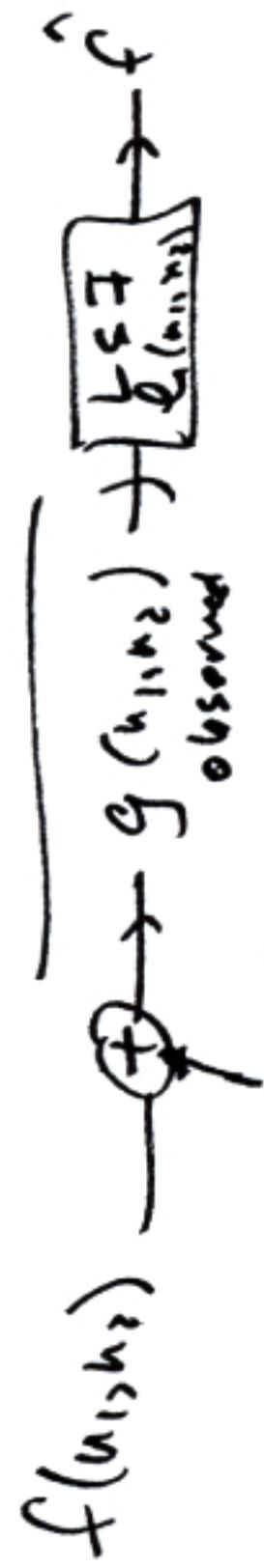


## Weinertter



more generally



Sample of a few mean stationary random process.

$w = " "$

$f, w$  are independent of each other

Define metric :  $\hat{f}$  as close as possible to  $f$ .

$$\mathbb{E} \left[ (f - \hat{f})^2 \right] \rightarrow \text{linear least squares error}$$

Orthogonality principle :

least square error is achieved when  
error orthogonal to "observation"

$$e = f - \hat{f} \perp g \Rightarrow$$

$f - \hat{f}$  must be uncorrelated with  $g$ .

$$g \neq h \Rightarrow f$$

Find  $\hat{g}$ .

$$\Rightarrow E \left[ \hat{f}(u_1, u_2) g(u_1, u_2) \right] = E \left[ \hat{f}(u_1, u_2) \hat{g}(u_1, u_2) \right]$$

$$f(u_1, u_2), \quad g(u_1, u_2).$$

$$E \left[ \left( f(u_1, u_2) - \hat{f}(u_1, u_2) \right) \cdot g(u_1, u_2) \right] = 0$$

(goal): Design  $\hat{g}$  so that  $f - \hat{f} \perp g$ .

$$f \xrightarrow{\oplus} g \rightarrow \hat{f}$$

$$E[f(n_1, n_2)g(m_1, m_2)] = \\ E \left[ \left( \sum_{k_1} h(k_1, k_2) g(k_1 - k_1, n_2 - k_2) \right) g(m_1, m_2) \right]$$

$\Rightarrow$  Cross Correlation =  $R$ .

$$\text{Cross Correlation} = R. \\ \text{Cross Correlation} = R_g(n_1 - m_1, n_2 - m_2) \\ = \sum_{k_1} \sum_{k_2} h(k_1, k_2) R_g(n_1 - k_1, n_2 - k_2)$$

$\downarrow$  auto-correlation of  $g$  with itself.

$$R_{fg}(n_1, n_2) = \sum_{k_1} \sum_{k_2} h(k_1, k_2) R_g(n_1 - k_1, n_2 - k_2)$$

$$R_{fg}(n_1, n_2) = h(n_1, n_2) * R_g(n_1, n_2)$$

$\downarrow F.T.$

$$P_{fg}(w_1, w_2) = \frac{H(w_1, w_2)}{P_g(w_1, w_2)}$$

$$\underbrace{H(w_1, w_2)}_{\text{filter}} = \frac{\underbrace{P_g(w_1, w_2)}_{\text{filter}}}{P_{fg}(w_1, w_2)}$$

$$R_{fg}(n_1, n_2) \stackrel{\Delta}{=} E \left[ f(k_1, k_2) g(k_1 - n_1, k_2 - n_2) \right]$$

$$g = f * w.$$

$$R_{fg}(n_1, n_2) = E \left[ f(k_1, k_2) \left( f(k_1 - n_1, k_2 - n_2) * w(k_1, n_1, k_2, n_2) \right) \right]$$

$$R_{fg}(n_1, n_2) = E \left[ f(k_1, k_2) f(k_1 - n_1, k_2 - n_2) \right] + \\ E \left[ f(k_1, k_2) w(k_1 - n_1, k_2 - n_2) \right]$$

$\partial$   $\nwarrow$   
 $f, w$  are i.i.d.

$$R_{fg}(n_1, n_2) = R_f(n_1, n_2)$$

$$R_f(n_1, n_2) = h(n_1, n_2) \xrightarrow{\text{F.T.}} \rho_g(n_1, n_2)$$

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$$\frac{P_f(\omega_1, \omega_2)}{H(\omega_1, \omega_2)} = \frac{P_g(\omega_1, \omega_2)}{P.g(\omega_1, \omega_2)}$$

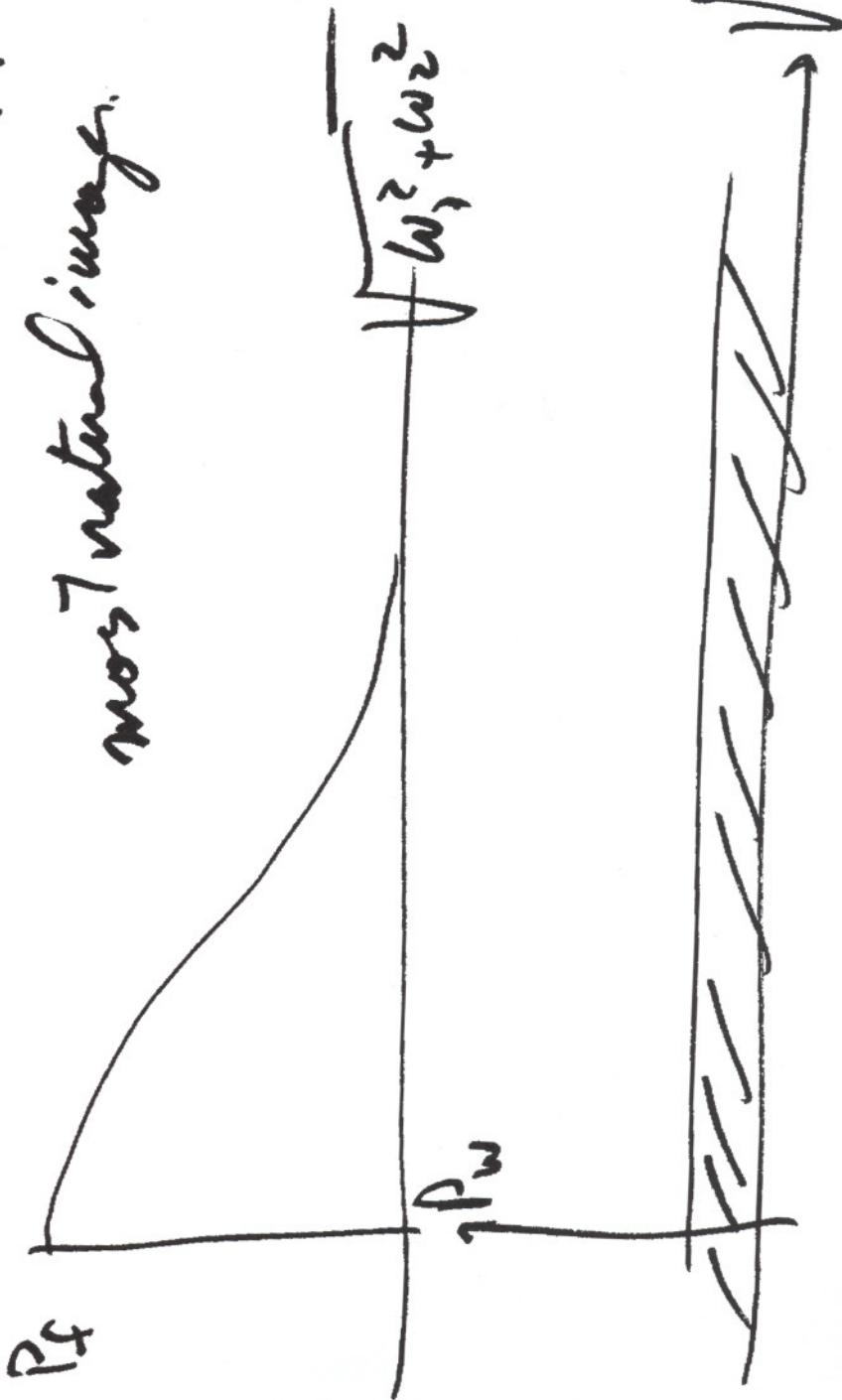
Weine

$$\begin{aligned}
R_g(n_1, n_2) &= E \left[ g(k_1, k_2) g(k_{1-n_1}, k_{2-n_2}) \right] \\
&= E \left[ f(k_1, k_1) + w(k_1, k_1) \right] \left( f(k_{1-n_1}, k_{1-n_1}) + \right. \\
&\quad \left. w(k_{1-n_1}, k_{2-n_2}) \right) \\
&= E \left[ f(k_1, k_1) f(k_1 - n_1, k_2 - n_2) \right] + \cancel{f_w} \\
&= E \left[ f(k_1, k_1) f(k_1 - n_1, k_2 - n_2) \right] + \cancel{f_w} \\
&= E \left[ f(k_1, k_2) \xrightarrow{\text{if } k_1 = k_2} 0 \right] + \cancel{f_w} \\
&\quad + E \left[ w(k_1, k_2) \cancel{f(k_1 - n_1, k_2 - n_2)} \right] + \cancel{f_w} \\
&\quad + E \left[ w(k_1, k_2) \cancel{f(k_1 - n_1, k_2 - n_2)} \right] + \cancel{f_w} \\
&\quad + E \left[ w(k_1, k_2) w(k_{1-n_1}, k_{2-n_2}) \right] + \cancel{f_w} \\
R_g &= P_f(n_1, n_2) + P_w(bw_1, bw_2)
\end{aligned}$$

$$H(\omega_1, \omega_2) = \frac{P_f(\omega_1, \omega_2)}{P_f(\omega_1, \omega_2) + P_W(\omega_1, \omega_2)}$$

bedieneinfacher.

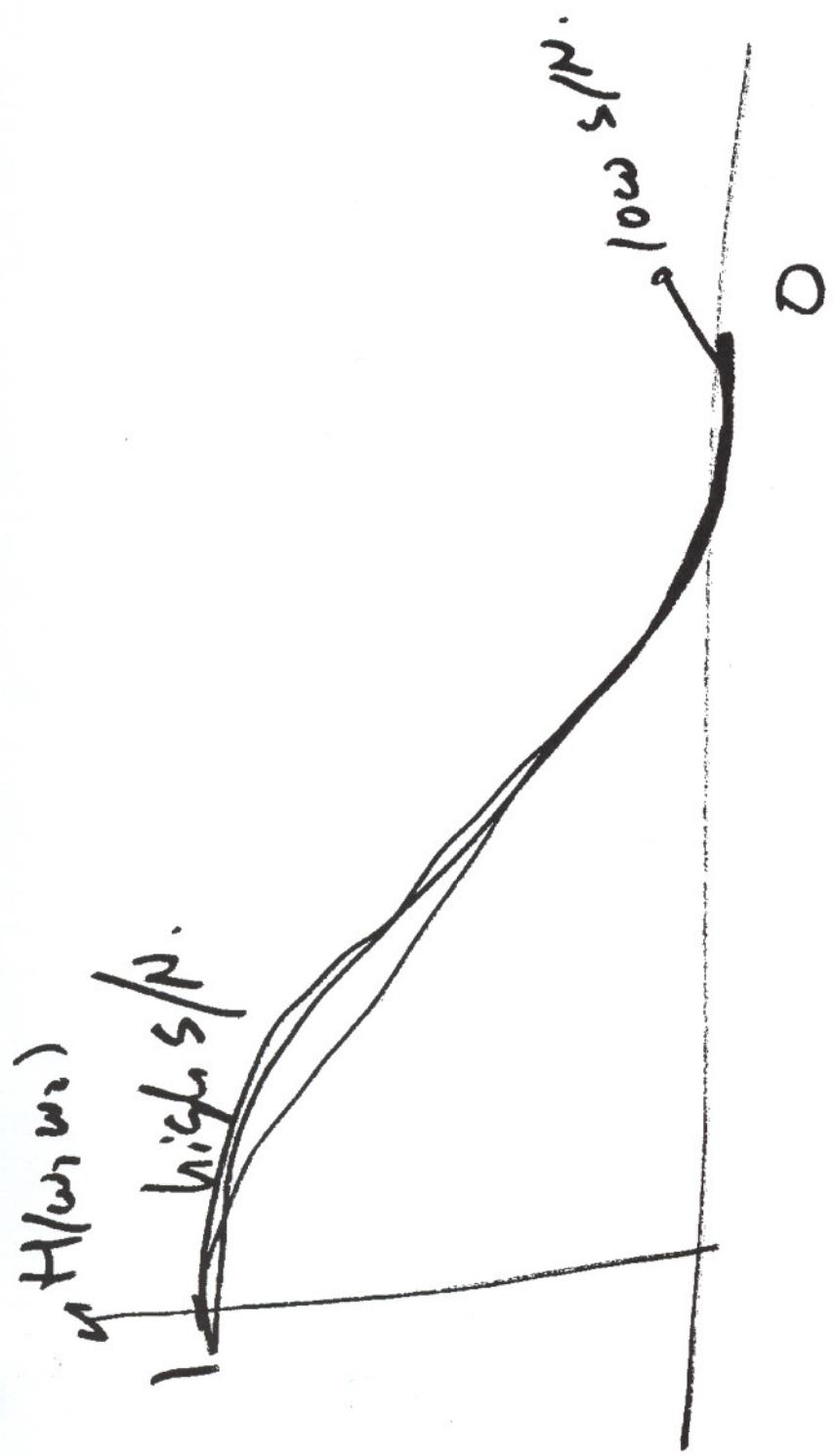
most natural image.



Consider 2 cases:

①  $P_f \gg P_w \Rightarrow H(w_1, w_2) \approx 1$   
denominator  $\approx P_f \Rightarrow$   
 $\Rightarrow$  signal gets thru.

②  $P_f \ll P_w \Rightarrow$   
 $H(w_1, w_2) \approx \frac{P_f}{P_w} \approx 0$   
 $\Rightarrow$  nothing gets thru.



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## Problem

How to find  $A$ ,  $P_w$ ?

(1)  $f$  is just a sample of P.P.

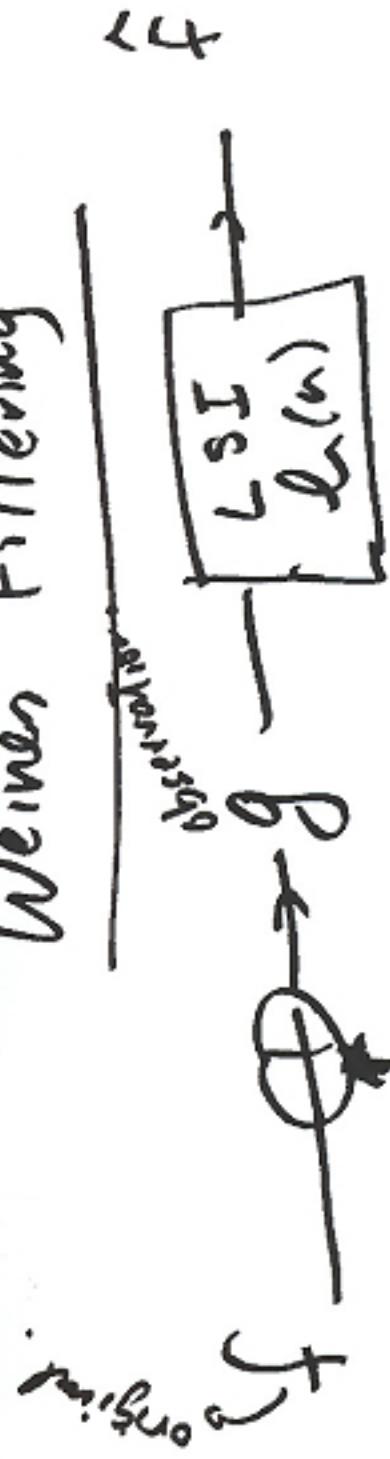
Averg.  $|F_i(w_1, w_2)|$  over a lot of natural images.



(2) Assume model  $P_f$ . est. univariate parameters of  $P_f$  by observing  $y$ .

$\Rightarrow$  Another problem: Images are not really locally stationary, locally stable,

## Weiner Filtering

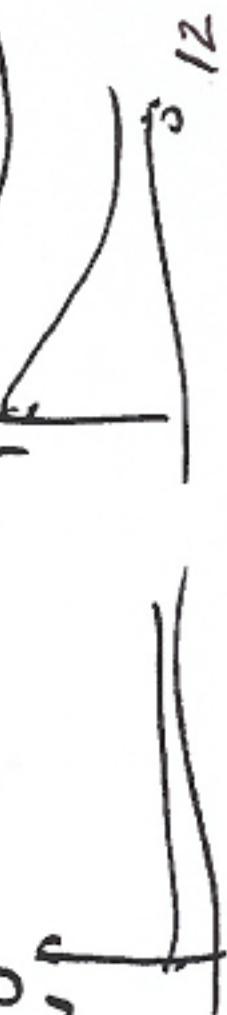


$\min_{L(m)} E[(f - \hat{f})^2]$  via minimizer  
 choose  $L(m)/S.N.$

$$H(\omega_1, \omega_2) = \frac{P_f(\omega_1, \omega_2)}{P_e(\omega_1, \omega_2) + P_w(\omega_1, \omega_2)}$$



Images are only locally & T autonomy  
 & not globally.

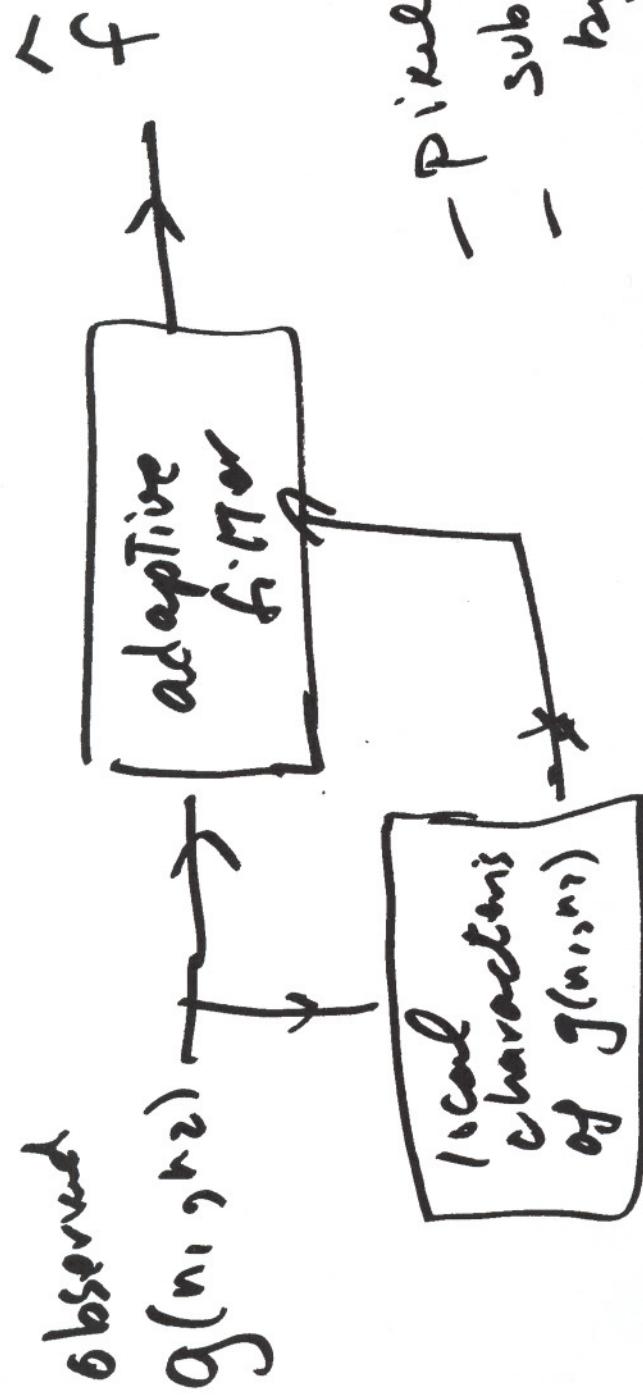


$$\sqrt{\omega_1^2 + \omega_2^2}$$

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## Adaptive Wiener Filter:

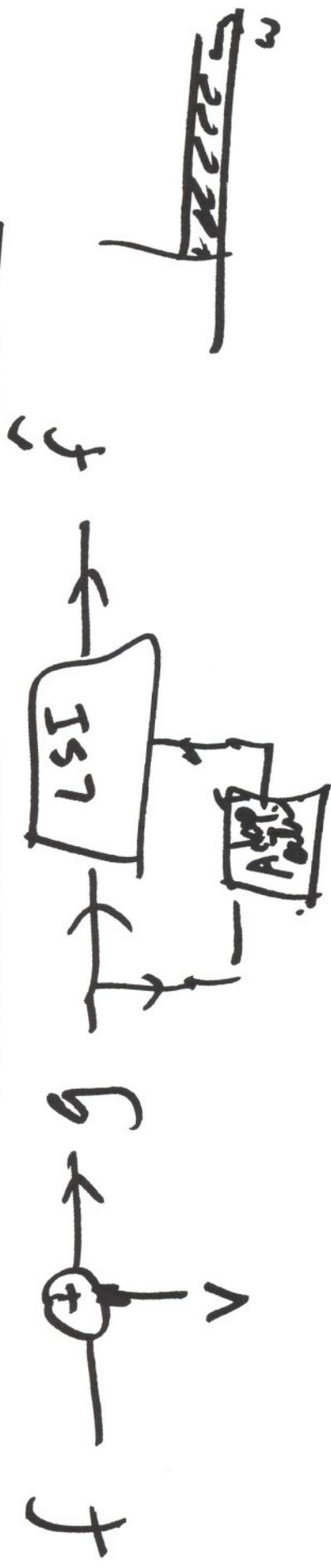
Basic idea :  $\rightarrow$  image stationary.



- Pixel by pixel
- subimage
- by subimage
- block by block.

## One possible Adaptive filter

---



- Assume, noise  $w$  is white, zero mean, variance  $\sigma_w^2$

$$P_w(\omega_1, \omega_2) = \sigma_w^2$$

- Assume signal satisfies the following model:

$$f(n_1, n_2) = m_f + g_f w(n_1, n_2)$$

- $w$  is a white noise, zero mean, unit variance process.

$\Rightarrow$  Wiener filter  $H(\omega_1, \omega_2) =$   
assuming  $g_f$  even input.

$$P_f(\omega_1, \omega_2)$$

$$\frac{P_f(\omega_1, \omega_2)}{P_f(\omega_1, \omega_2) + P_w(\omega_1, \omega_2)}$$

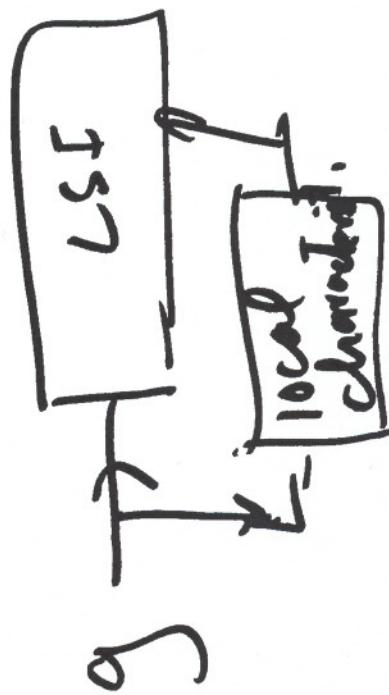
$$H(w_1, w_2) =$$

$$\Rightarrow h(w_1, w_2) = \frac{G_f}{G_f^2 + G_V^2} \delta(w_1, w_2)$$

~~$G_f$~~   ~~$G_V$~~

\* Taking care of  
"The mean"

$$f(w_1, w_2) = m_f + (g(w_1, w_2) - m_f) * \frac{\delta(w_1, w_2)}{\delta(w_1, w_2)}$$



both firs of  $(w_1, w_2)$   
 $\{ G_f, G_V \}$  are

$$\hat{f}(n_1, n_2) = m_f(n_1, n_2) + (g(n_1, n_2) - m_f(n_1, n_2)) \frac{\hat{\sigma}_f^2(n_1, n_2)}{\hat{\sigma}_f^2(n_1, n_2) + \hat{\sigma}_g^2(n_1, n_2)}$$

$$f(n_1, n_2) = m_f(n_1, n_2) + (g(n_1, n_2) - m_f(n_1, n_2)) \frac{\sigma_f^2(n_1, n_2)}{\sigma_f^2(n_1, n_2) + \sigma_g^2(n_1, n_2)}$$

2 cases:

$$\sigma_f^2 < \sigma_g^2 \iff f \approx g(n_1, n_2)$$

flat parts of image. For chance or

regions where image is "flat",  
 $\Rightarrow$  output is just mean.

$$\hat{f} \approx g(n_1, n_2)$$

$$\sigma_f^2 > \sigma_g^2 \iff f \approx g(n_1, n_2)$$

(2)

Textured part of image

How to estimate  $m_f$ ?

$$f \xrightarrow{\oplus} g$$

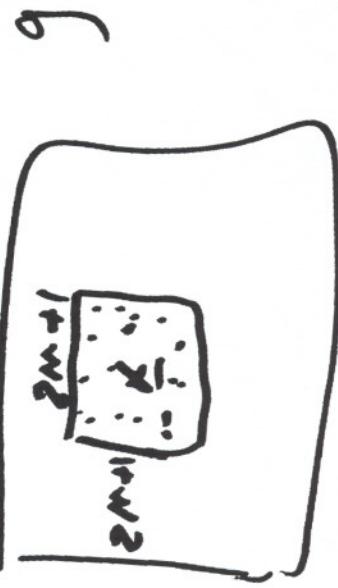
$$mg = m_v + m_f \quad \cdot m_v = 0$$

$$\hat{m}_f = \hat{m}_g = \cancel{\frac{\hat{m}_g}{2^{M+1}}}$$

$$\hat{m}_g = mg = \frac{1}{(2^M+1)^2} \sum_{i=1}^{2^M+1} g(k_1, k_2)$$

$k_1 = n_1 \dots n_1 \quad k_2 = n_2 \dots n_2$

plugging in  $\sqrt{m_f}$



$$f' = g * h'$$

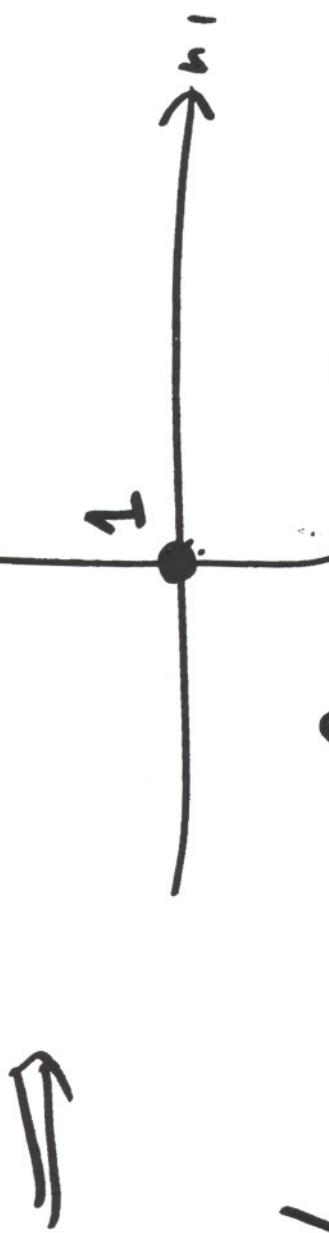
$$h'(n_1, n_2) = \left\{ \begin{array}{ll} \frac{6_f^2 + 6_v^2}{(2M+1)^2} & |n_1| \leq M \\ \frac{6_f^2 + 6_v^2}{(2M+1)^2} & |n_2| \leq M \\ 0 & \text{otherwise} \end{array} \right.$$

$$n_1 = n_2 = 0$$

except  $n_1 = n_2 = 0$

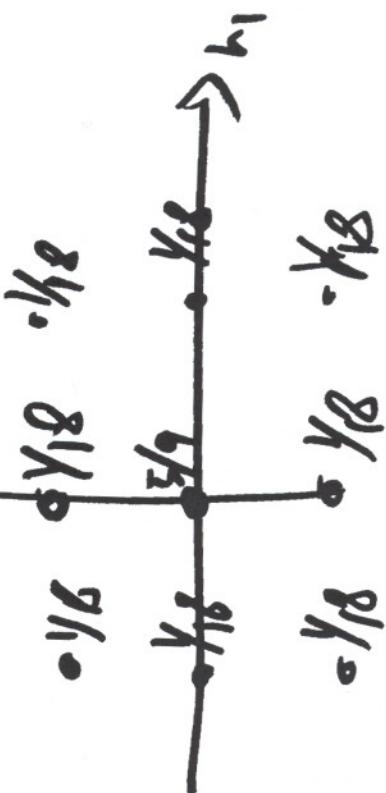
Otherwise

3 cases ①  $b^2_f \gg b^2_v$   $n_2 \downarrow \text{at } l_1'(n_1, n_2)$



If energy variance of  $f$  is much larger than noise, let the signal Dirac.

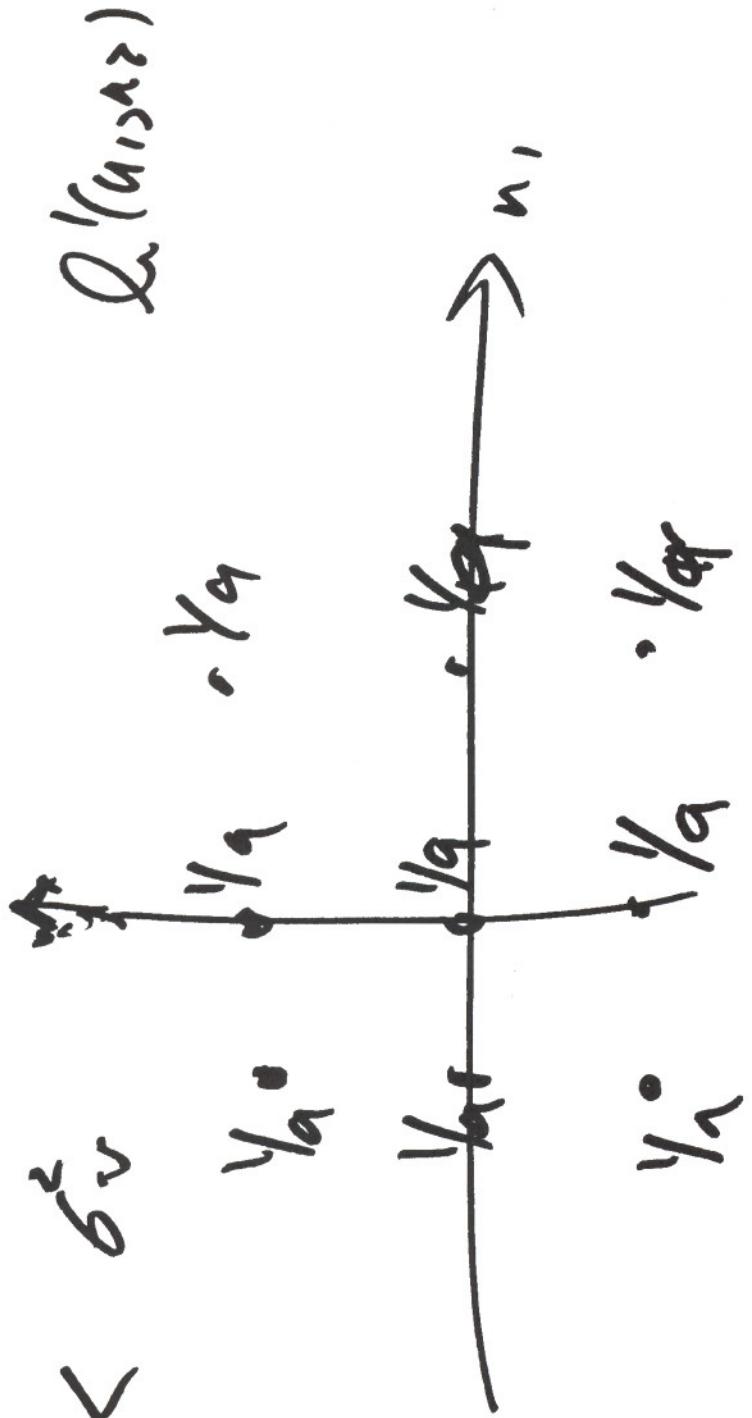
$$\textcircled{2} \quad b^2_f \approx b^2_v \quad , \quad n_2 \downarrow \text{at } l_1' \quad l_1' = 1 \Rightarrow$$



$$= \frac{1}{16} + \frac{5}{8} + \frac{1}{8}$$

$$(3) \quad 6f^2 < < 6v$$

$$M=1$$



$$f, v \text{ are independent}$$

$$f \xrightarrow{+} g$$

$$g = v + f$$

$$\Rightarrow \begin{aligned} 6g &= 6v + 6f \\ &\Rightarrow \cancel{\frac{6^2}{6}}_g = \cancel{\frac{6^2}{6}}_v - \cancel{\frac{6^2}{6}}_f \end{aligned}$$

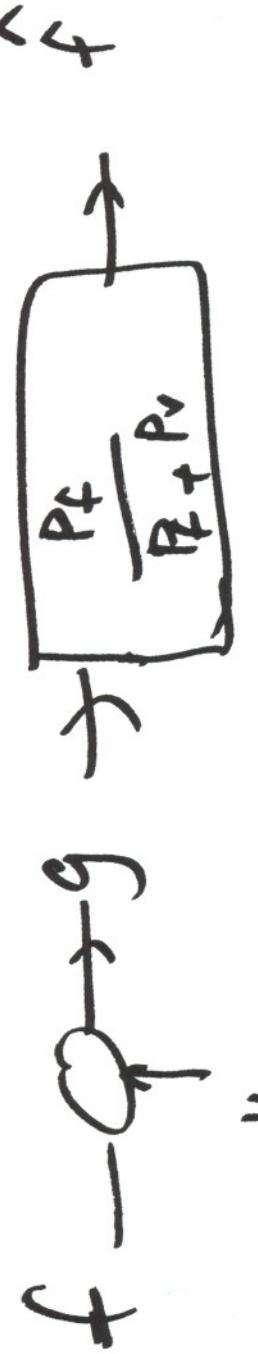
$$\langle \hat{G}_e^2 \rangle = \underbrace{\langle \hat{g}^2 \rangle}_{0} - \langle \hat{g}^2 \rangle$$

$$\text{only if } \hat{g}^2 > \hat{g}^2_{\text{av}}$$

e Theorie.

$$\langle \hat{g}^2 \rangle = \frac{1}{(2M+1)^2} \sum_{k_1=-M}^{M} \sum_{k_2=-M}^{M} [g(k_1, k_2) - \hat{m}_g(k_1, k_2)]^2$$

# Power Spectrum Filtering



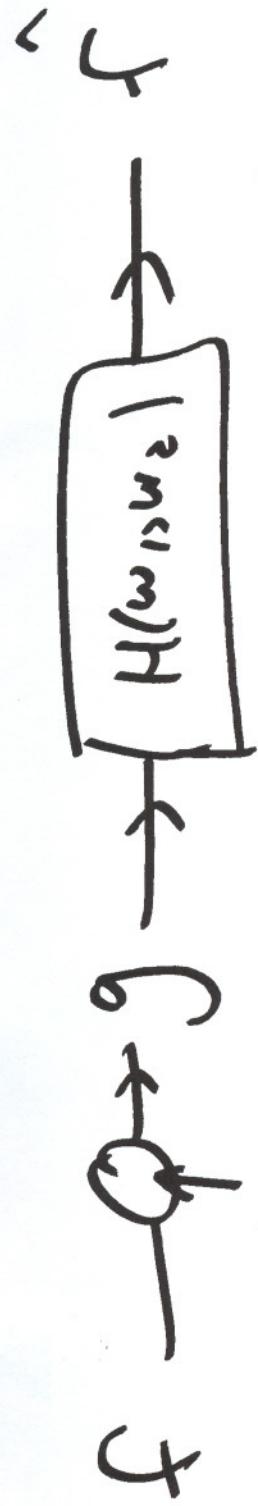
- Observations:

$$\hat{P}_f \neq \hat{P}_f^2$$

$$\hat{P}_f = P_g(\omega_1, \omega_2) = \frac{P_f^2(\omega_1, \omega_2)}{\left[ P_f(\omega_1, \omega_2) + P_v(\omega_1, \omega_2) \right]^2}$$

$$P_g(\omega_1, \omega_2) = P_f + P_v$$

$$\hat{P}_f = \frac{P_f(\omega_1, \omega_2) + P_v(\omega_1, \omega_2)}{\left[ P_f(\omega_1, \omega_2) + P_v(\omega_1, \omega_2) \right]^2} \neq P_f$$



Choose filter,

$$P_f^* = P_f$$

$H.$  s.t.  $H$ .

resulting filtering  
= "Power spectrum filtering".

$$\sqrt{|H(\omega_1, \omega_2)|^2}$$

$$P_g$$

$$P_f^* = P_f$$

$$P_f^*$$

$$= \left( \frac{P_f}{P_g} \right) Y_2$$

$$\frac{1}{2} \left( P_f(\omega_1, \omega_2) + P_g(\omega_1, \omega_2) \right)$$

$$= \left| H(\omega_1, \omega_2) \right|$$

→ zero phase.

## Parametric Weine's Theorem

"Traditional Weine f. He." =

$$\frac{P_f}{P_{f-1} P_V \beta}$$

Parametric:

$$\left( \frac{P_f}{P_f + \alpha P_V} \right) \beta$$

$\alpha = \beta = 1 \rightarrow$  Traditional and power spectrum.

$\alpha = 1, \beta = \gamma, \rightarrow$  ~~power spectrum~~

~~power spectrum~~  $\alpha = 1$