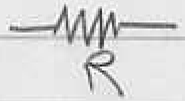
- Lumped Element circuits

We use symbols like resistors:



they are basically lump, or better, they can be considered very small.

Let's formalize this concept.

FACT 1) Electromagnetic waves propagates at speed of light $c = 3 \cdot 10^8$ m/s in the empty space.

[in a different propagation medium c is different.

It is defined as $c = \frac{1}{\sqrt{\mu \varepsilon}}$

μ = magnetic permeability

ε = electric permittivity]

If we have a circuit, we usually know the frequency of the signals that the circuit is elaborating and let's denote it with f_{max}.

We can say, then, that the smallest time interval that we are interested in is

FACT 2) $$t_{min} = \frac{1}{f_{max}}$$

Another thing that we know about our circuit is for sure its size. Let's denote its maximum lenght with L. The time required by the electromagnetic field to go from one point of the circuit to another is:

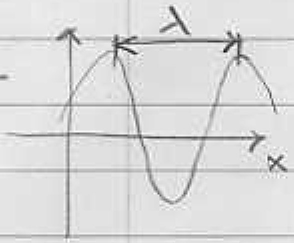
$$t \leq \frac{L}{c}$$ (space/velocity based on FACT 1)

in order to consider the lumped model valid, it must b:

$$t << t_{min} \Rightarrow \frac{L}{c} << \frac{1}{f_{max}}$$

$$\Rightarrow L << \frac{c}{f_{max}} = \lambda$$

λ is the signal wavelenght



Example:

An audio amplification circuit has a maximum lenght of 40 cm = = 0.4 m

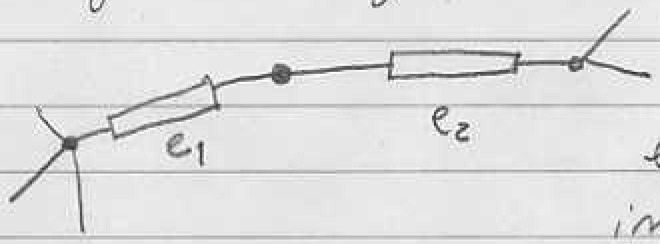
It elaborate music whose maximum frequency is 20 KHz = $= 20 \cdot 10^3$ Hz

$$\Rightarrow \lambda = \frac{3 \cdot 10^8}{20 \cdot 10^3} = \frac{3}{2} \cdot 10^4 = 15 \text{ Km}$$ (yes, fifteen Kilometers)

15 Km $\gg$ 0.4 m that's why we can use our lumped element model to design it.

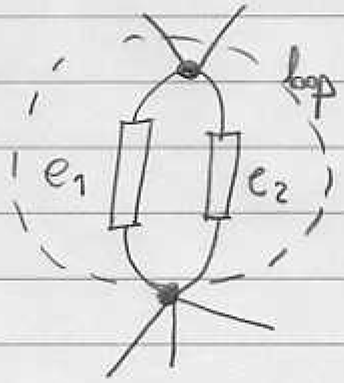
- Series an Paraller connections

We defined a node and a loop. When only two element are connected to a node then we say that they are in <u>series</u>



e_1 and e_2 are in series

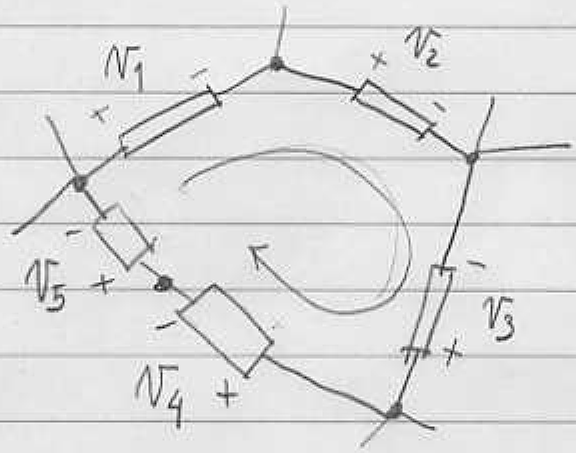
If two elements in a circuit form a loop then we say that they are in <u>parallel</u>



e_1 and e_2 are in parallel

- Kirchoff's voltage law (KVL)

"The algebraic sum of the voltages in a loop is equal to zero"



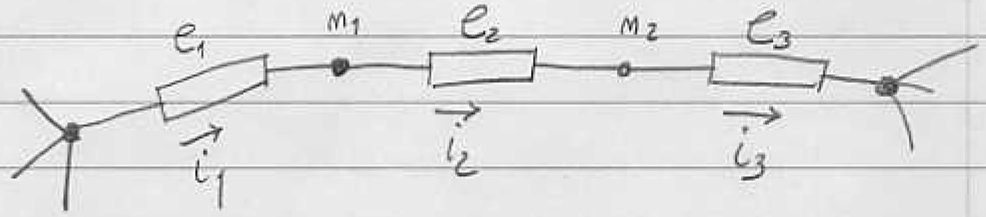
Fix an orientation for the loop (the arrow in figure)

Consider positive the voltages drops an negative the rises with respect to the orientation

$$v_1 + v_2 - v_3 + v_4 + v_5 = 0$$

- Easy consequences of KCL and KVL

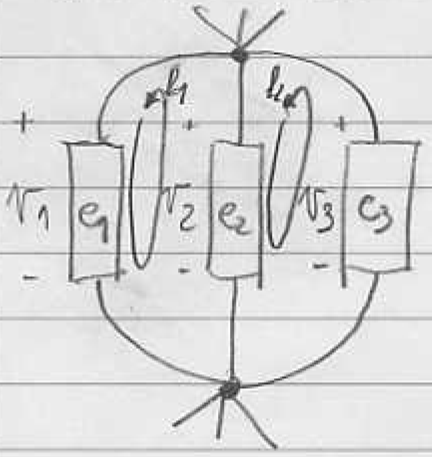
In a series connection, the same current flows in all the elements.



KCL at n_1 $\quad i_1 - i_2 = 0 \Rightarrow i_1 = i_2$

KCL at n_2 $\quad i_2 - i_3 = 0 \Rightarrow i_2 = i_3 \Rightarrow i_1 = i_2 = i_3$

In a parallel connection the voltage across all the elements is the same



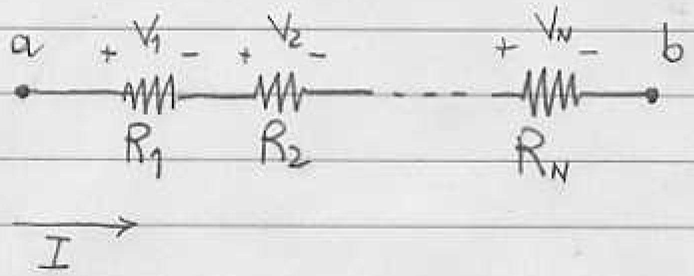
KVL for l_1 $\quad v_1 - v_2 = 0 \Rightarrow v_1 = v_2$

KVL for l_2 $\quad v_2 - v_3 = 0 \Rightarrow v_2 = v_3$

$\Rightarrow v_1 = v_2 = v_3$

Symbol : $e_1 \,//\, e_2 \,//\, e_3$

– Series, Parallel resistors



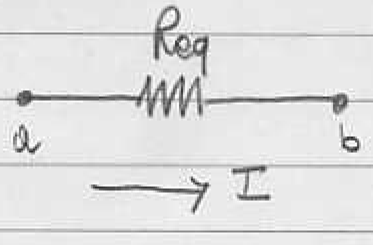
By KVL $\quad V_{ab} = V_1 + V_2 + \cdots + V_N$

Since the current is the same for all elements

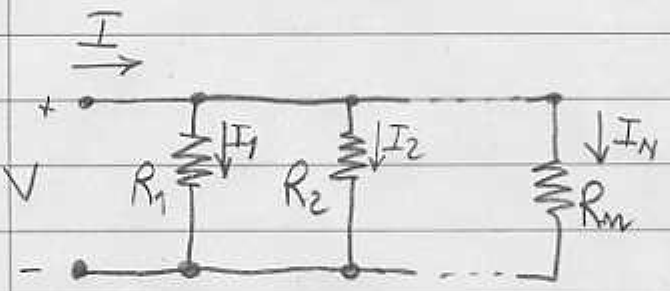
$$V_i = R_i I \quad \text{for } i = 1 \cdots N$$

$$V_{ab} = \sum_{i=1}^{N} V_i = \sum_{i=1}^{N} R_i I = I \sum_{i=1}^{N} R_i = I R_{eq}$$

So the circuit is electrically equivalent to



$$R_{eq} = R_1 + \cdots + R_N = \sum_{i=1}^{N} R_i$$



by KCL $I = I_1 + \cdots + I_N = \sum_{i=1}^{N} I_i$

but the voltage is the same for all resistor:

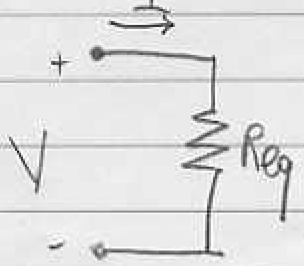
$$I_i = \frac{V}{R_i} \qquad i = 1 \cdots N$$

hence

$$I = \sum_{i=1}^{N} \frac{V}{R_i} = V \sum_{i=1}^{N} \frac{1}{R_i}$$

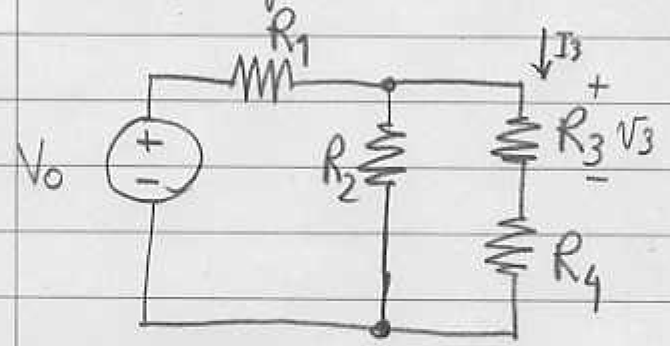
$$\Rightarrow V = \frac{I}{\sum_{i=1}^{N} \frac{1}{R_i}} = \frac{I}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}} = I R_{eq}$$

The circuit is electrically equivalent to:



$$R_{eq} = \frac{1}{\frac{1}{R_1} + \cdots + \frac{1}{R_N}}$$

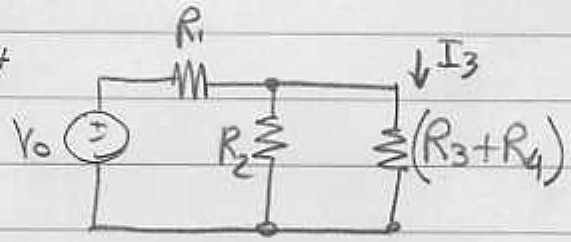
Example



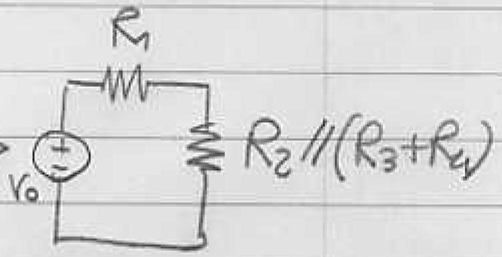
$V_0 = 1.5V$
$R_1 = 3.3K\Omega$
$R_2 = 1K\Omega$
$R_3 = 2.2K\Omega$
$R_4 = 1K\Omega$

V_3?

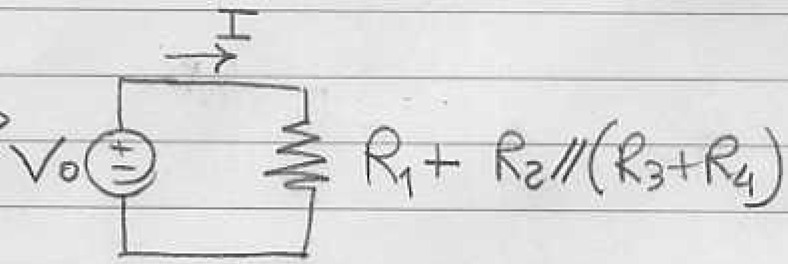
1^{st}



$\Rightarrow$

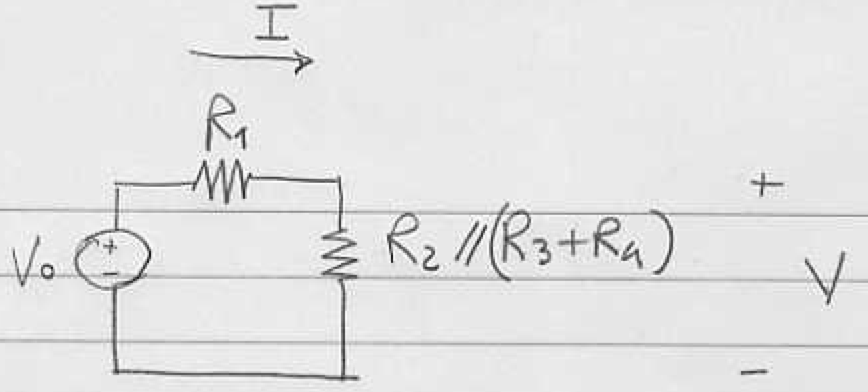


$\Rightarrow$

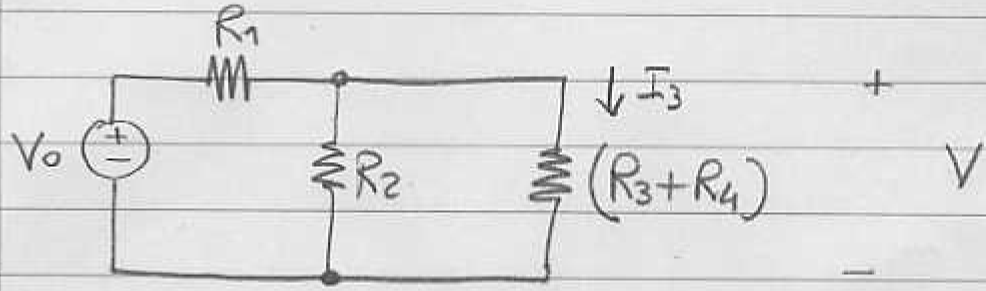


$$I = \frac{V_0}{R_1 + R_2 // (R_3+R_4)}$$

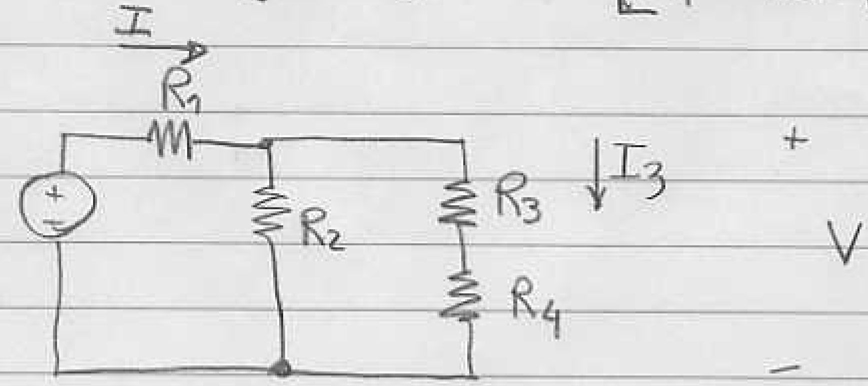
Now we have to roll back to find V_3



$$V = \left[R_2 // (R_3+R_4)\right] \frac{V_0}{R_1 + R_2 // (R_3+R_4)}$$

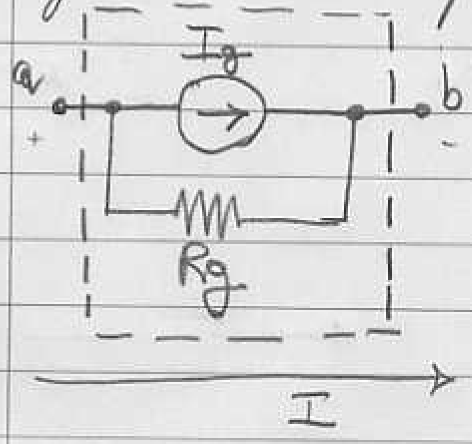


$$I_3 = \frac{V}{(R_3+R_4)} = V_0 \frac{R_2 // (R_3+R_4)}{\left[R_1 + R_2 // (R_3+R_4)\right](R_3+R_4)}$$



$$V_3 = \frac{R_3}{R_3+R_4} \; \frac{R_2 // (R_3+R_4)}{R_1 + R_2 // (R_3+R_4)} \; V_0$$

- Generators equivalence



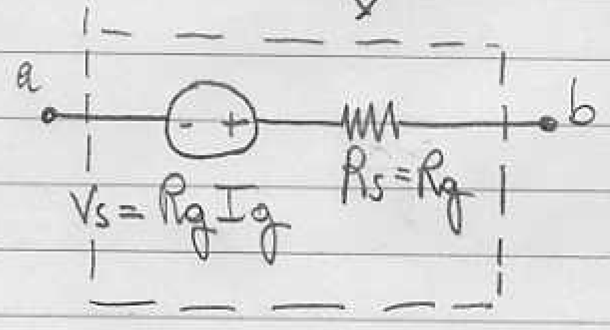
Current flowing in R_g:

$I - I_g$

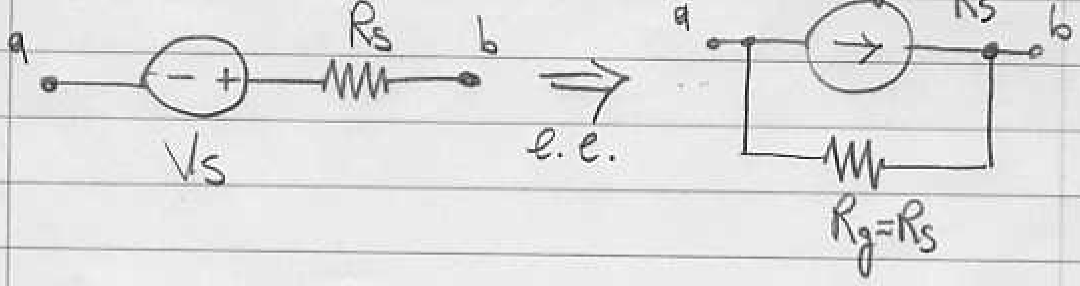
hence $V_{ab} = R_g(I - I_g)$

$V_{ab} = R_g I - R_g I_g$

$\Downarrow$ electrically equivalent to

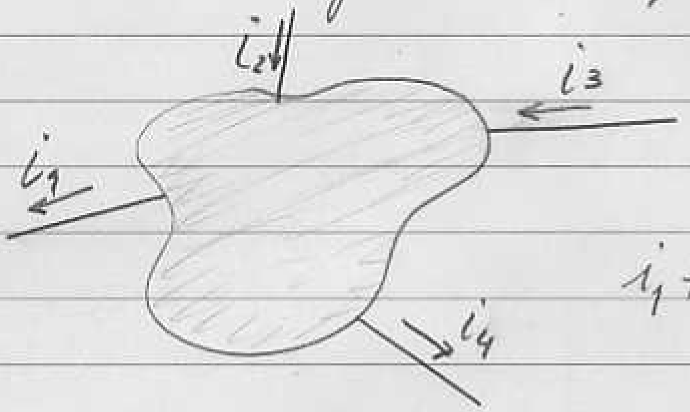


It is also true that:



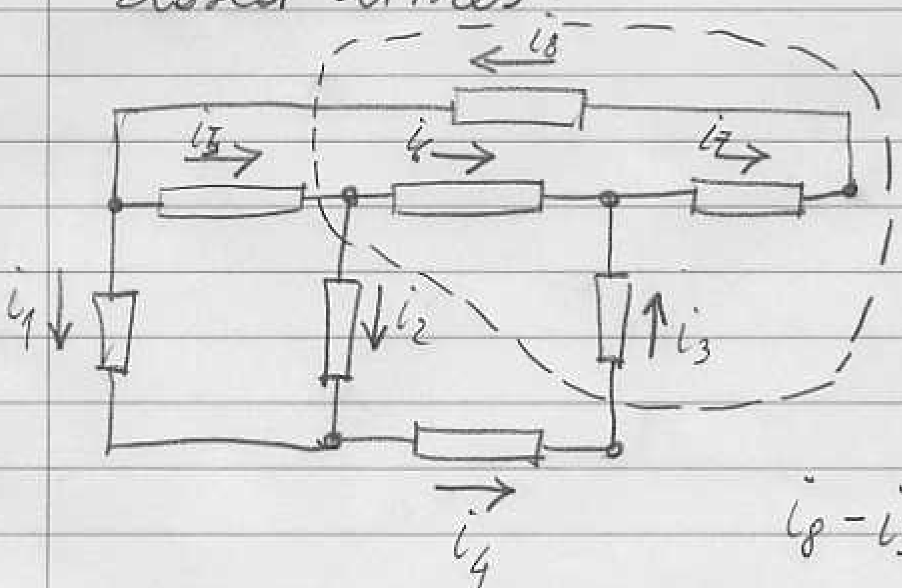
– Generalization of KCL

Given a surface S,
the algebraic sum of
all the currents flowing through
the surface is equal to zero



$$i_1 + i_4 - i_3 - i_2 = 0$$

talking about circuits on
a plane we can just use
closed lines



$$i_8 - i_5 + i_2 - i_3 = 0$$

We call this lines "cuts"

- Node-Voltage Analysis

It is a special case of a more general method that uses cuts.

1) Given a circuit fix a reference node n_0 with voltage V_0 (you need it because voltages are defined with respect to a reference point).

$\forall$ other nodes n_i, let V_i be the voltage with respect to V_0.

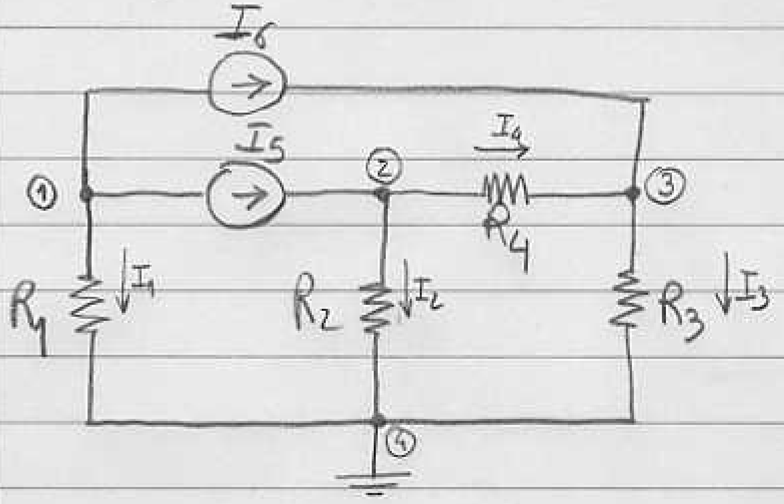
2) $\forall$ node n_i write the KCL they will have expressions like

$$\sum_j i_{ij} = \sum_k i_{ki}$$

$\forall j, k$ nodes that are connectet to nod i by some element.

3) Express each current using voltages across nodes: $i_{lm} = f(V_l, V_m)$

Example



$I_5 = 3\,A$
$I_6 = 2\,A$
$R_1 = R_3 = R_4 = 1\,\Omega$
$R_2 = 1/3\,\Omega$

Let node ④ be the reference node, denoted with the special symbol $\perp$ and called "ground".

Given a node ⓘ, let V_i to be V_{i4}, the voltage difference between node ⓘ and the reference node.
Also we fix the currents reference directions as in figure.

Let's write KCL for all nodes (but the reference):

$$\begin{cases} I_1 + I_5 + I_6 = 0 & (1) \\ I_2 + I_4 - I_5 = 0 & (2) \\ I_3 - I_4 - I_6 = 0 & (3) \end{cases}$$

We can now re-write the system of equations by substituting the known current and writing the unknown using Ohm's Law

$I_1 = V_1 / R_1$; $I_2 = V_2/R_2$; $I_3 = V_3/R_3$

$$I_4 = \frac{V_2 - V_3}{R_4}$$

The new system is:

$$\begin{cases} \frac{V_1}{R_1} + 3 + 2 = 0 & (1) \\ \frac{V_2}{R_2} + \frac{V_2 - V_3}{R_4} - 3 = 0 & (2) \\ \frac{V_3}{R_3} - \frac{V_2 - V_3}{R_4} - 2 = 0 & (3) \end{cases}$$

Now we have just to solve the system of equations:

$$\frac{V_1}{R_1} + 5 = 0 \Rightarrow V_1 = -5R_1 = -5V$$

$$\begin{cases} V_2\left(\frac{1}{R_2} + \frac{1}{R_4}\right) - \frac{V_3}{R_4} - 3 = 0 \\ V_3\left(\frac{1}{R_3} + \frac{1}{R_4}\right) - \frac{V_2}{R_4} - 2 = 0 \end{cases} \Rightarrow \begin{cases} 4V_2 - V_3 = 3 \\ 2V_3 - V_2 = 2 \end{cases}$$

By substitution:

$$V_2 = 2V_3 - 2 \Rightarrow 4(2V_3 - 2) - V_3 = 3$$

$$\Rightarrow 8V_3 - V_3 - 8 = 3 \Rightarrow 7V_3 = 11 \Rightarrow V_3 = 11/7 \text{ V}$$

$$V_2 = 2V_3 - 2 = 22/7 - 2 = \frac{22}{7} - 14/7 = 8/7 \text{ V}$$

$$\boxed{\begin{aligned} V_1 &= -5 \text{ V} \\ V_2 &= 8/7 \text{ V} \\ V_3 &= 11/7 \text{ V} \end{aligned}}$$

these voltages are indeed voltage differences across the nodes and the reference node!!!

where $f(\ )$ is a function representin the costitutive equation of the component.

If the component is a resistor R then

$$i_{\ell m} = \frac{V_\ell - V_m}{R}$$

If the component is a current generator with value i_g

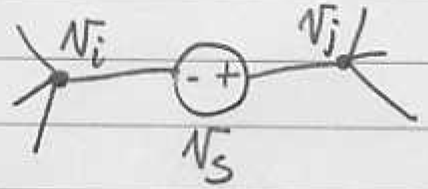
$$i_{\ell m} = i_g$$

If the component is a voltage source then?

We don't know the current!!

Let the current to be an unknow. But now we have more unknowns than equations.
Fortunately enough, the

Voltage source adds one more equation :

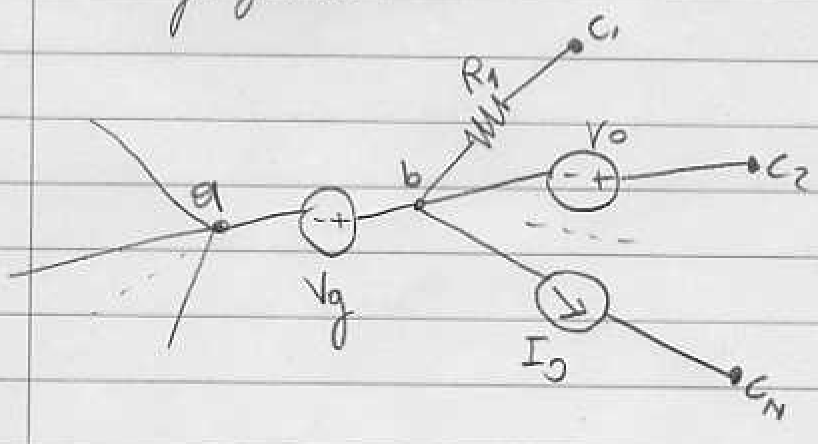


$V_j - V_i = V_S$

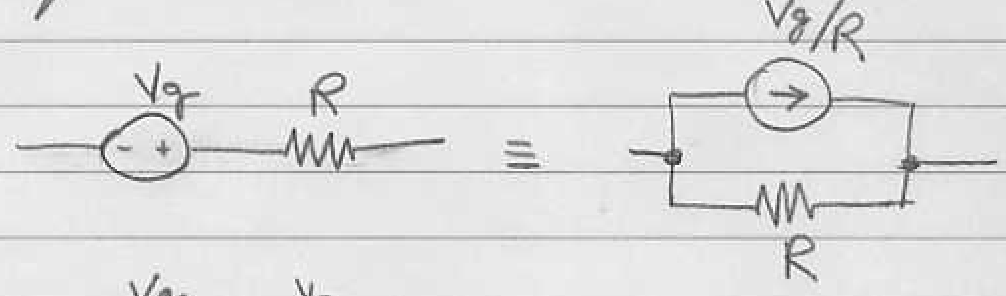
So we have the same number of unknowns and equations and we can still solve the circuit.

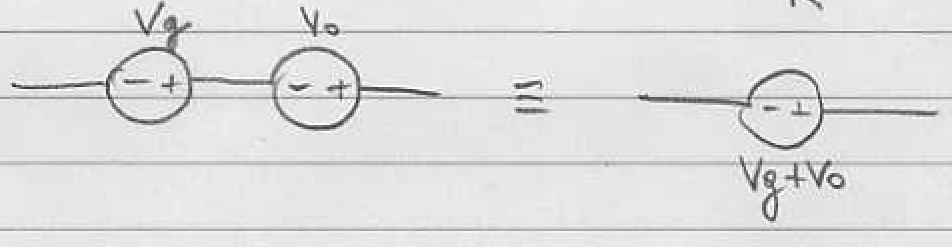
There is also a direct method which uses the generators tranformation.

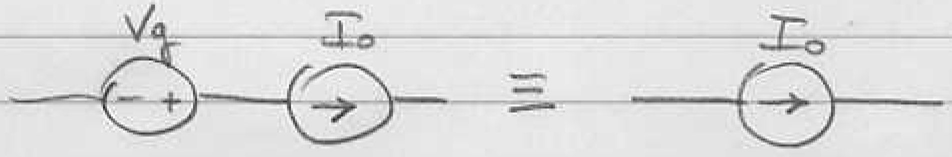
The only possible cases are summarized in the following figure:



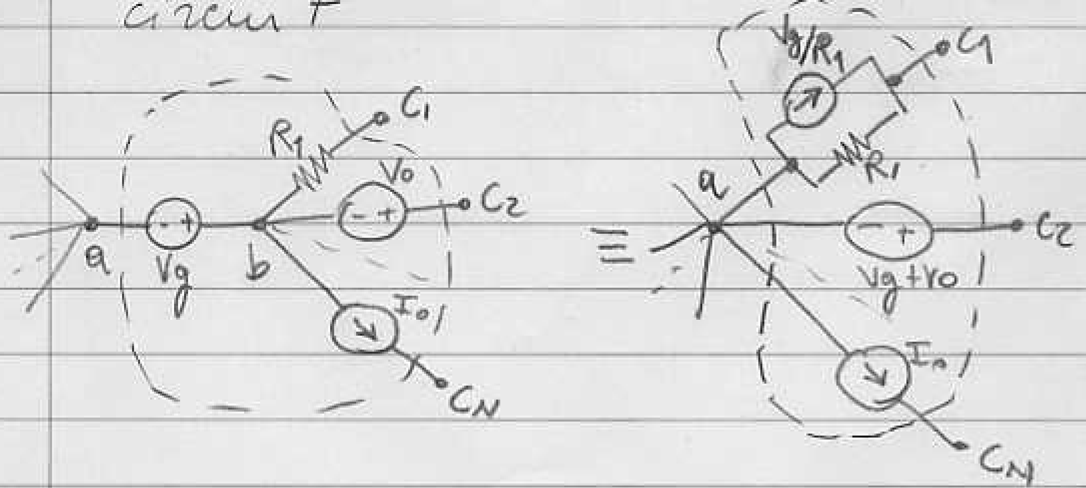
We can use the following equivalence



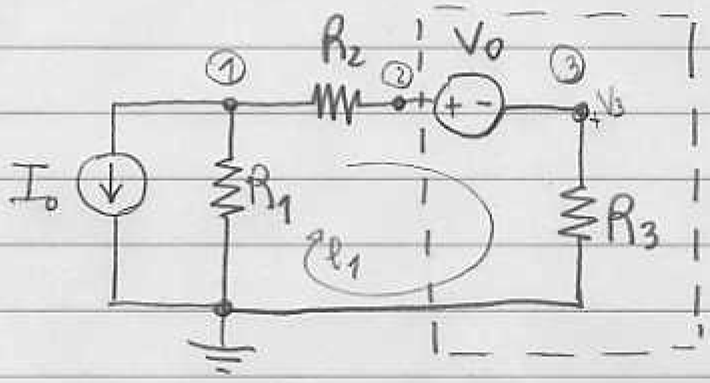




And modify the original circuit to obtain an electrically equivalent circuit

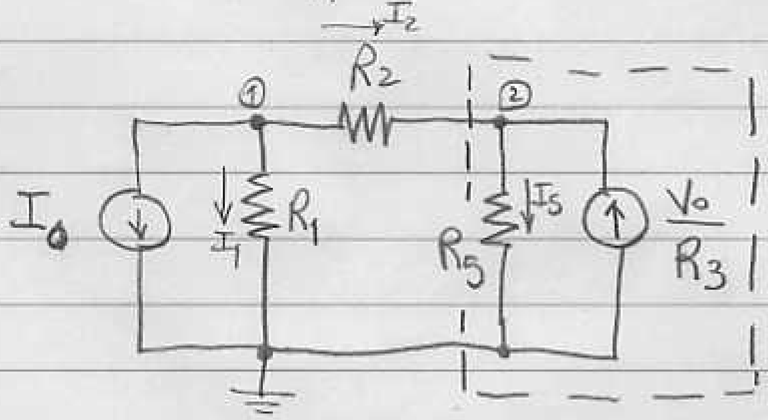


Example



$R_1 = R_2 = R_3 = 1\,\Omega$
$V_0 = 5\ V$
$I_0 = 1\ A$

|||



$$\begin{cases} I_0 + I_2 + I_1 = 0 & (1) \\ I_5 - I_2 - \dfrac{V_0}{R_3} = 0 & (2) \end{cases}$$

$$\begin{cases} 1 + \dfrac{V_1 - V_2}{R_2} + \dfrac{V_1}{R_1} = 0 & (1) \\ \dfrac{V_2}{R_5} - \dfrac{V_1 - V_2}{R_2} - 5 = 0 & (2) \end{cases}$$

$$\begin{cases} 2V_1 - V_2 = -1 \\ -V_1 + 2V_2 = 5 \end{cases} \quad \begin{cases} V_2 = 2V_1 + 1 \\ -V_1 + 2(2V_1 + 1) = 5 \end{cases}$$

$$\begin{cases} V_2 = 2V_1 + 1 \\ -V_1 + 4V_1 + 2 = 5 \end{cases} \quad \begin{cases} V_2 = 2V_1 + 1 \\ 3V_1 = 3 \end{cases} \quad \begin{cases} V_2 = 3V \\ V_1 = 1V \end{cases}$$

Now we want to know V_3. Just roll back your tranformation! I_2 an V_2 will stay the same because looking from node 2, the two circuits are electrically equivalent!

1) Using Ohm's law: R_2 is in series with V_0 an R_3, so the current in R_3 is I_2

$$\Rightarrow \quad V_3 = R_3 I_2 = R_3 \frac{V_1 - V_2}{R_2} = -2\ V$$

2) Using KVL on loop l_1

$$V_3 - V_1 + (V_1 - V_2) + V_0 = 0$$

$$V_3 = -V_0 + V_1 - (V_1 - V_2) = -5 + V_2 = -5 + 3 =$$
$$= -2V$$