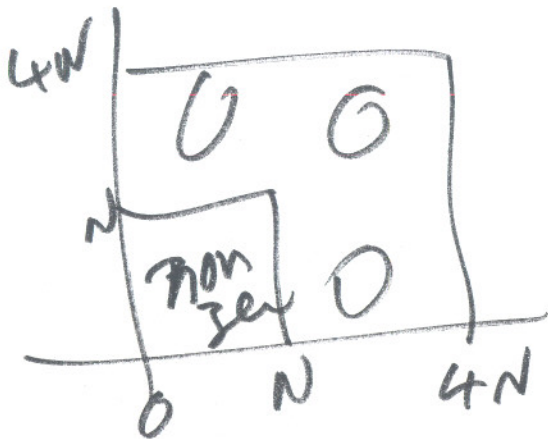
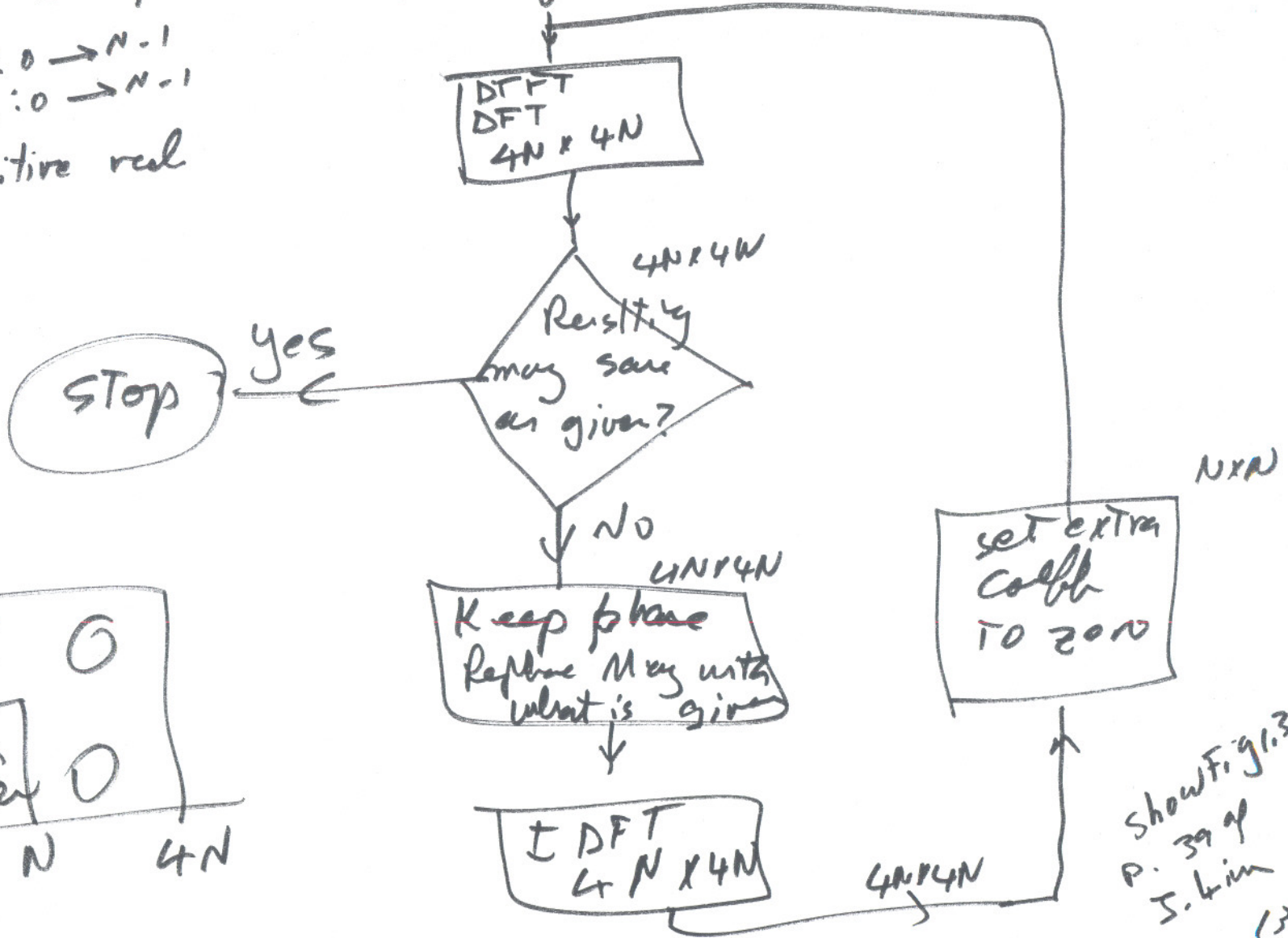
① ^x same / Real, ~~is~~ ~~ten~~ and positive.
Know extent.

FTM can be used to recover either x
or its reflection "uniquely"

Proposed iterative Alg

Assume know
 $4N \times 4N$ samples
of $|X(\omega_1, \omega_2)|^2$
 $N \times N$ $n_1: 0 \rightarrow N-1$
 $n_2: 0 \rightarrow N-1$
 X is positive real

Initial guess $N \times N$



Show Fig. 1.32
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S. Kim
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