Supplementary Reader I

EECS 40
Introduction to Microelectronic Circuits

Prof. C. Chang-Hasnain

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Chapter 1. Bode Plots

1.1 Introduction

Bode plots are widely used in various fields of engineering because they characterize the magnitude and the phase response of a system. In this section, we will present step-by-step analysis to create the Bode plots of a given transfer function. We analyze the trend of the transfer function at different frequency regimes based on the value of the break or resonant frequency. This approach helps to understand the frequency behavior of the circuit, and also works for first order, second and higher order circuits.

1.2 First Order Circuits

1.2.1 General Construction and Break Frequency

A Bode plot illustrates the behavior of a circuit by generalizing its response into *trends* and graphing it against a log scale of frequency. Given a transfer function, $H(\omega)$, we may produce a magnitude and phase Bode plot. In each plot, we break down the analysis of the transfer function into 3 regimes, depending on the frequency in question:

1. At $\omega = \omega_B$,
2. when $\omega << \omega_B$, or when ω is much less than the break frequency
3. when $\omega >> \omega_B$, or when ω is much greater than the break frequency

The break frequency, ω_B , is a property of the filter that can be found by examining the transfer function. It describes the frequency where the trends on the Bode plot are broken, where one trend (when $\omega << \omega_B$) ends and the next (when $\omega >> \omega_B$) begins.

For first-order circuits, the break frequency can be found by looking solving for the frequency at which the real and complex components are equal.

Example 1 – Find the Break Frequency

Given the transfer function $H(\omega) = \dfrac{1}{1 + j\omega RC}$. We find the break frequency by examining the denominator, because it consists of two terms, 1 and $j\omega RC$. The first term is real and has a ω^0 dependence. The second term $j\omega RC$ has a ω^1 dependence. The break frequency, ω_B, occurs when the MAGNITUDE two components are equal. Thus, the break frequency for a filter with this transfer function is at $\omega_B = \dfrac{1}{RC}$.

After finding the break frequency, we can examine the trends on either side of ω_B . When $\omega << \omega_B$, the real term (in the denominator of Exp. 1) dominates and we ignore the imaginary component. When $\omega >> \omega_B$, ω dominates real term, so we work with only the imaginary term, as will be shown next.

Note: We can only disregard terms that are added and subtracted and not those that are multiplied or divided when making approximations.

e.g For a<<b, a<c and c<<d, $a \times \frac{(c+d)}{b} \approx \frac{ad}{b}$

Try to plug in any number you like and see if it is true.

1.2.2 Bode Magnitude Plot

For the magnitude plot, we plot on the y-axis: the 'square' of the magnitude of the transfer function, $H(\omega)$ in decibels (unit dB). This is because the transfer function describes the output to input voltage ratio. Since power is proportional to V^2, we plot $|H(\omega)|^2$, which in dB is:

$$\begin{aligned} |H(\omega)|_{dB} &= 10\log|H(\omega)|^2 \\ &= 20\log|H(\omega)| \end{aligned} \qquad (1)$$

Both of these expressions can be useful, depending on the form of the transfer function. We plot $|H(\omega)|_{dB}$ on the y-axis, against a logarithmic scale of ω on the x-axis. One thing to keep in mind about logarithmic functions is the ability to pull multiplicative factors out, for instance:

$$20\log(A \times B) = 20\log A + 20\log B$$

$$10\log(\frac{C}{D}) = 10\log C - 10\log D$$

The key to our analysis is that we identify and examine the dominant terms of the transfer function in each of the 3 regimes in the following.

(1) At $\omega = \omega_B$, we simply plug ω_B in $|H(\omega)|_{dB}$ and get the actual value.

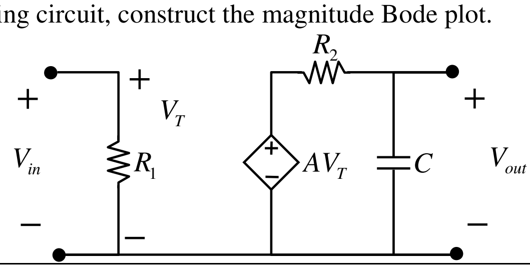
(2) When $\omega << \omega_B$. Examine SEPERATELY in the numerator and denominator all terms containing ω (e.g. ω^0 , ω^1). The one with the lowest power (e.g. ω^0) dominates in this frequency range. Hence we keep only the dominating term, one each for the numerator and denominator, respectively. The resulting formula will be used to determine the asymptotic behavior , or "trend".

(3) When $\omega >> \omega_B$. Again, we examine the numerator and denominator separately. Leave only the term with the highest power (ω^1 in the previous example), one each for numerator and dominator. The resulting formula will be used to determine the asymptotic behavior , or "trend".

By analyzing these three regimes, we can construct the magnitude Bode plot. We start at the break frequency, $\omega = \omega_B$, and then plot the trends for $\omega << \omega_B$ and $\omega >> \omega_B$.

Example 2 – Find the Transfer Function, $H(\omega)$, and plot the magnitude Bode Plot.

Given the following circuit, construct the magnitude Bode plot.



$A = 100$
$R_1 = 200\Omega$
$R_2 = 1k\Omega$
$C = 10\mu F$

We can see $V_T = V_{in}$ By voltage divider, we have: $V_{out} = \frac{Z_c}{Z_c + R_2} A V_{in}$

where $Z_c = \frac{1}{j\omega C}$

Thus, the transfer function is given by:

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{Z_c}{Z_c + R_2} A$$
$$= \frac{1/j\omega C}{1/j\omega C + R_2} A = \frac{A}{1 + j\omega R_2 C}$$

The magnitude plot will be the transfer function in dB, or:

$$|H(\omega)|_{dB} = 20\log|H(\omega)|$$
$$= 20\log\left|\frac{A}{1 + j\omega R_2 C}\right|$$

Plotting this function would yield the exact behavior of our filter, but we only need the asymptotic behavior for the Bode plot. Now we begin a 3-part analysis.

Step 1: Break Frequency

Setting equal the real and imaginary components in the denominator, we find the break frequency:

$$\omega_B = \frac{1}{R_2 C} = \frac{1}{(1\,\text{k}\Omega)(10\,\mu\text{F})}$$
$$= \frac{1\,\text{rad}}{10^{-2}\,\text{s}} = 100\,\text{rad/s}$$

Step 2: Asymptotes

(1) At $\omega = \omega_B$,

$$|H(\omega_B)|_{dB} = 10\log\left|\frac{A}{1+j}\right|^2 = 10\log\left(\frac{A}{\sqrt{2}}\right)^2 = 10\log\left(\frac{A^2}{2}\right)$$
$$= 20\log A - 10\log 2 = 20\log A - 3dB$$

With A = 100, $|H(\omega_B)|_{dB} = 37$ dB. The first line of the above steps shows that at ω_B, the output is at half the maximum power (-3dB is half in linear scale); the transfer function (voltage) has a value of $A/\sqrt{2}$, and power is proportional to the square of voltage. The break frequency for a first-order circuit is also referred to as the half-power frequency, where the output is one-half the maximum power. The factor of 1/2 in the logarithm can be expanded out to -20 log 2:

$$10\log\left(\frac{1}{2}\right) = 10\log 1 - 10\log 2 = 0 - 10\log 2 \cong -3\,\text{dB}$$

The half-power frequency is -3dB below the maximum power, so we also call ω_B the -3dB frequency.

(2) For $\omega << \omega_B$, the constant 1 dominates in the denominator and is the only term to be kept.. The transfer function can be approximated as: $H(\omega) \cong \frac{A}{1+0} = A$

which is purely real, so $|H(\omega)|_{dB} = 20\log A$ and $|H(\omega)| = |A| = A$

Substituting the gain factor, A = 100, we obtain $|H(\omega)|_{dB} = 20\log 100 = 20\times 2 = 40$ dB.

(3) Next, for $\omega >> \omega_B$, the ω term dominates in the denominator term, and the transfer function can be approximated as:

$$H(\omega) \cong \frac{A}{j\omega R_2 C}$$

$$|H(\omega)|_{dB} = 20\log\left|\frac{A}{j\omega R_2 C}\right| = 20\log\left|\frac{A}{R_2 C}\right| - 20\log|j\omega|$$
$$= 20\log\frac{A}{R_2 C} - 20\log\omega = C' - 20\log\omega$$

In the asymptote, the filter's magnitude decreases at a rate of 20 dB/decade (a 10x increase in ω). Here C' is a constant and we can simply connect the curve for $\omega >> \omega_B$ with the point $|H(\omega_B)|_{dB} = 37$ dB. The magnitude $|H(\omega_B)|_{dB}$ decreases by 20 dB at 10 ω_B, 40 dB at 100 ω_B, and so on.

With these three regions analyzed, the magnitude Bode plot is complete and shown in Fig. 1.

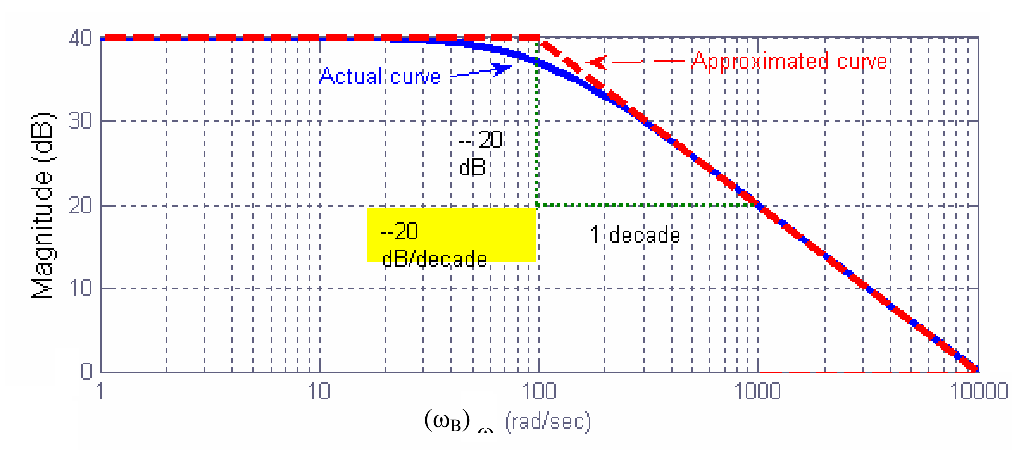


Fig. 1. Magnitude Bode Plot for H(ω) of Example 2, with the gain factor A = 100.

The dotted straight lines in Fig. 1. form the Bode plot. Note how it approximates the behavior of the actual function, shown in blue.

NOTE: This is an example of low-pass filter. The low frequencies are kept by the filter and the high frequencies are filtered out. First-order circuits are either high-pass or low-pass filters.

Exercise 1:

What if the voltage across the resistor R_2 (Example 2) is taken as the output? Is the total energy conserved in each frequency?

1.2.3 Bode Phase Plot

The Bode phase plot is constructed in a similar manner to the magnitude plot except we use limits (instead of trends). We first find the expression for phase, $\angle H(\omega)$, from the transfer function, $H(\omega)$. For $\omega << \omega_B$, we take the limit of $\angle H(\omega)$ as $\omega \to 0$. At $\omega = \omega_B$, we use the complete expression for $\angle H(\omega)$, as both the real and imaginary components have equal magnitude. For $\omega >> \omega_B$, we find the phase by taking the limit of $\angle H(\omega)$ as $\omega \to \infty$. We then construct the Bode phase plot by performing the following:

Plot lower-frequency limit value (from x $\omega << \omega_B$ to $\omega_B/10$),

Plot upper-frequency limit value (from x=$10\omega_B$ to $\omega \to \infty$)

Plot the value for $\omega = \omega_B$

Connect the extremes by curve (arctan) lines.

Example 3 – Find the Phase, $\angle$, of a Transfer Function, $H(\omega)$

Using the transfer function of Example 2,

$$H(\omega) = \frac{A}{1 + j\omega R_2 C}$$

We find the net phase, $\angle H(\omega)$, by subtracting the phase of the denominator from the phase of the numerator:

$$\angle H(\omega) = 0 - \tan^{-1}\left(\frac{\omega R_2 C}{1}\right)$$
$$= -\tan^{-1}(\omega R_2 C)$$

To plot the expression for phase in Exp. 3, we examine our 3 regions. Taking the limit of $\angle H(\omega)$ as $\omega \to 0$, we find the lower-frequency asymptote, $\omega << \omega_B$.

$$\lim_{\omega \to 0} \angle H(\omega) = -\tan^{-1}(0) = 0$$

The upper-frequency asymptote, $\omega >> \omega_B$ is found by taking the limit as $\omega \to \infty$:

$$\lim_{\omega \to \infty} \angle H(\omega) = -\tan^{-1}(\infty) = -90° \; or \; -\frac{\pi}{2}$$

At the break frequency, $\omega_B = \dfrac{1}{R_2 C}$, we find the phase to be:

$$\angle H(\omega_B) = -\tan^{-1}(1) = -45° \; or \; -\frac{\pi}{4}$$

This value is in agreement with a line drawn between the upper and lower asymptotes; ω_B is halfway between $\omega_B/10$ and $10\omega_B$, and -45° is halfway in between 0° and 90°. The phase Bode plot of Exp. 3 is shown in Fig. 2.

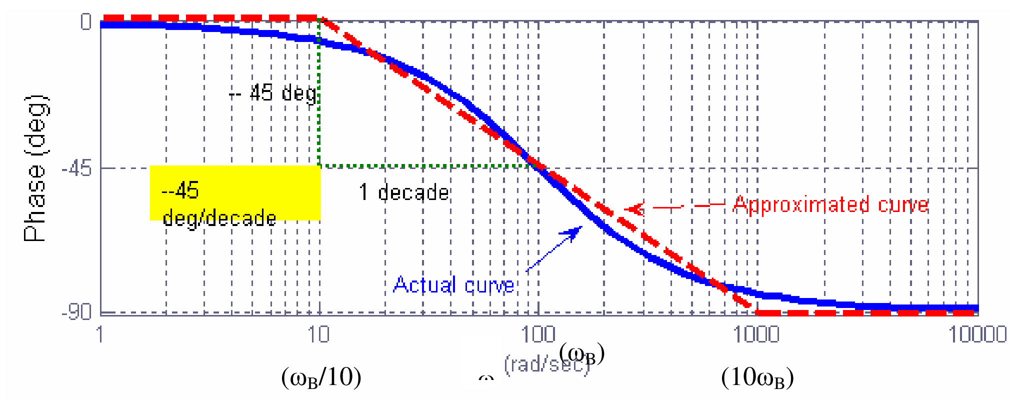
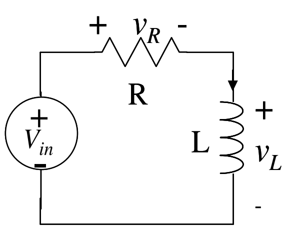


Fig. 2. Phase Bode Plot for H(ω) of Example 2.

As before, the dotted lines in Fig. 2 outline the phase Bode plot for our transfer function, but the actual values are shown by the solid line. Note that with our approximation, in the intermediate region between $\omega_B/10$ and $10\omega_B$, the phase decreases by 45 degrees per decade of angular frequency, and outside this region, the phase is constant (the asymptote).

Exercise 2: Find the Transfer Function, $H(\omega)$, and plot Bode magnitude and phase plots for (a) $V_{out}=V_L$ and (b) $V_{out}=V_R$. In both cases, do we have high-pass or low-pass filters?



1.3 Second Order Circuits

1.3.1 General Construction and Resonant Frequency

As with Bode Plots for first-order circuits, the transfer function for second-order circuits can be found by writing an equation relating V_{out} and V_{in}. Once this equation is written, an expression for $H(\omega)$ is easily obtained. You can use this transfer function to find its magnitude and phase.

Similar to first-order circuits, we depict the behavior of second-order circuits by plotting trends of the transfer function at three frequency regimes relative to a certain resonance frequency, ω_0:

1. At $\omega = \omega_o$
2. For $\omega << \omega_o$, or in other words, the limit as $\omega \to 0$
3. For $\omega >> \omega_o$, or in the limit as $\omega \to \infty$

Around the resonant frequency, the magnitude of the transfer function $|H(\omega)|$ ***changes trend*** and ***often reaches the maximum or minimum value***. Though whether it reaches the maximum or minimum depends greatly on the Q value, which will be discussed next, the trend is always changed at the resonance frequency.

The resonant frequency, ω_0, can be found by setting the complex part of either numerator or denominator of the transfer function to zero. This is because the magnitude of $a + jb$ reaches minimum when b=0. If we have a complex term $a + jb$ on the numerator, by letting b=0, we reached minimum, whereas if this complex term is in the denominator, we reached maximum.

Example 4: Finding the resonant frequency

For a circuit with transfer function as $H(\omega) = \dfrac{R}{j\omega L + R - j\dfrac{1}{\omega C}}$

What is ω_o, the resonant frequency?

Solution:

The numerator is R, which has no frequency dependence term and hence irrelevant for this question. We look at the denominator. The resonant frequency is when

$j\omega L - j\dfrac{1}{\omega C} = 0$ Hence, $\omega_o = \dfrac{1}{\sqrt{LC}}$

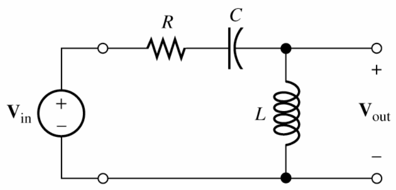
1.3.2 Bode Magnitude Plot

Constructing Bode Plots for second-order circuits is the same as for first-order circuits, so analyzing trends consists of the same process mentioned earlier – finding the dominant term amongst now typically three terms (ω^{-1}, ω^0, ω^1) in both numerator and denominator ***separately***.

Example 5 below shows how to construct a Bode magnitude plot. Note, that in most Bode Plot problems, you will be asked the slope at which lines are ascending and descending. By using the techniques of Example 2, you will be able to answer such questions very easily.

Example 5: Constructing the transfer function and Bode magnitude Plot

Construct the magnitude Bode Plot for the circuit shown in the diagram below. Also, label the slopes of any lines in the plot.



First, write the output voltage in terms of the values given. Notice that all the electrical components of the circuit are in series, so applying the voltage-divider technique gives the following:

$$V_{out} = \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}} V_{in} \qquad (1)$$

So, the transfer function becomes:

$$H(\omega) = \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}} \qquad (2)$$

Step 1: Finding the Resonance Frequency

The magnitude for (2) is the magnitude of the numerator divided by the magnitude of the denominator. Remembering that the magnitude for a complex number, $a + bi$, is $\sqrt{a^2 + b^2}$, we obtain the following:

$$|H(\omega)| = \frac{\omega L}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \qquad (3)$$

This is an example that we cannot simply use the definition of $|H(\omega)|$ reaching maximum or minimum, because we have a ω dependence on the numerator. However, we can use the definition of breaking trends. The denominator has three terms, ω^{-1}, ω^0 and ω^1. The trend is either dominated by ω^{-1} or ω^1 at very low or high frequencies, respectively. At the resonant frequency, only the ω^0 term will dominate.

Hence, $\omega_0 L - \frac{1}{\omega_0 C} = 0$.

And $\omega_0 = \sqrt{\frac{1}{LC}}$ (4)

Note: This resonant frequency is the same formula we got using second-order differential equation on slide 177 of EE40 Reader! (Surprise or not a surprise?)

Step 2: Asymptotes

Now, we must analyze this magnitude function in the three regimes mentioned above.

(1) For $\omega << \omega_o$, the numerator has only one term, so we leave it alone. For the denominator, we compare the three terms and keep only the dominant one. In this case, we will keep the $(\omega C)^{-1}$ term.

So, we obtain the following equation:

$$|H(\omega)| = \frac{\omega L}{(\frac{1}{\omega C})} = \omega^2 LC$$

$$Or \qquad (5)$$

$$|H(\omega)|_{dB} = 10\log(\omega^2 LC)^2 = 40\log(\omega) + 20\log(LC)$$

Note that as $\omega \to 0$, $|H(\omega)|$ goes to 0, which means that $|H(\omega)|_{dB}$ goes to $-\infty$. Note that the x-axis on a Bode Plot is $\log(\omega)$ and the y-axis is $|H(\omega)|_{dB}$, so equation 5 is of the form:

$y = 40x + B$ where B is a constant

This says that the slope of the Bode magnitude plot for $\omega << \omega_o$ is 40 *dB/decade*.

(2) For $\omega >> \omega_o$, we will keep only the ωL in the denominator of the transfer function. This gives the magnitude as:

$$|H(\omega)| = 1$$

$$OR \qquad (6)$$

$$|H(\omega)|_{dB} = 10\log(1) = 0dB$$

This means that the output remains unchanged for large values of ω and the slope is 0 *dB/decade*.

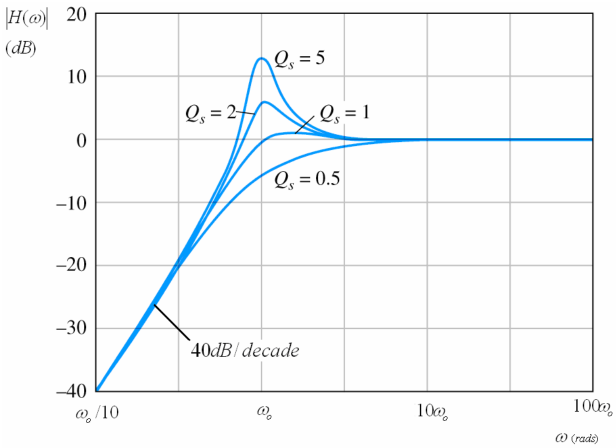
(3) At $\omega = \omega_o$,

$$|H(\omega)| = \frac{\sqrt{(\omega_o L)^2}}{\sqrt{R^2 + (\omega_o L - \frac{1}{\omega_o C})^2}} = \frac{\sqrt{(\omega_o L)^2}}{\sqrt{R^2}} = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} = Q_S$$

$$OR \qquad (7)$$

$$|H(\omega)|_{dB} = 20\log(Q_S)$$

The Bode magnitude Plot for this circuit is shown below.



This is a high-pass filter which rejects low frequencies while leaving the high frequency content unchanged. Note, this is better than the first-order high-pass filter because the slope of this filter in the low-frequency region is steeper, $40dB/decade$, rather than the $20dB/decade$ slope of the first-order filter. The increased slope provides stronger discrimination against the unwanted low frequencies.

Also note, a high value of Q_S means a large hump in the figure below. As Q_S increases, so does the value of $|H(\omega_o)|_{dB}$. At $Q_S = 0.5$, $|H(\omega_0)|_{dB} = 20\log(Q_S) = -3dB$; this curve resembles a first-order filter the most with only exception that the slope of filtered frequency is steeper.

Note:

$$Q_S = \frac{1}{2\varsigma}$$

(Surprise or not a surprise?)

Hence when $Q_s=0.5$, $\zeta=1$, we have critically damped circuit.

When $Q_s>0.5$, $\zeta<1$, we have under-damped circuit, whose signal, in the time domain, oscillates a great deal (slide 178, EE40 Reader). This can also be seen with the frequency response, we see $|H(\omega_o)|>1$ or $|H(\omega_o)|_{dB} > 0dB$. Likewise, we have an over-damped circuit with $Q_s<0.5$.

Exercise 1 Find the Transfer Function, $H(\omega)$, and plot Bode magnitude plot for $V_{out}=V_C+V_R$.in Example 5. Do we have high-pass or low-pass filter? Is the energy conserved at ω_0?

Exercise 2 Repeat the exercise for $V_{out}=V_C$.in Example 5. What kind of filter is this?

Exercise 3 Repeat the exercise for $V_{out}=V_C+V_L$. in Example 5. What kind of filter is this?

1.3.3 Bode Phase Plot

Constructing the phase plot is done in a similar manner to the magnitude plot. After finding the transfer function, you can find its phase. Remember that the phase of a function is the phase of the numerator minus the phase of the denominator, and the phase of a complex number, $a+bi$, is $\arctan(b/a)$. After finding the phase function, you must analyze the function in the same 3 regimes as was done for the magnitude plot. Follow the next example, which constructs a phase plot for the circuit in Example 2.

Example 6: Constructing the Phase Plot:

Construct the Phase Plot for the circuit shown in Example 5.

Solution:

The transfer function of example 2 is reproduced below:

$$H(\omega) = \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}} \qquad (6\text{-}1)$$

The phase of the above transfer equation is:

$$\angle H(\omega) = 90° - \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \qquad (6\text{-}2)$$

Now, we must analyze this function in the same 3 regimes as before.

(1) For $\omega << \omega_o$ or as $\omega \to 0$, we can disregard the ωL term in the numerator of the second term since this value is approaching 0 and will not make a significant effect when added to other values. So, the above function reduces to:

$$\angle H(\omega) = 90° - \arctan\left(\frac{-\frac{1}{\omega C}}{R}\right) = 90° + \arctan\left(\frac{1}{\omega RC}\right) = 90° + \arctan\left(\frac{\omega_0}{\omega}\frac{1}{Q_S}\right) \qquad (6\text{-}3)$$

Notice that as $\omega \to 0$, the argument to arctan in equation 3 becomes very large, so:

$$\angle H(\omega) = 90° + 90° = 180° \qquad (6\text{-}4)$$

(2) For $\omega >> \omega_o$ or as $\omega \to \infty$, in equation 2, we can disregard the $(-1/\omega C)$ term in the numerator because it becomes very small and has a negligible effect when it is added to other values. So, equation 2 reduces to:

$$\angle H(\omega) = 90° - \arctan\left(\frac{\omega L}{R}\right) = 90° - \arctan\left(Q_S \frac{\omega}{\omega_0}\right) \qquad (6\text{-}5)$$

As $\omega \to \infty$, in equation 5, the argument to arctan becomes very large, so:

$$\angle H(\omega) = 90° - 90° = 0° \qquad (6\text{-}6)$$

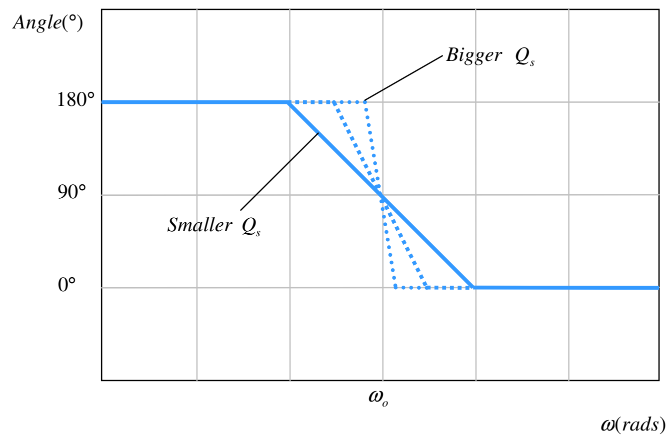
(3) At $\omega = \omega_o$, equation 2 becomes:

$$\angle H(\omega) = 90° - \arctan\left(\frac{0}{R}\right) = 90° \qquad (6\text{-}7)$$

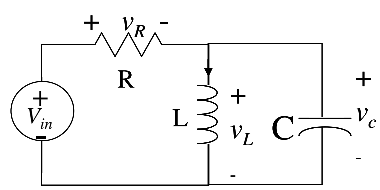
The figure below shows the phase plot for equation 2. As seen in the plot, for low frequencies, the phase completely changes, yet for high frequencies, it remains unchanged.

Note that as Q_S increases, the downward sloping line becomes steeper. This can be seen using either equation 6-3 for $\omega < \omega_0$ and 6-5 for $\omega > \omega_0$, respectively.

If Q_S could reach ∞, the downward sloping line would become vertical which approaches an ideal high-pass filter.



Exercise. Find the Transfer Function, $H(\omega)$, and plot Bode magnitude and phase plots for (a) $V_{out}=V_L$ and (b) $V_{out}=V_R$. In both cases, what filters do we have? Show the definition of Q is different for a parallel LC than a series LC circuit. Explain how is Q related to the damping ratio in this case.



1.3.4 Definitions

Table 1. Symbol Table for ω_o, Q_s and Q_p

Symbol	Definition
$\omega_o = 1/\sqrt{LC}$	The resonant frequency is defined to be the frequency at which the impedance is purely resistive.
$Q_s = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$	The quality factor for a series resonant circuit is defined to be the ratio of the reactance of the inductance at the resonant frequency (e.g. $\omega_o L$) to the resistance.
$Q_p = \frac{R}{\omega_0 L} = \omega_0 RC$	The quality factor for a parallel resonant circuit is defined to be the ratio of the resistance to the reactance of the inductance at the resonant frequency (e.g. $\omega_o L$).