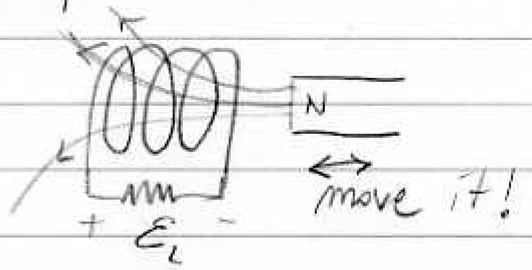
INDUCTORS

To understand current-voltage relation in an inductor we need to go back to Faraday an Lenz.

Faraday made the following experiment



He moved a magnet in and out a solenoid (coil).

He observed an induced voltage whose value was proportional to the change in magnetic flux:

$$\mathcal{E}_L \propto \frac{d\Phi_B}{dt}$$

Lenz's law says that the voltage that gets generated is such that the current generats a magnetic field which is opposite to the one generating the voltage this law introduces a minus sign in the equation

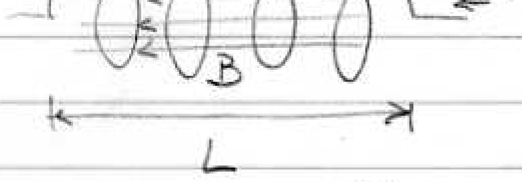
$$\mathcal{E}_L = -\frac{d\Phi_B}{dt}$$

The flux is proportional to the current so the induced voltage is proportional to the current

$$\mathcal{E}_L = -L\frac{di}{dt}$$ L is called inductance

and its unit of measure is Henry = H.

The solution of Maxwell's equations for a solenoid gives the following magnetic field



$$B = \mu_0 n I = \mu_0 \frac{N}{L} I$$

when N is the number of turns. The flux is

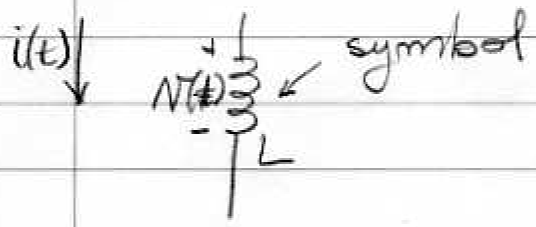
$$\Phi_B = NBA = N\mu_0\frac{N}{L} I A$$

where A is the area of one turn.

$$\mathcal{E}_L = -\frac{d\Phi_B}{dt} = -\mu_0 \frac{N^2}{L} A \frac{dI}{dt}$$

$$\Rightarrow L = \mu_0 \frac{N^2}{L} A$$

- <u>Current an voltage in an inductor</u>



$$v(t) = L \frac{di(t)}{dt}$$

In steady state $\frac{di(t)}{dt} = 0 \Rightarrow v(t) = 0$

an inductor behaves like a short circuit.

Note that the relation is linear!

Integrating the expression we have

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$$

- Power and energy in an inductor

$$p(t) = v(t)\, i(t) =$$

$$= L\, i(t) \frac{d\, i(t)}{dt}$$

The energy is the integral.

$$e(t) = \int_{t_0}^{t} p(t)\, dt = L \int_{t_0}^{t} i(t)\, d\, i(t) =$$

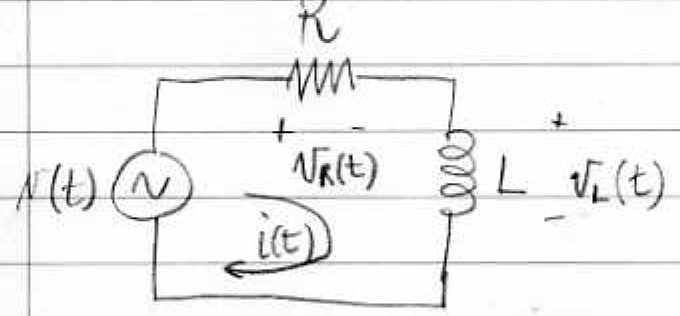
$$= \frac{1}{2} L\, i^2(t) - \frac{1}{2} L\, i^2(t_0)$$

This is the energy stored in the inductor at time t.

– ANALYSIS OF CIRCUITS IN PRESENCE OF INDUCTORS

As in the case of capacitors, we can apply node-voltage or mesh-current analysis to obtain a system of differential equations.

Cosider the following circuit:



$$v(t) = \sin(2\pi f_0 t) = \sin(\omega_0 t)$$

Using KVL:

$$Ri(t) + v_L(t) = v(t)$$

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$

Again we can compute $i(t)$

homogeneous equation:

$$R\,i(t) + L\frac{d\,i(t)}{dt} = 0$$

considering $i(t) = e^{\alpha t}$

$$Re^{\alpha t} + L\alpha e^{\alpha t} = 0 \Rightarrow R + \alpha L = 0$$
$$\Rightarrow \alpha = -\frac{R}{L}$$

$$i(t) = e^{-\frac{R}{L}t}$$

Considering $\eta(t) = A\sin(w_o t) + B\cos(w_o t)$

$RA\sin(w_o t) + RB\cos(w_o t) - LAw_o\cos(w_o t) +$

$+ LBw_o\sin(w_o t) = \sin(w_o t)$

$$\begin{cases} RA + LBw_o = 1 \\ RB - LAw_o = 0 \end{cases} \quad \begin{cases} RA + \dfrac{L^2Aw_o^2}{R} = 1 \\ B = \dfrac{L}{R}Aw_o \end{cases}$$

$$\begin{cases} A = \dfrac{1}{R + \dfrac{w_o^2L^2}{R}} = \dfrac{R}{R^2 + w_o^2L^2} \\ B = \dfrac{L}{R}\,\dfrac{R}{R^2 + w_o^2L^2}\,w_o = \dfrac{w_o L}{R^2 + w_o^2L^2} \end{cases}$$

call $L/R = \tau$

$$A = \frac{1}{R(1+\omega_0^2\tau^2)} \qquad B = \frac{\omega_0\tau}{R(1+\omega_0^2\tau^2)}$$

a particular solution of the differential equation is:

$$i(t) = \frac{1}{R(1+\omega_0^2\tau^2)} \sin(\omega_0 t) + \frac{\omega_0\tau}{R(1+\omega_0^2\tau^2)} \cos(\omega_0 t) =$$

$$= \sqrt{A^2+B^2}\, \sin\left(\omega_0 t + \operatorname{arctg}\left(-\frac{B}{A}\right)\right)$$

$$\sqrt{A^2+B^2} = \frac{1}{1+\omega_0^2\tau^2} \sqrt{1/R^2 + \omega_0^2\tau^2/R^2} =$$

$$= \frac{1}{R\sqrt{1+\omega_0^2\tau^2}}$$

$$i(t) = \frac{1}{R\sqrt{1+\omega_0^2\tau^2}} \sin(\omega_0 t + \alpha)$$

phase

$$v_L(t) = L\frac{di(t)}{dt} = -\frac{L}{R}\frac{\omega_0}{\sqrt{1+\omega_0^2\tau^2}} \cos(\omega_0 t + \alpha)$$

$$v_L(t) = \frac{\omega_0 \tau}{\sqrt{1+\omega_0^2\tau^2}} \sin(\omega_0 t + \alpha')$$

Again the amplitude depends on the frequency.

If $\omega_0 = 0 \Rightarrow$

$$\lim_{\omega_0 \to 0} \frac{\omega_0 \tau}{\sqrt{1+\omega_0^2\tau^2}} = 0$$

$$\lim_{\omega_0 \to \infty} \frac{\omega_0 \tau}{\sqrt{1+\omega_0^2\tau^2}} = 1$$

This is an high-pass filter whose cut-off frequency is (ω_0^*):

$$\frac{\omega_0^* \tau}{\sqrt{1+\omega_0^{*2}\tau^2}} = \frac{1}{\sqrt{2}} \Rightarrow \frac{\omega_0^{*2}\tau^2}{1+\omega_0^{*2}\tau^2} = \frac{1}{2}$$

$$2\omega_0^{*2}\tau^2 = 1 + \omega_0^{*2}\tau^2 \Rightarrow \omega_0^{*2}\tau^2 = 1 \Rightarrow \omega_0^* = \frac{1}{\tau}$$

Mead

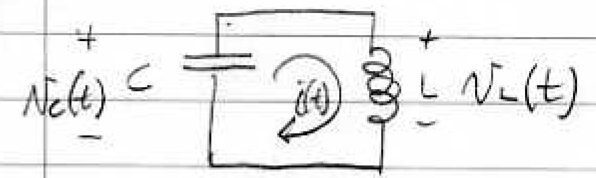
- SECOND ORDER CIRCUITS (ONLY FEW THINGS)

We can combine inductors an capacitor in circuits. Now since

$$i_c(t) = C \frac{d v_c(t)}{dt} \qquad v_L(t) = L \frac{d i(t)}{dt}$$

it may happen that part (or all) the capacitor current flows into the inductor giving a double derivative.

Consider the following circuit



with $v_c(0) = 1$
$i(0) = 0$

we have $v_c(t) = v_L(t) = L \frac{d\,i(t)}{dt} =$

$$= LC \frac{d^2 v_c(t)}{dt^2}$$

$$\frac{d^2 v_c(t)}{dt^2} - \frac{1}{LC} v_c(t) = 0$$

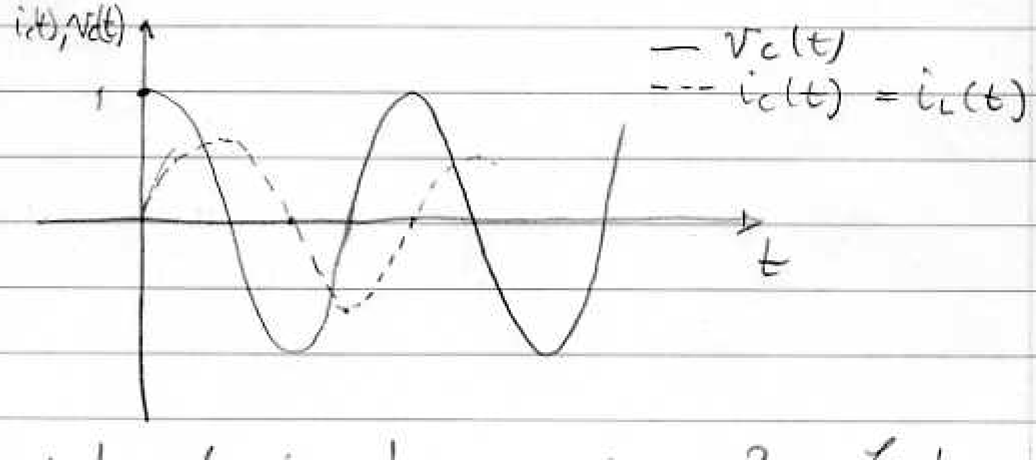
consider the solution $v_c(t) = A \sin(\omega t + \varphi)$

$$-A\omega^2 \sin(\omega t + \varphi) - \frac{1}{LC} A \sin(\omega t + \varphi) = 0$$

$$\omega^2 = \frac{1}{LC} \Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

Since $v_c(0) = 1$ and $i_c(0) = 0$
then it must be $\varphi = \pi/2$ and $A = 1$

The voltage across the capacitor is a sinusoid:



what is happening? Look at the energy:

$$w_c(t) = \frac{1}{2} C v_c^2(t)$$

$$w_L(t) = \frac{1}{2} L i_L^2(t) = \frac{1}{2} L i_c^2(t)$$

so when the energy stored in the capacitor is maximum the energy in the inductor is zero and vice-versa.
The two objects are exchanging energy tha was originally stored in the capacitor.

This circuit is lossless because C an L do not dissipate (absorb) energy. To see this, compute the total energy absorbed by, for instance, C

$$P = \lim_{T\to\infty} \frac{1}{T} \int_0^T v(t)\, i(t)\, dt$$
$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T \sin(\ )\cos(\ )\, dt$$

and the integral is known to be 0 !!