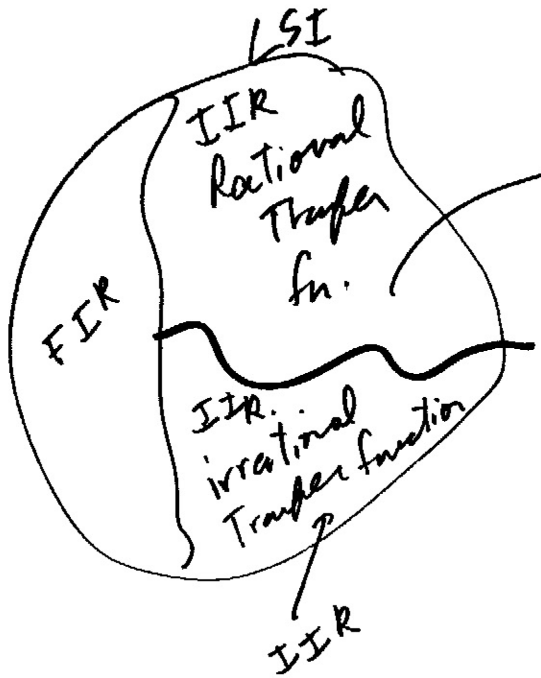
Filters (LSI systems)

Feb 8, 06

LSI $\longrightarrow$ Impulse response.



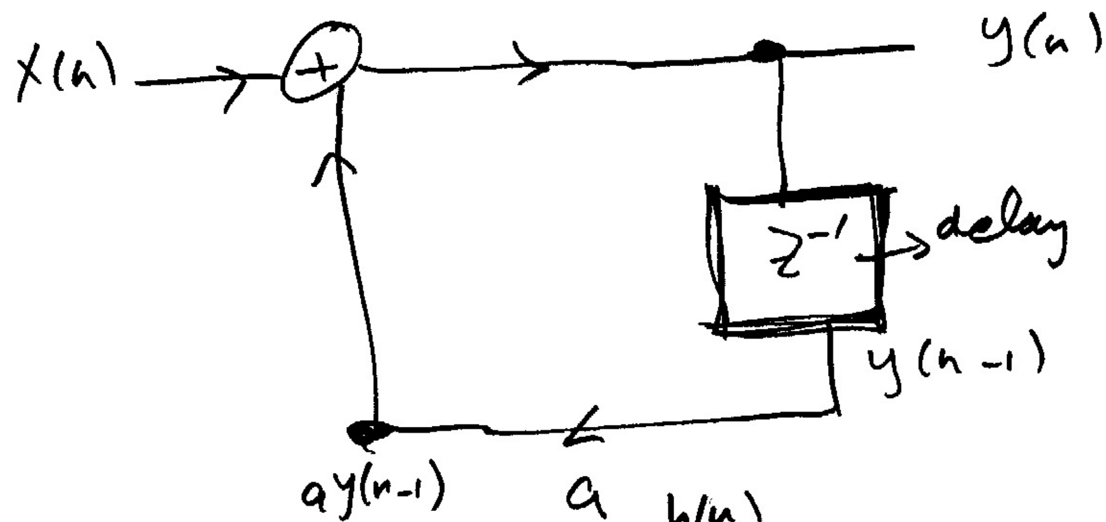
$$H(z) = \frac{P(z)}{Q(z)}$$ $\longrightarrow$ polynomial in z

Can be implemented using Difference Equ.
D.E.

Ex $H(z) = \frac{1}{1 - a z^{-1}} = \frac{Y(z)}{X(z)} \Longrightarrow X(z) = Y(z)\left[1 - a z^{-1}\right]$

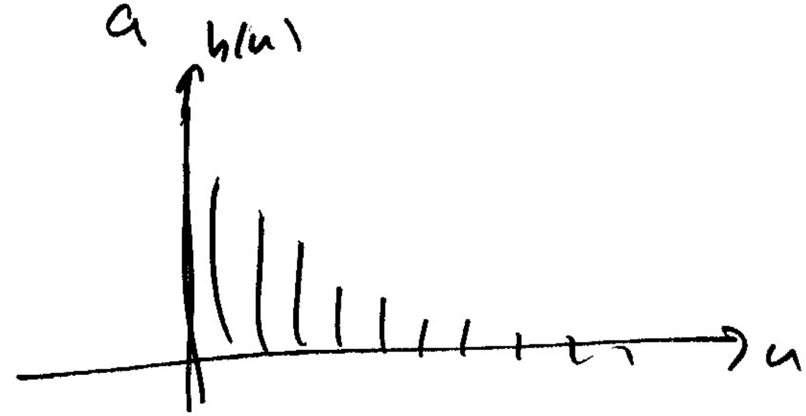
D.E $\longrightarrow y(n) = x(n) + a\, y(n-1)$

flow graph.



$$h(n) = a^n u(n)$$

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$x(n) \longrightarrow X(z)$$

$$x(n-1) \longrightarrow z^{-1} X(z)$$

2D- D.E.

$$H(z_1, z_2) = \frac{\text{Poly in } z_1, z_2}{\text{Poly in } z_1, z_2} = \frac{P(z_1, z_2)}{Q(z_1, z_2)}$$

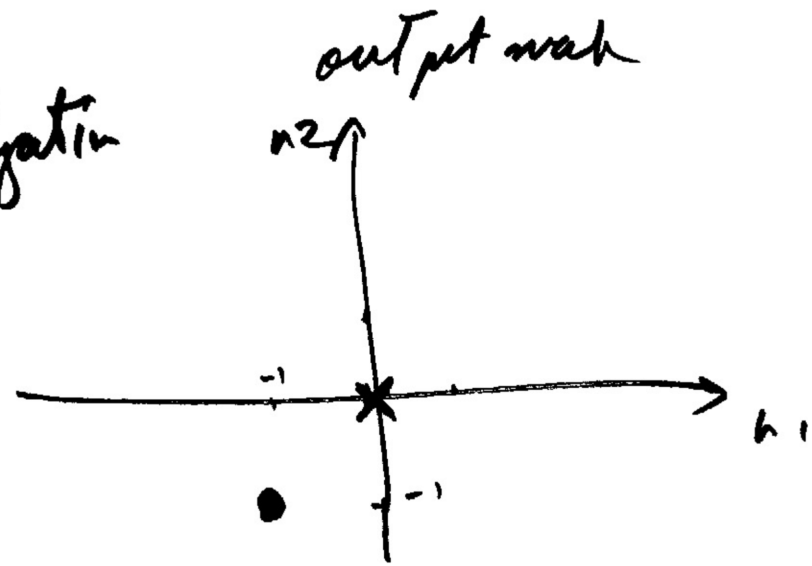
<u>Ex</u> $$H(z_1, z_2) = \frac{1}{1 - a z_1^{-1} z_2^{-1}} = \frac{Y(z_1, z_2)}{X(z_1, z_2)}$$

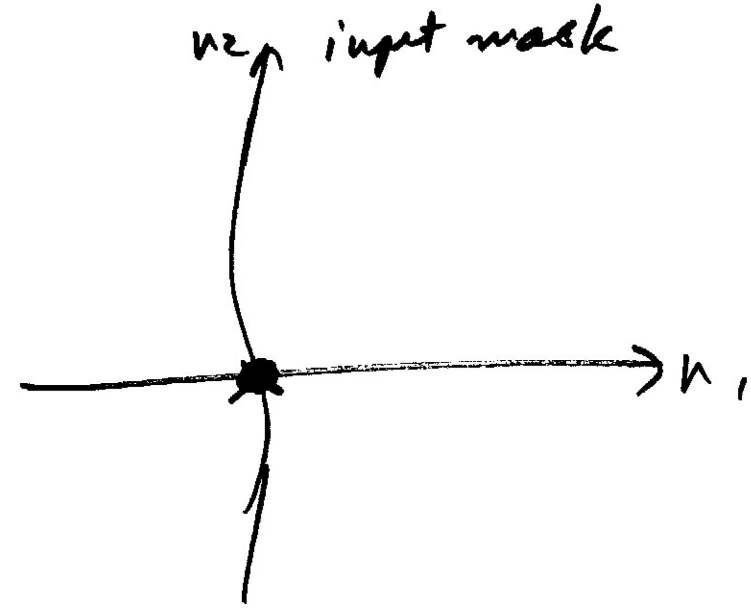
$$x(n_1, n_2) = y(n_1, n_2) - a\, y(n_1 - 1, n_2 - 1)$$

$$y(n_1, n_2) = x(n_1, n_2) + a\, y(n_1 - 1, n_2 - 1)$$

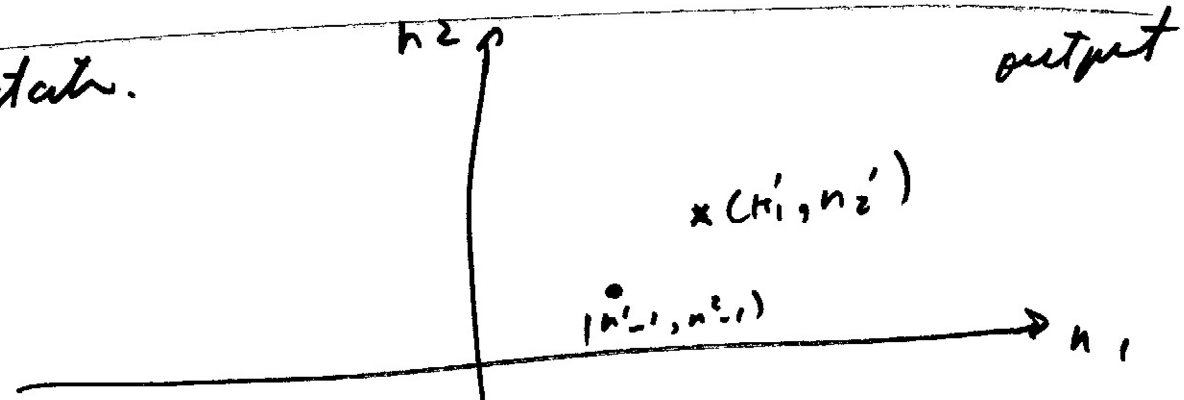
<u>Define</u> Input mask , output mask.

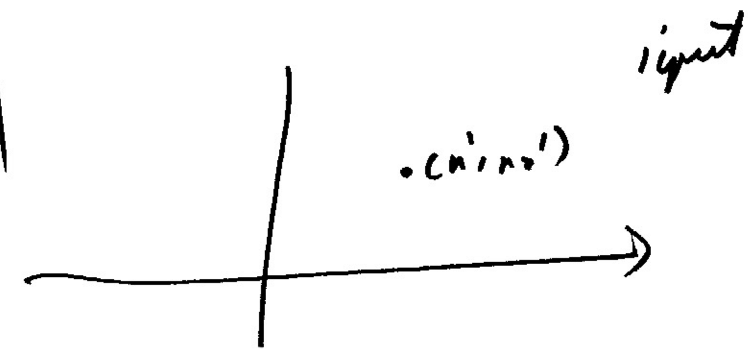
Dualgation (1)





Real life Computation.



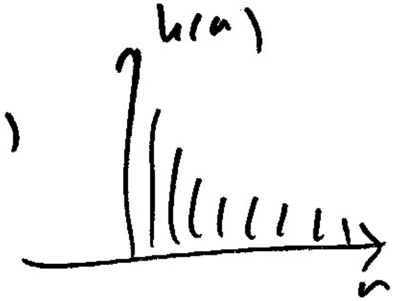


Different Realization of same. D.E.

$y(n) = x(n) + a\, y(n-1) \leftarrow$ D.E.

$H(z) = \dfrac{1}{1 - a z^{-1}}$

ROC $|z| > |a|$ $\rightarrow$ causal $h(n) = a^n u(n)$



ROC $|z| < |a|$ $\rightarrow$ anticausal system $h(n) = -a^n u(-n-1)$

$\boxed{H(z) + \text{ROC}} \rightarrow \boxed{h(n)}$

① Causal way

$y(n) \leftarrow x(n) + a\, y(n-1)$

Realization of D.E.

② Anticausal realization

$a\, y(n-1) \leftarrow y(n) - x(n)$

$y(n) \leftarrow \frac{1}{a} y(n+1) - \frac{1}{a} x(n+1)$

$$H(z) \xrightarrow[|z|>|a|]{ROC} y(n) \leftarrow x(n) + a\,y(n-1)$$

$$H(z) \xrightarrow[|z|<|a|]{ROC} y(n) \leftarrow \frac{1}{a} y(n+1) - \frac{1}{a} x(n+1)$$

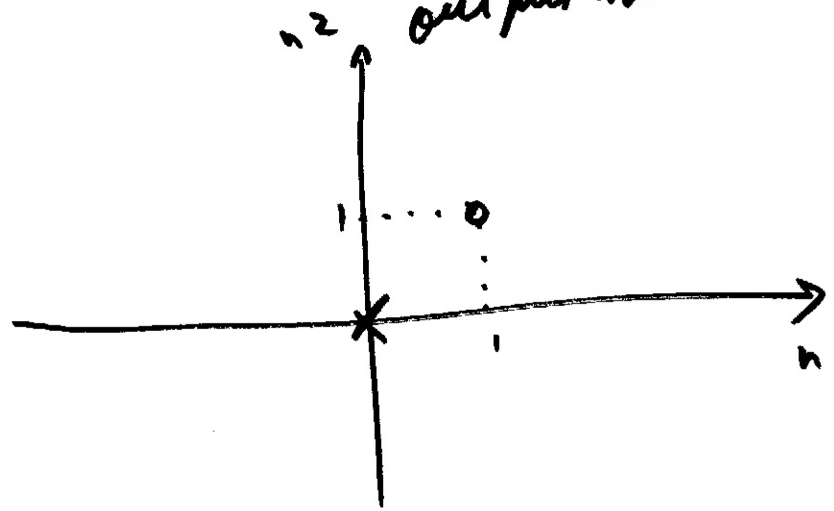
2D: 2 different realization of D.E.

① $y(n_1, n_2) \leftarrow x(n_1, n_2) + a\, y(n_1 - 1, n_2 - 1)$

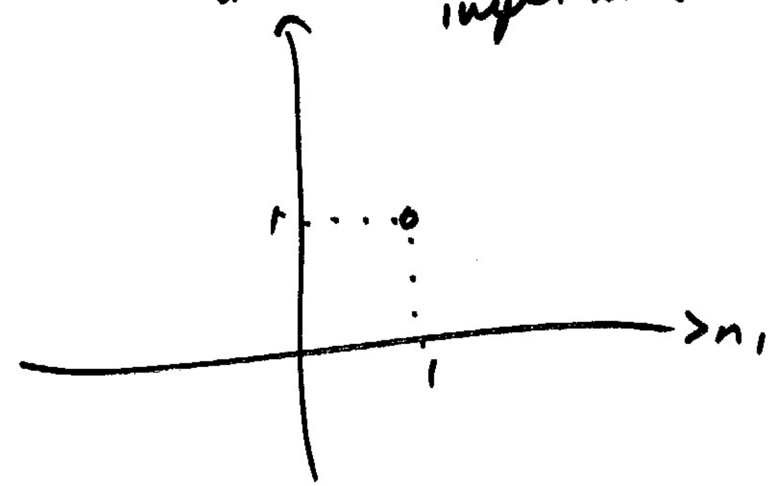
② $y(n_1, n_2) \leftarrow \frac{1}{a} y(n_1 + 1, n_2 + 1) - \frac{1}{a} x(n_1 + 1, n_2 + 1)$

②

output mask



input mask



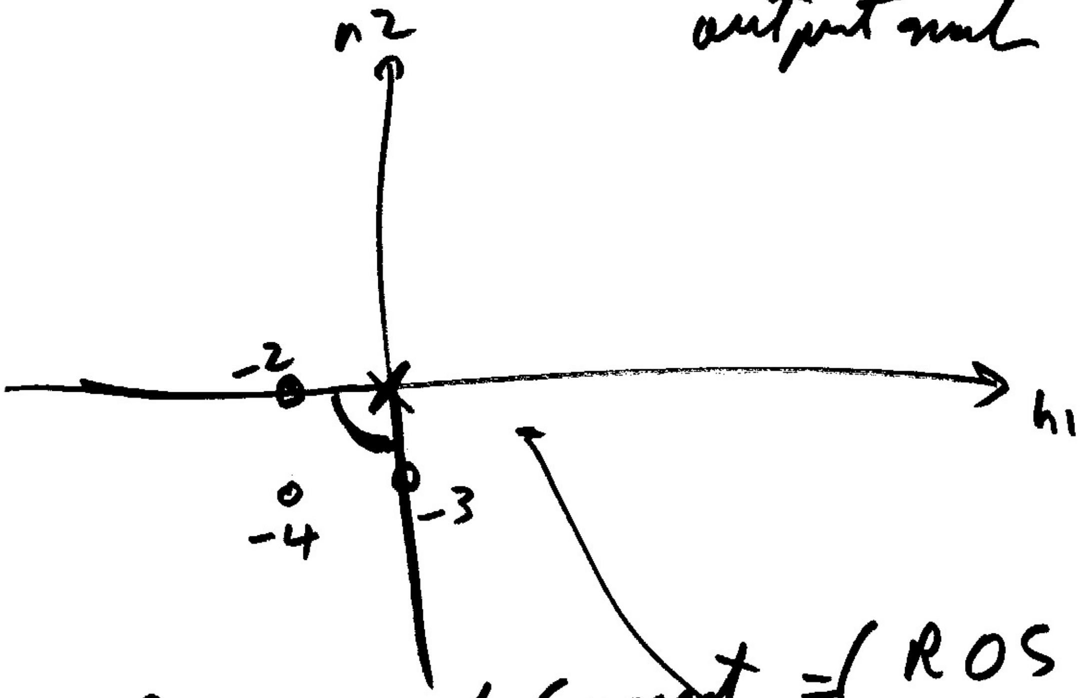
Recursive Computability

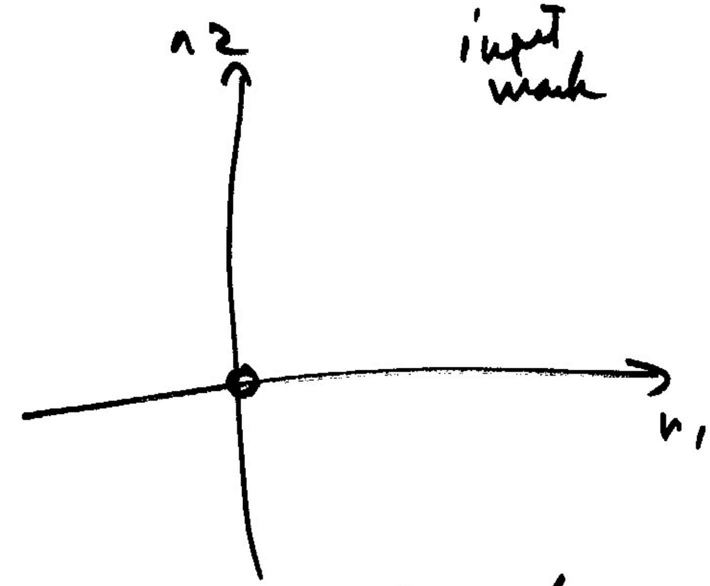
<u>Def</u>: sys is recursively computable if there is a sequential way of computing output points one after another

<u>Ex</u> $$y(n_1,n_2) + 2y(n_1-1,n_2) + 3y(n_1,n_2-1) + 4y(n_1-1,n_2-1) = x(n_1,n_2)$$

4 possible realization of D.E.

(1) $$y(n_1,n_2) \longleftarrow -2y(n_1-1,n_2) - 3y(n_1,n_2-1) - 4y(n_1-1,n_2-1) + x(n_1,n_2)$$

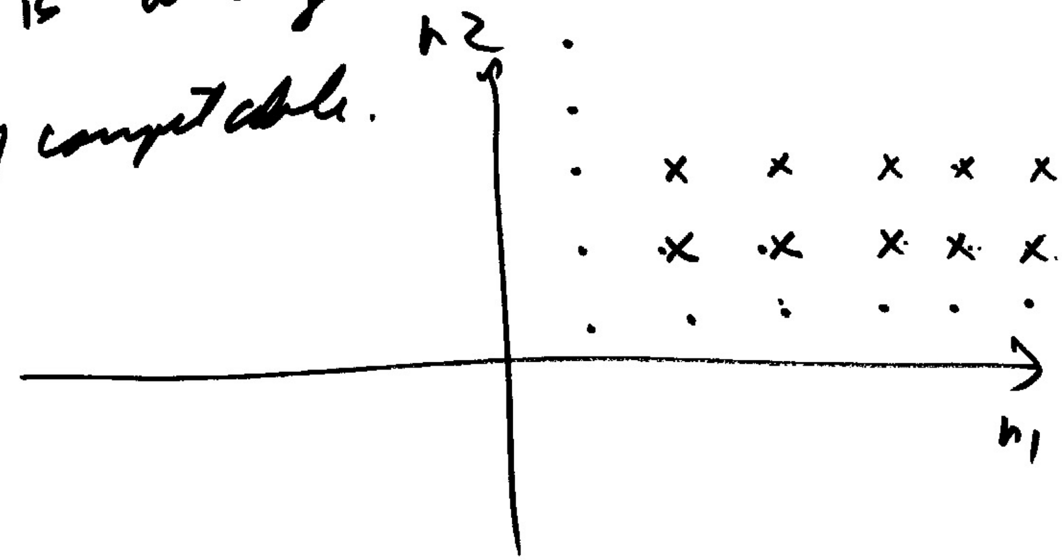




Region of Support = (ROS) of output mask.

ROS for this is wedge $\angle = 90 < 180°$

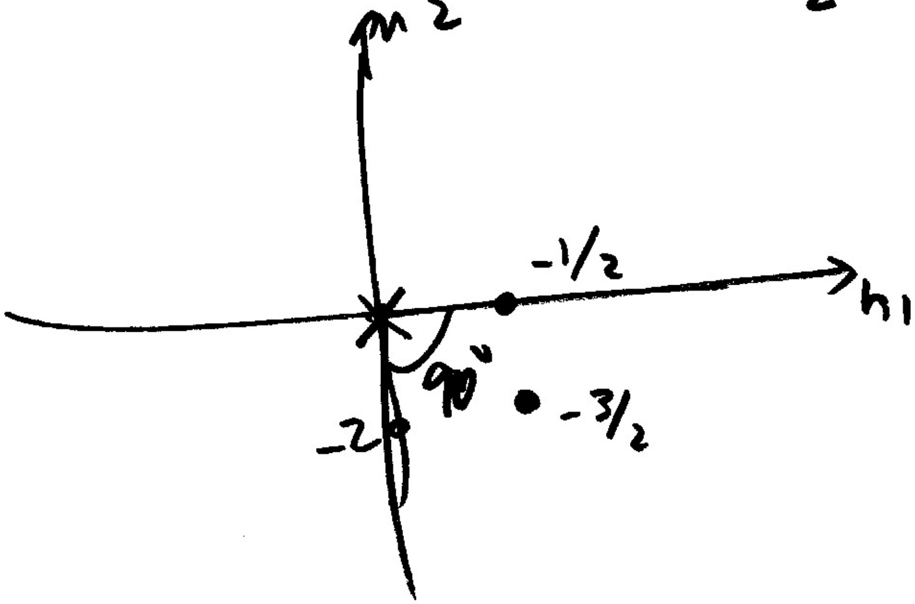
$\Rightarrow$ Recursively computable.



(2) Another realization of same D.E.

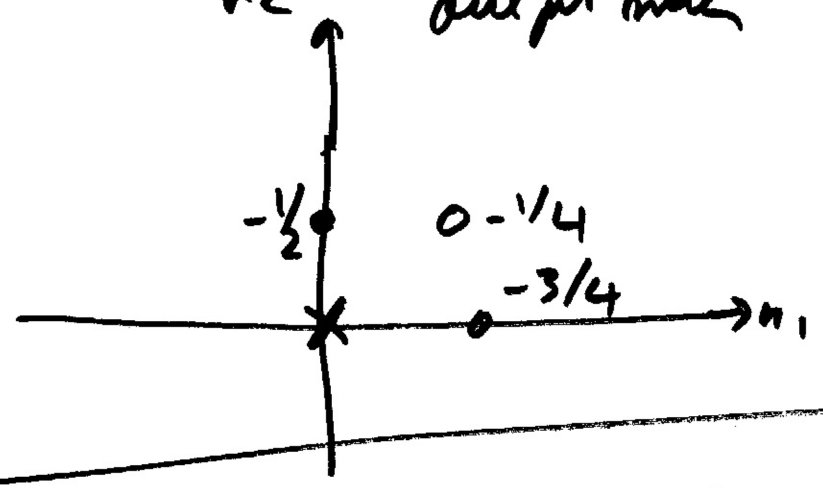
$$y(n_1,n_2) \longleftarrow -\frac{3}{2}y(n_1+1,n_2-1) - \frac{1}{2}y(n_1+1,n_2) - \frac{4}{2}y(n_1,n_2-1) + x(n_1+1,n_2)$$

output mask



(3) $y(n_1, n_2) \longleftarrow -\frac{1}{4} y(n_1+1, n_2+1) - \frac{2}{4} y(n_1, n_2+1) - \frac{3}{4} y(n_1+1, n_2) + x(n_1+1, n_2+1)$

output mask



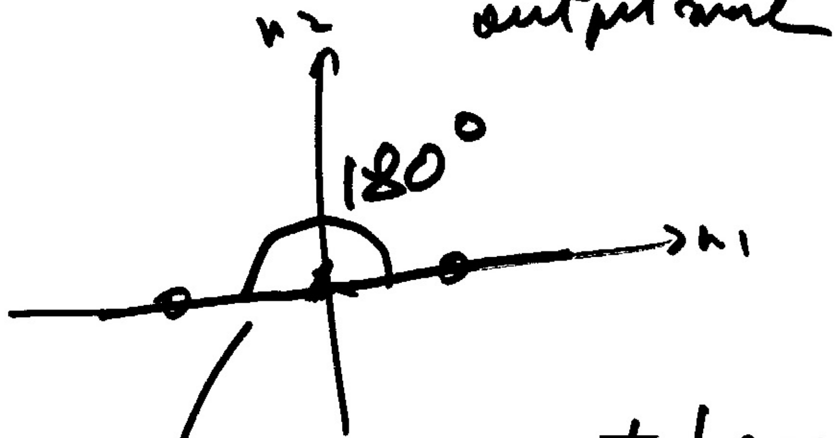
D.E

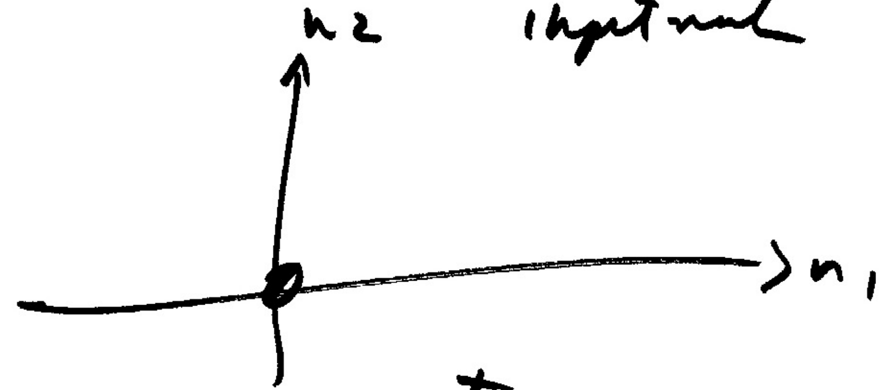
$$1\, y(n_1, n_2) + 2y(n_1+1, n_2) + 4y(n_1-1, n_2) = x(n_1, n_2)$$

Realization:

$$y(n_1, n_2) \longleftarrow -2y(n_1+1, n_2) - 4y(n_1-1, n_2) + x(n_1, n_2)$$

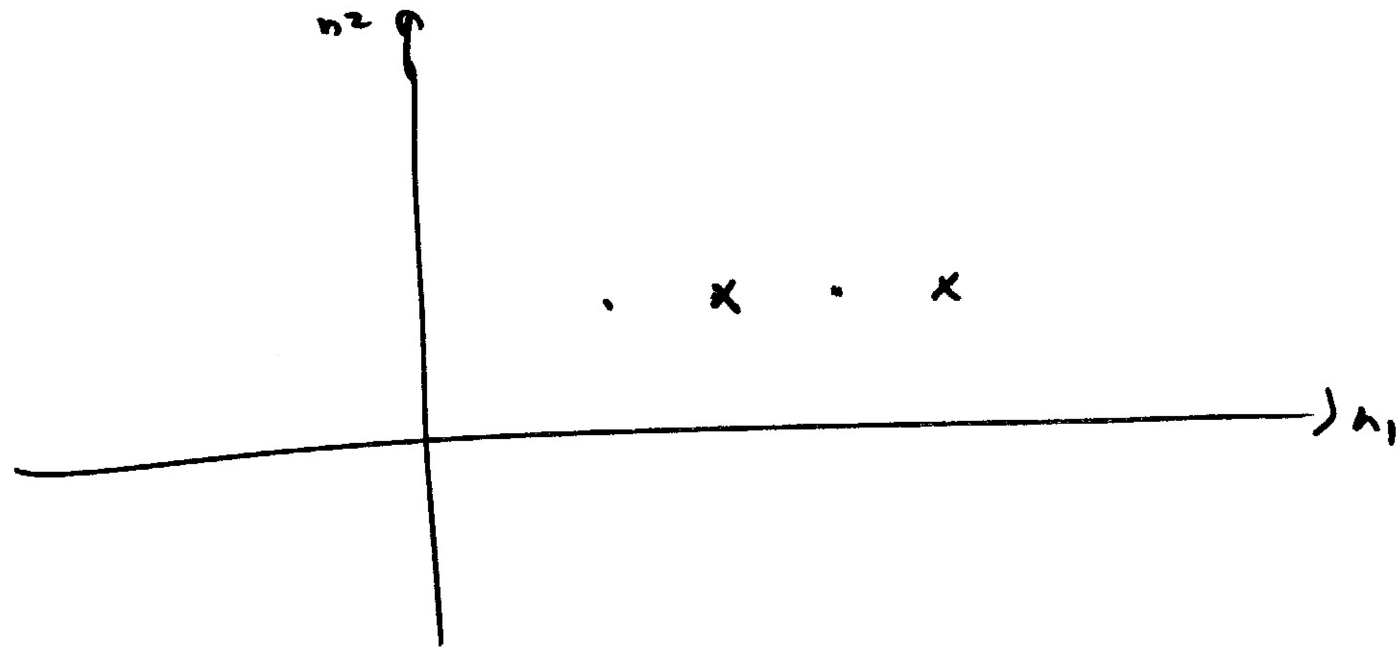
//

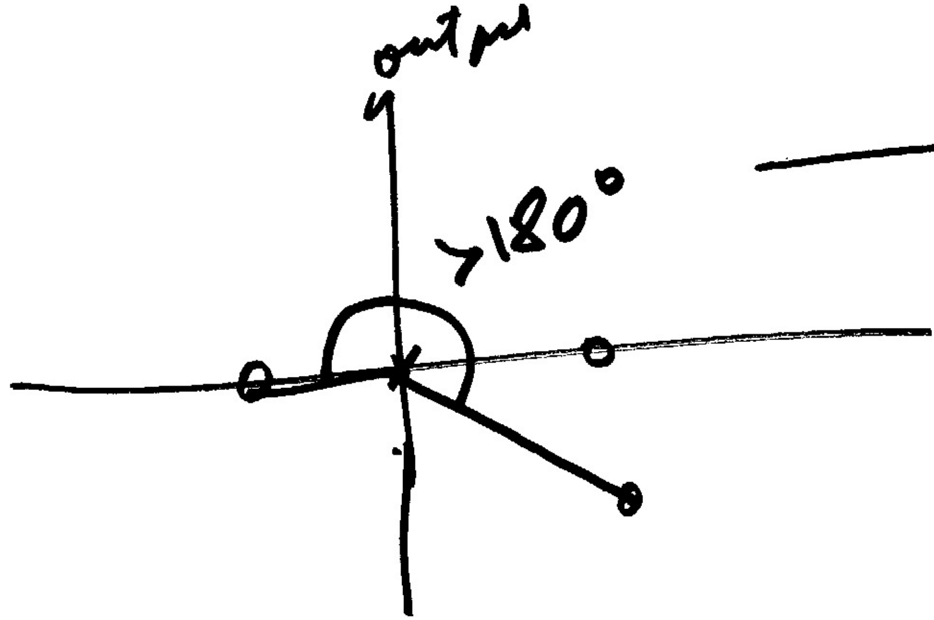




Does not have wedge support

$\Longrightarrow$ NOT recursively computable.





→ not recursively computable

If ROS of output mask is wedge ($<180°$)

⟹ Recursively computable

⟹ possible to find set of I.C. to compute output points one after another.

Choosing B.C.

Given D.E + Realization, how to choose B.C.?

Ex $y(n) = a\,y(n-1) + b\,x(n)$. $\qquad x(n) = \delta(n)$

Q Is This system?

A No: unique input $\not\Rightarrow$ unique output.

$y_1(n)$ is a soln.

$y_2(n) = y_1(n) + k a^n$ is also a soln.

Fix Add I.C. I.C. $y(-1) = y_0$

$n \geq 0 \qquad y(0) = a\,y(-1) + b\,\delta(0)$

$y(1)$:

$\vdots$

$$\boxed{n \geq 0 \qquad y(n) = a^n (a y_0 + b)}$$

$n \leq -2$

$$y(n-1) = \frac{1}{a} y(n) - \frac{b}{a} \delta(n)$$

$$y(-2) = \frac{1}{a} y(-1) - \frac{b}{a} \delta(-1)$$

$$\vdots$$

$$y(-3)$$

$$\vdots$$

$$\boxed{n \leq -1 \qquad y(n) = a^{n+1} y_0}$$

$$y(n) = a^n (a y_0 + b)\, u(n) + a^{n+1} y_0\, u(-n-1)$$

$$\boxed{y(n) = a^{n+1} y_0 + a^n b\, u(n)} \quad \forall n$$

Conclusion: Make I.C. to be zero

$$y_0 = 0$$

$$y(n) = a^n b\, u(n)$$

Linear Const. coeff. D.E.

$\Big\downarrow$ need I.C.

to make it correspond to a syst

$\Big\downarrow$ IC $= 0$

linear system

$\Big\downarrow$ SI?

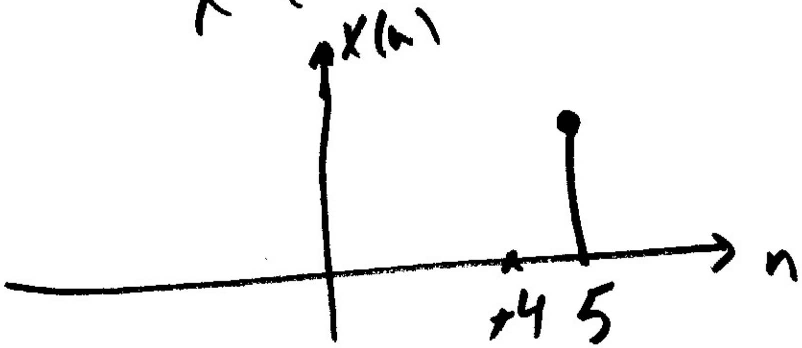
yes only if I.C. is chosen in accordance with the input.

$x(n) = b\delta(n) \longrightarrow$ IC. $y(-1) = 0$

$\downarrow$ results in S.D.

$x(n) = b\delta(n-5) \longrightarrow$ I.C. $y(-1) = 0$

$\downarrow$ Does not result in S.I.



IC: $y(4) = 0$

LCCDE + I.R.C $\longleftrightarrow$ LTI + Causal.

Initial Rest condi

LCCDE + F. RC. $\longleftrightarrow$ LTI anti causal

Final Rest cond

$\underline{Ex}$ $y(n) = a\,y(n-1) + b\,x(n)$

$x(n) = \delta(n-5)$

~~Causal~~

To make this LSI system

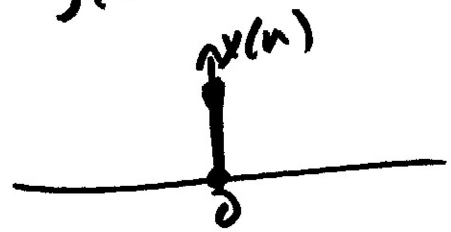
I.C. $y(4) = 0$ — To make it linear

S.I.

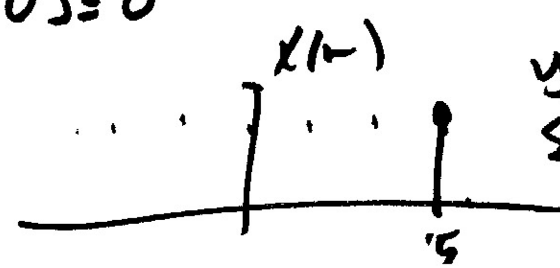
$y(n) \leftarrow a\,y(n-1) + b\,x(n)$

~~Anticausal.~~ $y(n-1) \leftarrow -\frac{1}{a}y(n) - \frac{b}{a}x(n)$

$x(n) = \delta(n)$

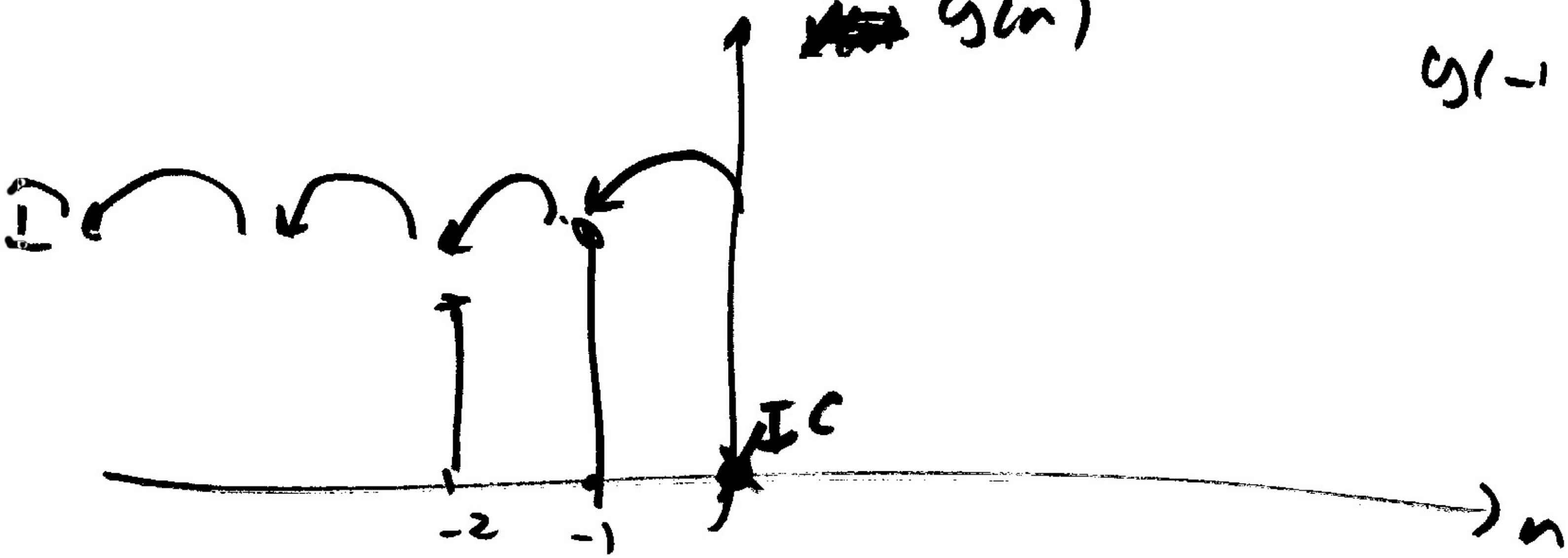


$y(0) = 0$



$y(5) = 0$

$\delta(n+5)$



$$y(-1) = \frac{1}{a}y(0) - \frac{b}{a}x(0)$$