Compressed Sensing Meets Machine Learning
- Classification of Mixture Subspace Models via Sparse Representation

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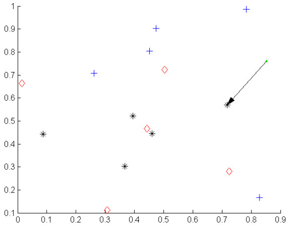
Mini Lectures in Image Processing (Part II), UC Berkeley

Nearest Neighbor Algorithm



1. **Training**: Provide labeled samples for K classes.
2. **Test**: Present a new sample
   - Compute its distances with all training samples.
   - Assign its label as the same label of the nearest neighbor.

Nearest Subspace

Estimation of single subspace models

- Suppose $R = [\mathbf{w}_1, \cdots, \mathbf{w}_d]$ is a basis for a d-dim subspace in $\mathbb{R}^D$.
- For $\mathbf{x}_i \in \mathbb{R}^D$, its coordinate in the new coordinate system: $\mathbf{w}^T \mathbf{x}_i = y_i \in \mathbb{R}$.

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$$\text{svd}(A) = USV^T, \text{ where } U \in \mathbb{R}^{D \times D}, S \in \mathbb{R}^{D \times n}, V \in \mathbb{R}^{n \times n}.$$

Denote $U = [U_1 \in \mathbb{R}^{D \times d}; U_2 \in \mathbb{R}^{D \times (D-d)}]$. Then $R = U_1^T$.

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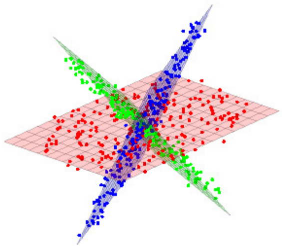
  Denote $U = [U_1 \in \mathbb{R}^{D\times d}; U_2 \in \mathbb{R}^{D\times(D-d)}]$. Then $R = U_1^T$.
- **Eigenfaces** If $\mathbf{x}_i$ are vectors of face images, the principal vectors $\mathbf{w}_i$ are then called Eigenfaces.

Nearest Subspace Algorithm



1. **Training**: For each of K classes, estimate its d-dim subspace model $R_i = [\mathbf{w}_1, \cdots, \mathbf{w}_d]$.
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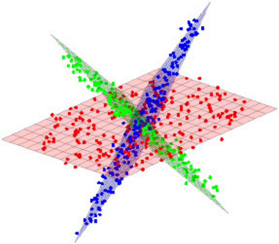

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Question

- Equation for computing distance from $\mathbf{y}$ to R_i?
- Why NS likely outperforms NN?

Noiseless ℓ^1-Minimization is a Linear Program

Recall last lecture: Compute **sparsest** solution **x** that satisfies

$$\tilde{\mathbf{y}} = \tilde{A}\mathbf{x} \in \mathbb{R}^d$$

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Formulate as **linear programming**:

1. Problem statement:

$$(P_1): \quad \mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ subject to } \tilde{\mathbf{y}} = \tilde{A}\mathbf{x} \in \mathbb{R}^d$$

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2. Denote $\Phi = (\tilde{A}, -\tilde{A}) \in \mathbb{R}^{d\times 2n}$, $\mathbf{c} = (1, 1, \cdots, 1)^T \in \mathbb{R}^{2n}$. We have the following linear program

$$\begin{aligned} \mathbf{w}^* \quad = \quad & \arg\min_{\mathbf{w}} \mathbf{c}^T\mathbf{w} \\ \text{subject to} \quad & \tilde{\mathbf{y}} = \Phi\mathbf{w} \\ & \mathbf{w} \geq 0 \end{aligned}$$

ℓ^1-Minimization Routines

- **Matching pursuit** [Mallat 1993]
  1. Find most correlated vector $\mathbf{v}_i$ in $\tilde{A}$ with $\mathbf{y}$: $i = \arg\max \langle \mathbf{y}, \mathbf{v}_j \rangle$.
  2. $\tilde{A} \leftarrow \tilde{A}^{\hat{i}}$, $x_i \leftarrow \langle \mathbf{y}, \mathbf{v}_i \rangle$, $\mathbf{y} \leftarrow \mathbf{y} - x_i \mathbf{v}_i$.
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  4. If $\|\mathbf{y} - B_m \mathbf{x}_m\|_2 < \epsilon$, stop.

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Matlab Toolboxes

- **SparseLab** by Donoho at Stanford.
- **cvx** by Boyd at Stanford.

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$$\tilde{\mathbf{y}} = \tilde{A}\mathbf{x}_0 + \mathbf{z} \in \mathbb{R}^d, \text{ where } \|\mathbf{z}\|_2 < \epsilon$$

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- Matlab toolboxes:
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Recall last lecture...

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$$\mathbf{x}_0 = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ s.t. } \tilde{\mathbf{y}} = \tilde{A}\mathbf{x}.$$

$\|\cdot\|_0$ simply counts the number of nonzero terms.

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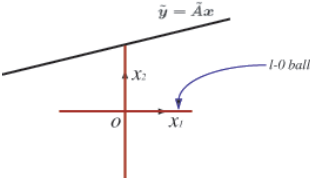
2. ℓ^0-Ball

- ℓ^0-ball is not convex.
- ℓ^0-minimization is NP-hard.



ℓ^1/ℓ^0 Equivalence

1. **Compressed sensing**: If $\mathbf{x}_0$ is *sparse enough*, ℓ^0-minimization is equivalent to

$$(P_1) \quad \min \|\mathbf{x}\|_1 \text{ s.t. } \tilde{\mathbf{y}} = \tilde{A}\mathbf{x}.$$

$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.

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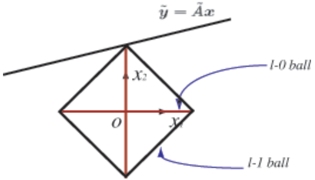
- Solution equal to ℓ^0-minimization.



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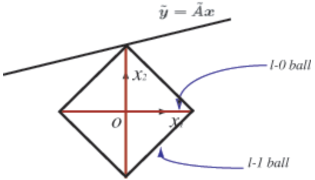

3. ℓ^1/ℓ^0 Equivalence: [Donoho 2002, 2004; Candes et al. 2004; Baraniuk 2006]
Given $\tilde{\mathbf{y}} = \tilde{A}\mathbf{x}_0$, there exists **equivalence breakdown point** (EBP) $\rho(\tilde{A})$, if $\|\mathbf{x}_0\|_0 < \rho$:
   - ℓ^1-solution is unique
   - $\mathbf{x}_1 = \mathbf{x}_0$

ℓ^1/ℓ^0 Equivalence in Noisy Case

Reconsider ℓ^2-bounded linear system

$$\tilde{\mathbf{y}} = \tilde{A}\mathbf{x}_0 + \mathbf{z} \in \mathbb{R}^d, \text{ where } \|\mathbf{z}\|_2 < \epsilon$$

Is corresponding ℓ^1 solution stable?

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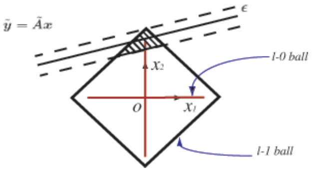

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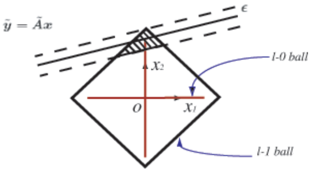

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$$\|\mathbf{x}_1 - \mathbf{x}_0\|_2 \leq C \cdot \epsilon$$

Compressed Sensing in the View of Convex Polytopes

For the rest of the lecture, investigate the estimation of EBP ρ.

To simplify notations, assume underdetermined system $\mathbf{y} = A\mathbf{x} \in \mathbb{R}^d$, where $A = \mathbb{R}^{d \times n}$.

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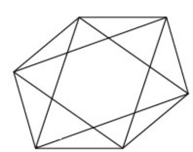
Definition (Quotient Polytopes)

Consider the convex hull P of the $2n$ vectors $(A, -A)$. P is called the **quotient polytope** associated to A.



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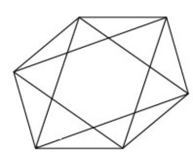
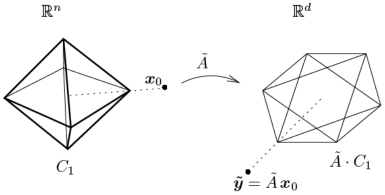


Definition (k-Neighborliness)

A quotient polytope P is called k-**neighborly** if whenever we take k vertices not including an antipodal pair, the resulting vertices span a face of P.
(Above example is 1-neighborly.)

ℓ^1-Minimization and Quotient Polytopes

Why ℓ^1-minimization is related to quotient polytopes?



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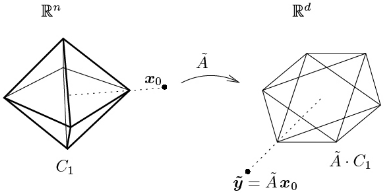

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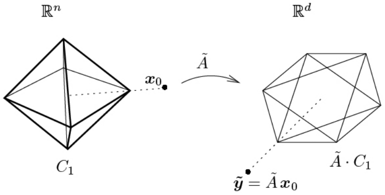

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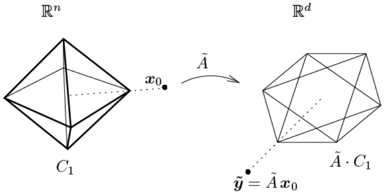

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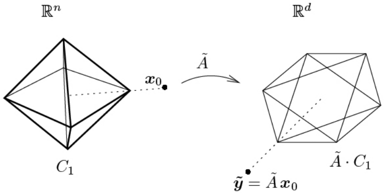

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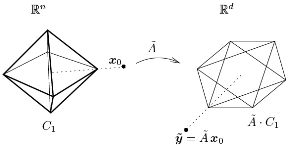
Theorem (ℓ^1/ℓ^0 equivalence condition)

If the quotient polytope P associated with A is k-neighborly, for $\mathbf{y} = A\mathbf{x}_0$ with $\mathbf{x}_0$ to be k-sparse, then $\mathbf{x}_0$ is the unique optimal solution of the ℓ^1-minimization.

Let's prove the theorem together



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- **vertices v** $\in$ vert(P).

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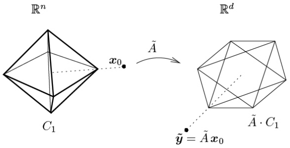

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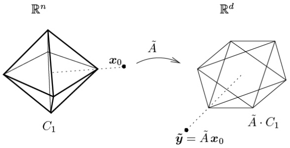

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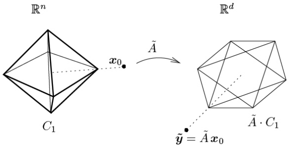

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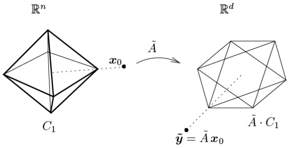
(1) vert$(P) = \mathcal{F}_0(P)$. (2) $P = \text{conv}(\text{vert}(P))$

- $F \in \mathcal{F}_k(P)$ is a **simplex** if $\#\text{vert}(F) = k + 1$.

Properties

$$\text{vert}(AC) \subset A\text{vert}(C); \quad \mathcal{F}_l(AC) \subset A\mathcal{F}_l(C).$$

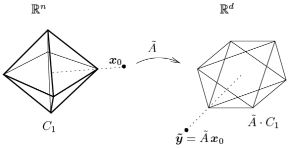
Two Fundamental Lemmas



Lemma (Alternative Definition of k-neighborliness)

*Suppose a centrosymmetric polytope $P = AC$ has $2n$ vertices. Then P is k-**neighborly** iff for any $l = 0, \cdots, k-1$ and $F \in \mathcal{F}_l(C)$, $AF \in \mathcal{F}_l(AC)$.*

Two Fundamental Lemmas



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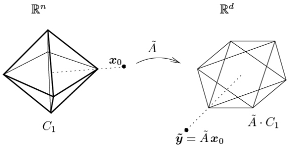
Lemma (Unique Representation on Simplices)

Consider an l-simplex $F \in \mathcal{F}_l(P)$. Let $\mathbf{x} \in F$. Then

1. $\mathbf{x}$ *has a* **unique representation** *as a linear combination of the vertices of P.*
2. *This representation places* **only nonzero weight** *on vertices of F.*

Proof of the Theorem



Suppose P is k-neighborly, and $\mathbf{x}_0$ is k-sparse. WLOG, scale and assume $\|\mathbf{x}_0\|_1 = 1$.

1. $\mathbf{x}_0$ is k-sparse $\Rightarrow \exists F \in \mathcal{F}_{k-1}(C)$, $\mathbf{x}_0 \in F$ and $\mathbf{y} \doteq A\mathbf{x}_0 \in AF$.
2. $P = AC$ is k-neighborly $\Rightarrow AF \in \mathcal{F}_{k-1}(AC)$ is a simplex.
3. By (1) and (2), $\mathbf{y} \in AF$ has a unique representation with at most k nonzero weights on the vertices of AF.
4. Hence, $\mathbf{x}_1$ given by ℓ^1-minimization is unique, and $\mathbf{x}_1 = \mathbf{x}_0$.

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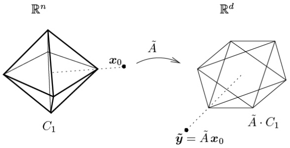

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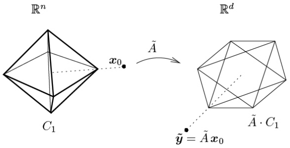
Corollary [Gribonval & Nielsen 2003]

Assume for all columns of matrix A, $\|\mathbf{v}_i\|_2 = 1$, and for all $i \neq j$, $\langle \mathbf{v}_i, \mathbf{v}_j \rangle \leq \frac{1}{2k-1}$, then $P = AC$ is k-neighborly.

Last question: Why random projection works well in ℓ^1-minimization?



Revisit the above corollary

Define **coherence** $M \doteq \max_{i \neq j} |\langle \mathbf{v}_i, \mathbf{v}_j \rangle|$, then $\text{EBP}(A) > \frac{M^{-1}+1}{2}$.

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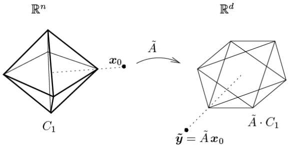

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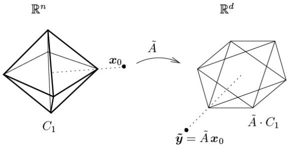

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1. in HD space $\mathbb{R}^d$, two randomly generated unit vectors have small coherence M.
2. Further define **coherence** of two dictionaries $M(A, B) = \max_{\mathbf{u} \in A, \mathbf{v} \in B} |\langle \mathbf{u}, \mathbf{v} \rangle|$.
    - $\frac{1}{\sqrt{d}} \leq M(A, B) \leq 1$.
    - Let T be the spike basis in time domain, F be the Fourier basis, then $M(T, F) = \frac{1}{\sqrt{d}}$. **Max incoherence!**
    - Random projection R in general is not coherent with most traditional bases.

Conclusion

1. Classical classifiers: NN & NS.
2. Linear and quadratic ℓ^1 solvers.
3. Stability of ℓ^0/ℓ^1 equivalence with bounded error.
4. Computation of equivalence breakdown point (EBP) via quotient polytopes.