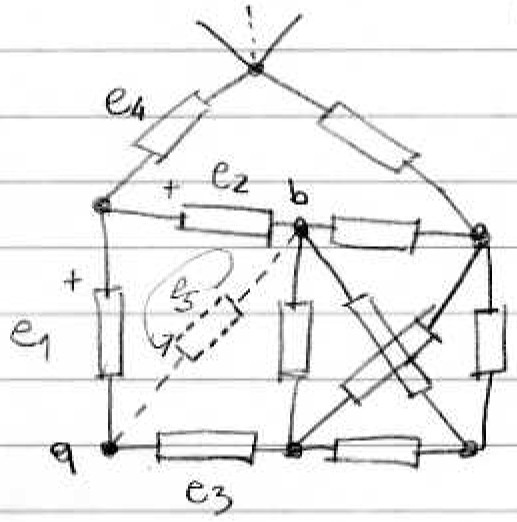
- Few things from last lecture

- About voltage across two points

Consider a complex circuit:



let V_i to be the voltage across e_i (the i-th element).

We want to express V_{ab}.

Of course we are going to apply KVL. Depending on which loop we choose we are going to obtain different expressions (they are all valid).

If there are two nodes that are not connected though, we can use the following procedure.

First of all let's define a new

quantity called CONDUCTANCE

$$G = \frac{I}{V} = \left[\frac{A}{V} = \text{SIEMENS}(S)\right]$$

It is relate to resistance : $G = \frac{1}{R}$

We do this because $R = \infty$ doesn't really make sense while $G = 0$ does!!

A perfect insulator is characterized by $G = 0$ S (vacuum for instance).

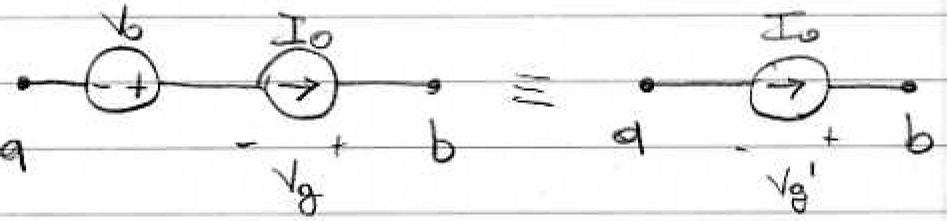
Given a circuit we can add this king of component

$G=0$

across every two modes without altering the circuit. So basically now we can add e_5 to the circuit such that its conductance $G_5 = 0$ an write

$$V_{ab} - V_2 + V_1 = 0 \Rightarrow V_{ab} = V_2 - V_1 \quad \text{(for instance)}$$

- About equivalences



If in a complex circuit we have a branch like the one on the left, we can take it out and substitute it with the branch on the right and nothing changes for THE REST of the circuit.

For the branch on the left

$V_{ab} = -V_0 - V_g \qquad I_{ab} = I_0$

Since V_g is determined by the rest of the circuit, so is V_{ab}

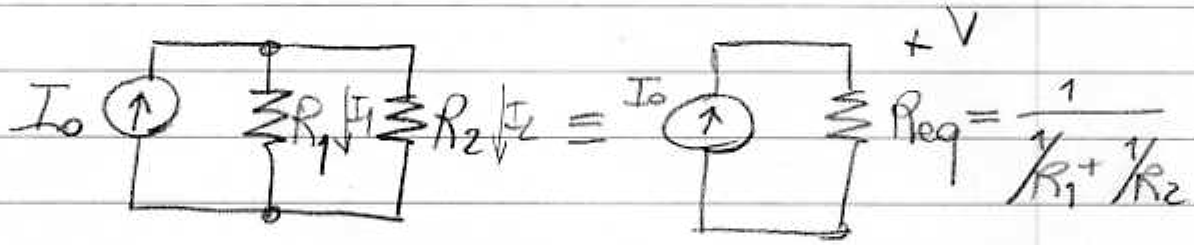
For the circuit on the righ

$V_{ab} = -V_g' \qquad I_{ab} = I_0$

Also here V_{ab} is determined by the rest of the circuit. So the two branches are electrically equivalent.

- Current divider

Consider a current generator a two resistors in paraller



$$V = R_{eq} I_0 = I_0 \frac{1}{1/R_1 + 1/R_2} = I_0 \frac{R_1 R_2}{R_1 + R_2}$$

$$\Rightarrow I_1 = \frac{V}{R_1} = I_0 \frac{R_2}{R_1 + R_2}$$

$$I_2 = \frac{V}{R_2} = I_0 \frac{R_1}{R_1 + R_2}$$

$$\frac{I_1}{I_2} = \frac{\cancel{I_0} \, R_2 / \cancel{(R_1+R_2)}}{\cancel{I_0} \, R_1 / \cancel{(R_1+R_2)}} = \frac{R_2}{R_1}$$

$I_1 \gg I_2$ if $R_2 \gg R_1$

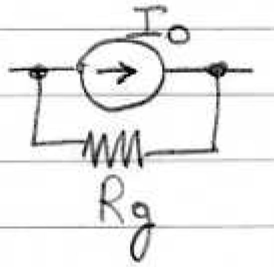
$I_2 \gg I_1$ if $R_1 \gg R_2$

(pretty obvious, isn't it?)

• About current generators

How can we build a current generator?

In fact, it is not possible to build a ideal current generator but only a non-ideal one:



The "art" in building one is to make R_g as big as possible. Why?

We are going to connect a load to this generator (a device). A model for the load is (by first approximation) a resistor. So when we connect our device, the circuit looks like

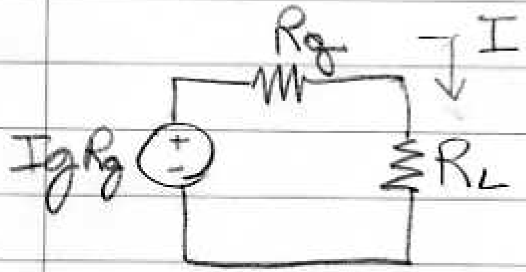
In order to be $I_L \gg I_g$ (and hence $I_L \simeq I_o$) it must be

$R_g \gg R_L$, so R_g must b Huge!!

So, now use the equivalence

(for exercice, check the equivalence for $I_{ab}=0$)

We can buil a current generator using a voltage generator with a big resistor in series:



← device

$$I = \frac{R_g I_g}{R_g + R_L}$$

but $R_g \gg R_L$

$\Rightarrow R_g + R_L \simeq R_g$

$$\Rightarrow I \simeq \frac{\cancel{R_g} I_g}{\cancel{R_g}} = I_g$$

Which is the current source nominal value !

- Summary of Node-Voltage Analysis

Problem: given a circuit as interconnection of resistors, voltage sources an current sources find currents/voltages through/across all elements

Fix a reference node n

↓

Consider voltages at all node n_i with respect to n as UNKNOWNS

↓ (1) to the next page

from the next page (2) → ∀ node i write KCL as

$$\sum_{\substack{\forall j \text{ connected} \\ \text{to } i \text{ and current from } i \text{ to } j}} i_{ij} = \sum_{\substack{\forall k \text{ connected} \\ \text{to } i \text{ on current from } k \text{ to } i}} i_{ki}$$

↓

Rewrite each current as function of voltages : $i_{lm} = f(V_l, V_m)$ wher f depends on the element connectet across l-m

For voltage sources consider the current as unknown, for current sources the current is known (it's a number)

↓ (1) to the next page

(1) from the previous page

↓

two ways now

$\forall$ voltage of value V_0 connected across node l-m write a new equation

$$V_l - V_m = V_0$$

↓

Solve the system of equations

transform each voltage source into a current source using the method in lec. 3, pag 19

↓

(2) to the previous page

Why have I introduced such complicated transformation?
It may seem complicated to you but it is very easy to implement on a computer!

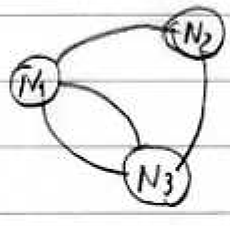
The method (we call it indirect) without transformation is perfectly fine by I personally like the other one (direct) because is very elegant.

- Tree an Co-Tree of a circuit

We have defined a graph associated with a circuit.

[quick graph definition:

$G = (N, E)$ N set of vertices
$E \subseteq N \times N$ set of edges



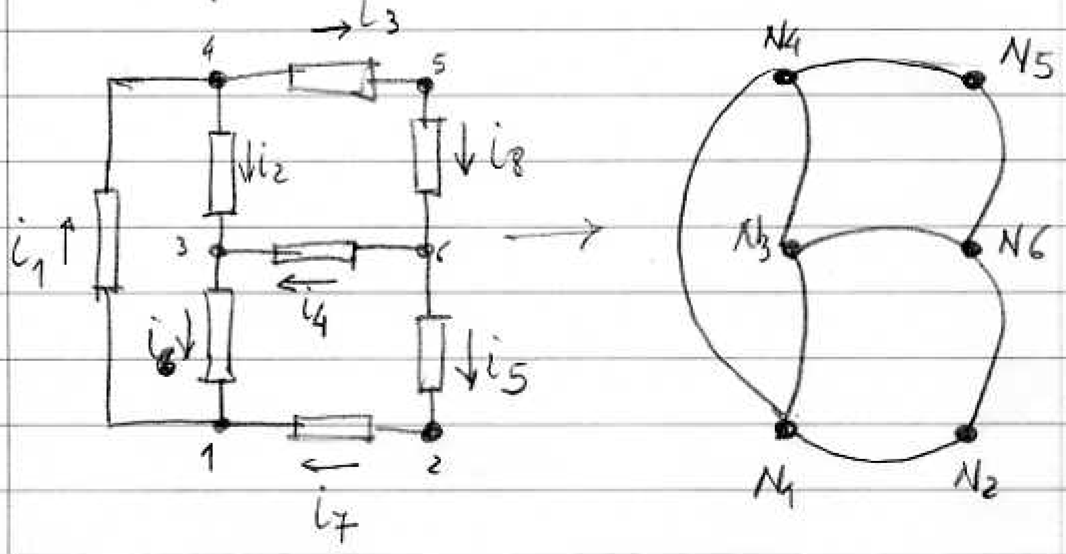
$N = \{N_1, N_2, N_3\}$

$E = \{(N_1, N_3), (N_1, N_3), (N_2, N_3), (N_1, N_2)\}$]

Given a circuit C, build the graph G_C in the following way:

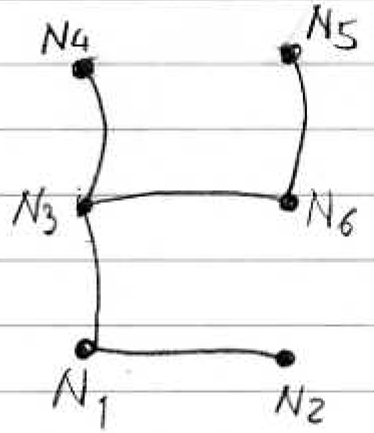
- $\forall$ node i in C create a vertex N_i in G
- $\forall$ branch in (i, j) in C create an edge (N_i, N_j) in G

Example



Definition: A spanning tree of a graph is an acyclic subset of edges $T \subseteq E$ that touches all the nodes

Example



$$T = \{(N_1, N_3), (N_1, N_2), (N_3, N_6), (N_6, N_5), (N_3, N_4)\}$$

$CT = E \setminus T$ (all the edges but the edges of the tree)
is called co-tree.

Ex: $CT = \{(N_1, N_4), (N_4, N_5), (N_6, N_2)\}$

Let's use the currents in the edges of the co-tree to express all the othe currents:

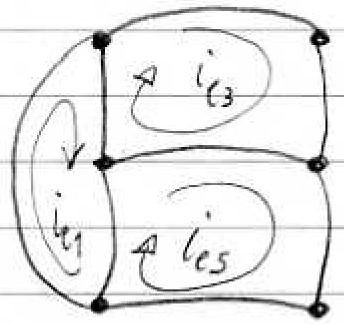
$$I_2 = I_1 - I_3$$
$$I_8 = I_3$$
$$I_7 = I_5 \qquad (1)$$
$$I_6 = I_1 - I_7 = I_1 - I_5$$
$$I_4 = I_8 - I_5 = I_3 - I_5$$

You can prove the the currents of the edges of the co-tree are a set of independent variables!!

(on the other hand voltages across the edges of the tree are independent variables!!)

(as an excercise try to prove it)

Let's define now some fictitious currents



$\forall$ edges of the co-tree, define a loop current in the loop gotten by adding the edge to the tree.

The orientation of the current in the loop determined by the orientation of the current in the edge of the co-tree.

Now, write the currents in the edges by superposition of the fictitious currents:

$$i_3 = i_{e_3}$$
$$i_2 = i_{e_1} - i_{e_3}$$
$$\vdots$$

the result is exactly the same as system 1

- Mesh-current analysis

This method is the dual of the node-voltage analysis.

Consider the case where only voltage sources and resistors are present.

The method works in the following way:

1) Pick a tree and consider the currents of the edges of the co-tree as fictitious mesh currents

2) $\forall$ loops l_i write the KVL, each equation will look like

$$\sum_{\substack{j \text{ element} \\ \text{in loop } i}} V_j = 0$$

3) Express each voltage using the loop currents

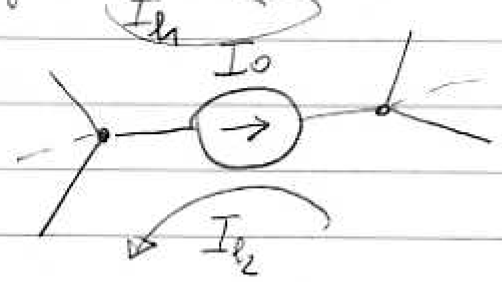
$$V_j = f\left(\sum_{\substack{k \text{ s.t. element} \\ j \text{ is in loop } k \\ \text{and current } i_k \text{ has the} \\ \text{same direction of } i_j}} i_k - \sum_{\substack{m \text{ s.t. element} \\ j \text{ is in loop } m \text{ and} \\ \text{current } i_m \text{ has opposite} \\ \text{direction w.r.t. } i_j}} i_m \right)$$

4) Solve the system of equations

In 3) if you have a voltage source, V_j is just a number so it is a known quantity.

The solution of the system of equations gives the value of all loop currents. Then we can compute all other quantities.

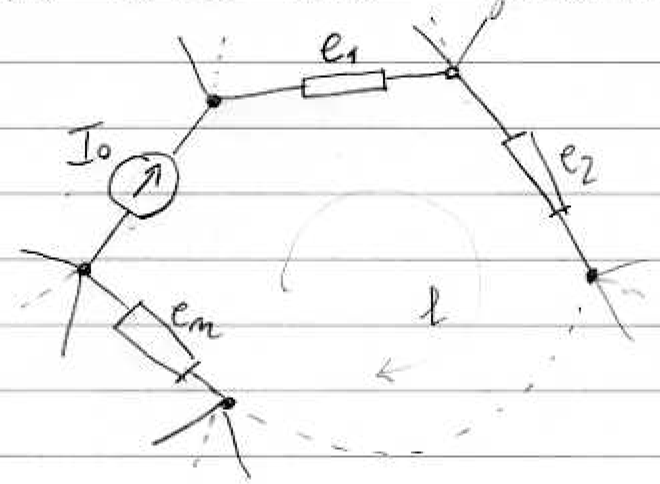
What happens if there are also current generators? We don't know V_j in that case and this quantity has to be considered unknown (it is determined by the rest of the circuit).
Fortunately enough, the current source gives us one more equation:



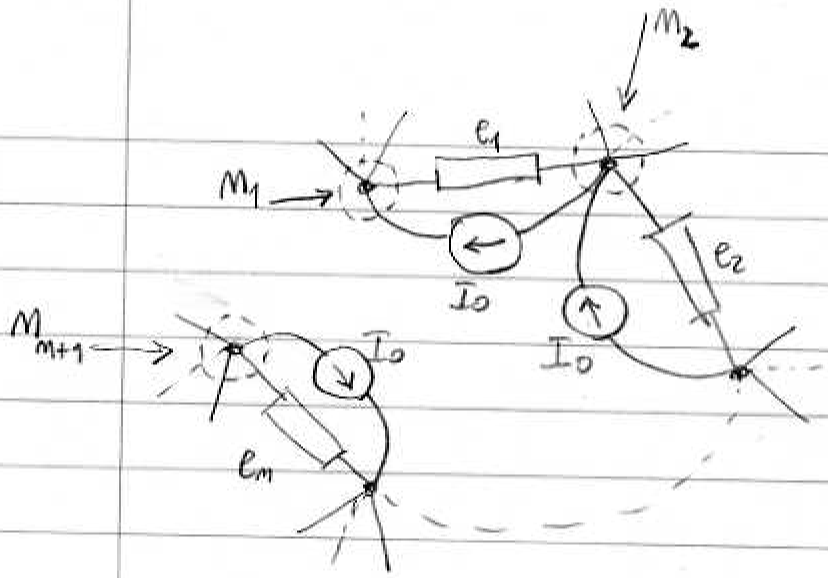
$$I_{l_1} - I_{l_2} = I_0$$

So we add as many more equations as current sources in order to have as many equations as unknowns (remember that each current source adds an unknown which is the voltage across it).

What I like better is a "direct method". We transform the circuit in order to avoid current sources. A current source will belong to one ore more loops. Just considere one of these loops, say l. The situation is like the following:



We now put the current source in parallel to each element in the loop:

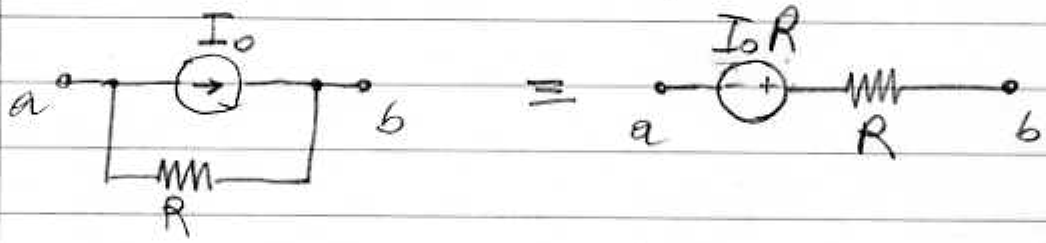


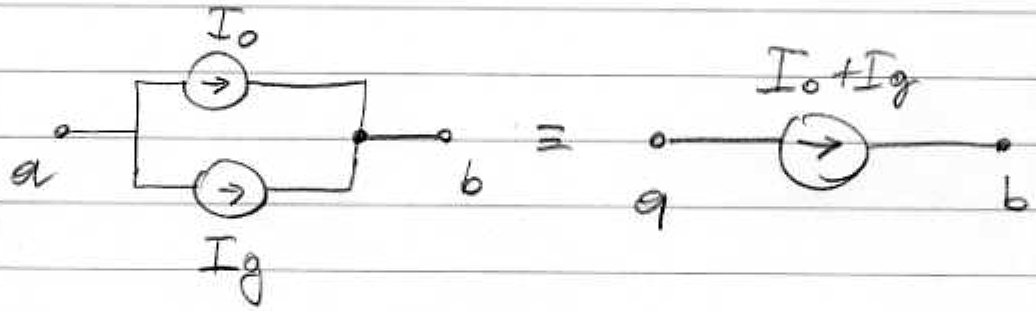
You can check that nothing changes from currents/voltages point of view.
For each node different from n_1 an n_{m+1} there is I_0 coming in an another I_0 going out and they cancel each other. (For node n_2 for instance).

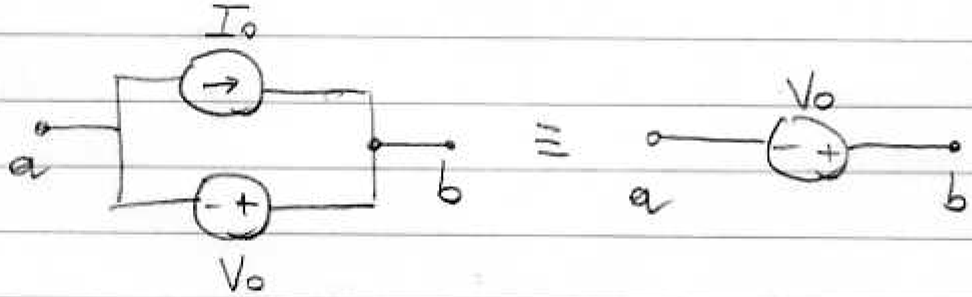
For n_1 there is a current I_0 entering like the original circuit, and for n_{m+1} there is a current I_0 leaving as in the original circuit.

Also, the voltage across each current source will be determined by the circuit and, since currents don't change, it will be the same as before.

Now we use the transformations:



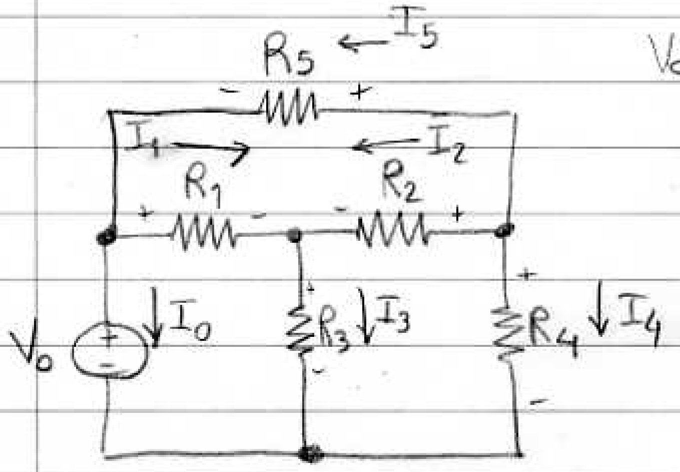




After the transformations, the circuit has only voltage sources and the application of the method is straightforward.

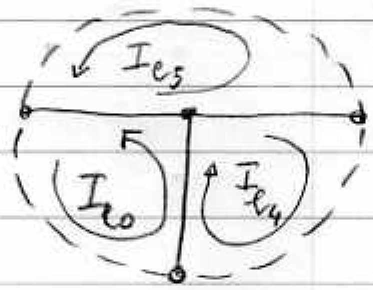
Example

(no current sources)



$V_0 = 2V$; $R_1 = R_4 = 2\Omega$; $R_2 = R_3 = R_5 = 1\Omega$

— tree , --- co-tree



Write KVL

(ℓ_0) $\quad V_0 - V_3 - V_1 = 0$

(ℓ_4) $\quad V_4 - V_3 - V_2 = 0$

(ℓ_5) $\quad V_5 + V_1 - V_2 = 0$

which is

$$V_0 - R_3 I_3 - R_1 I_1 = 0$$

$$R_4 I_4 - R_3 I_3 - R_2 I_2 = 0$$

$$R_5 I_5 + R_1 I_1 - R_2 I_2 = 0$$

Rewrite each current using the loop currents:

$I_3 = -I_{l0} - I_{l4}$ $\qquad$ $I_0 = I_{l0}$

$I_1 = I_{l5} - I_{l0}$ $\qquad$ $I_5 = I_{l5}$

$I_2 = -I_{l5} - I_{l4}$ $\qquad$ $I_4 = I_{l4}$

Rewrite the system

$$\begin{cases} V_0 - R_3(-I_{l0} - I_{l4}) - R_1(I_{l5} - I_{l0}) = 0 \\ R_4 I_{l4} - R_3(-I_{l0} - I_{l4}) - R_2(-I_{l5} - I_{l4}) = 0 \\ R_5 I_{l5} + R_1(I_{l5} - I_{l0}) - R_2(-I_{l5} - I_{l4}) = 0 \end{cases}$$

$$\begin{cases} 2 + I_{l0}(R_3 + R_1) + I_{l4} R_3 - I_{l5} R_1 = 0 \\ I_{l0} R_3 + I_{l4}(R_4 + R_3 + R_2) + I_{l5} R_2 = 0 \\ -R_1 I_{l0} + R_2 I_{l4} + I_{l5}(R_1 + R_5 + R_2) = 0 \end{cases}$$

$$\begin{cases} 3I_{l0} + I_{l4} - 2I_{l5} = -2 \\ I_{l0} + 4I_{l4} + I_{l5} = 0 \\ -2I_{l0} + I_{l4} + 4I_{l5} = 0 \end{cases}$$

$$\begin{bmatrix} 3 & 1 & -2 \\ 1 & 4 & 1 \\ -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} I_{l0} \\ I_{l4} \\ I_{l5} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

Now just solve it!

$$I_{\ell_0} = -\frac{10}{7} \qquad I_{\ell_4} = \frac{4}{7} \qquad I_{\ell_5} = -\frac{6}{7}$$

Now we can compute the original currents and voltages:

$$I_1 = I_{\ell_5} - I_{\ell_0} = -\frac{6}{7} + \frac{10}{7} = \frac{4}{7} \quad A$$

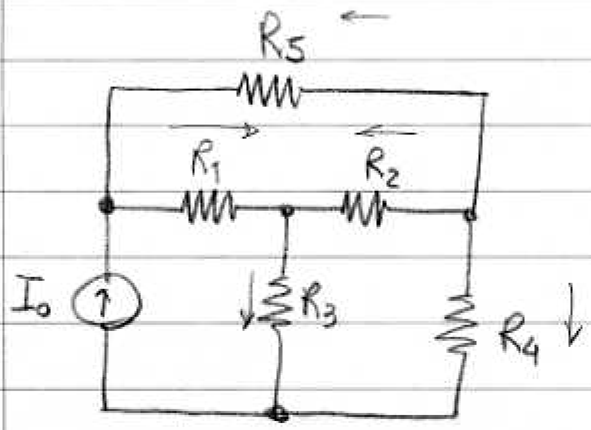
$$I_2 = -I_{\ell_5} - I_{\ell_4} = \frac{6}{7} - \frac{4}{7} = \frac{2}{7} \quad A$$

$$I_3 = -I_{\ell_0} - I_{\ell_4} = \frac{10}{7} - \frac{4}{7} = \frac{6}{7} \quad A$$

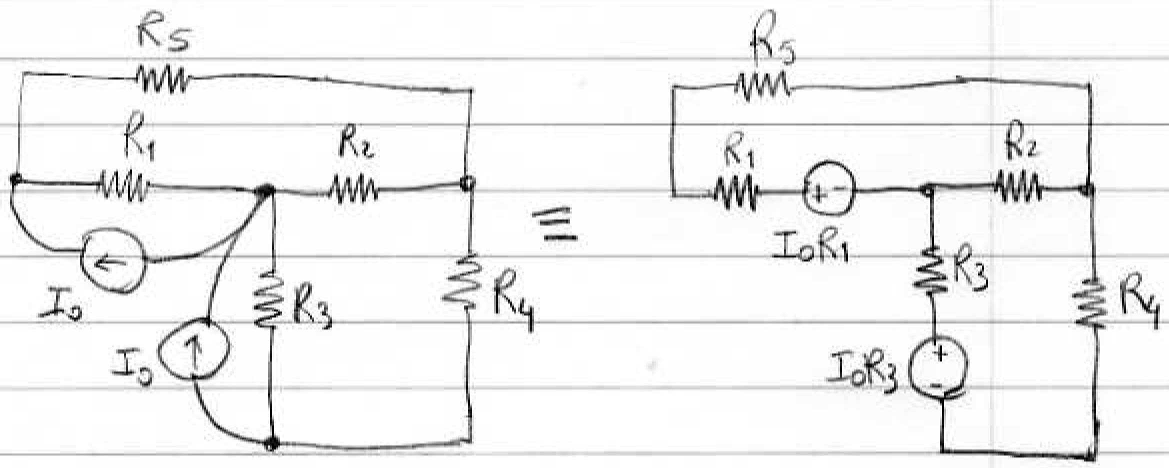
And for instance

$$V_1 = R_1 I_1 = 2 \cdot 4/7 = 8/7 \quad V$$

Consider instead this circuit



Then we can use the transformation



Now there are only voltage generators and resistors and the method applies as in the previous example.