EE40: Introduction to μelectronic Circuits
Lecture Notes

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July 10, 2004

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1 First Order Circuits

Electronic components that can store energy are extremely important. Energy can be stored in two forms: electric or magnetic. In this lecture we introduce capacitors, which can store electric energy, and inductors, which can store magnetic energy.

We are going to gently introduce both components, then we are going to analyze circuits that use capacitors and inductors and finally we are going to give some basic notions of frequency analysis that turns out to give us the possibility of analyzing circuits with capacitors and inductors using the same methods what we have studied for resistors.

1.1 Capacitors

Consider the structure in figure 1. It is built with two parallel plates of

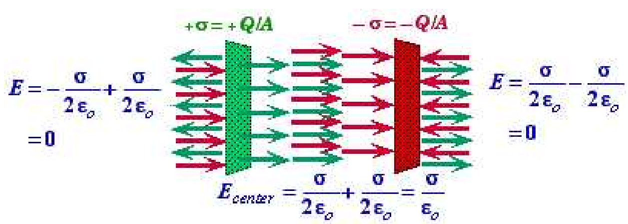

Figure 1: Electric field in an ideal parallel plate capacitor

area A that are charged with an opposite amount of charge Q Coulomb (one plate with Q and the other with $-Q$. The arrows represent the electric field lines. We define a new quantity which is the charge density $\sigma = Q/A$.

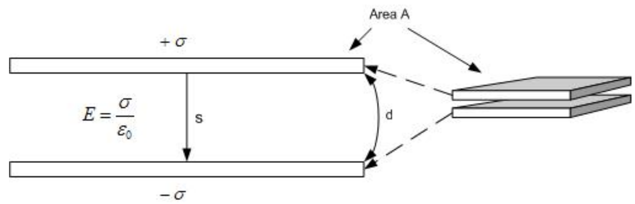

Figure 2: Setting for capacitance computation of a parallel plate capacitor.

It is a well known result that the electric field of a plane plate has lines that are orthogonal to the plate and the electric field value, close to the surface, is $E = \sigma/\epsilon_0$ where ϵ_0 is the dielectric constant of the vacuum.

We now put two parallel plates close to each other with opposite charge. Using the superposition principle (it is valid for electric field because Maxwell's equations are linear) we can compute the electric field inside and outside this structure. Since the plates are oppositely charged, the electric fields lines have opposite directions and the filed adds up inside and vanishes outside. This structure is called parallel plate capacitor.

We can relate the charge with the electric field and hence with the potential difference (voltage) across the two plates. To do that we compute voltage starting from the electric field. If you remember the relation between them, the voltage difference between two points p_1 and p_2 is the integral of the electric field along a line connecting the two points. In our case we can use the line denoted by s in figure 2:

$$V = \int_s E ds = \frac{\sigma d}{\epsilon_0}$$

The result is that $V = (qd)/(A\epsilon_0)$. The relation between voltage and total charge is $Q = CV$ where C is a constant that we call *capacitance*. For our structure the capacitance is $C = \epsilon_0 A/d$ which is a well know formula. The unit of measure is *Farad* (honoring Michael Faraday, b. Sept. 22, 1791, d. Aug. 25, 1867) which is a very big value so we usually use μF, nF and so on.

We have done the computation for the parallel plate capacitor but capacitance is defined in general for any two (and actually for N) bodies. It represents the capacity of a structure to generate a voltage difference between its bodies when their are charged.

1.2 Current and Voltage relation in capacitors

We have the relation between voltage and charge in a capacitor (they are related by a quantity that is the capacitance). We can now apply derivative to the equation and find the current-voltage relation:

$$q = cv \Rightarrow \frac{dq}{dt} = i(t) = c\frac{dv(t)}{dt}$$

Current flowing in a capacitor depends on rate of voltage change across the capacitor. In particular if the voltage is constant the current is zero. This is intuitive since if the voltage stays constant the charge is also constant and hence there is no current.

If we integrate the previous expression we get:

$$i(t) = c\frac{dv(t)}{dt} \Rightarrow v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)dt + v(t_0)$$

where $v(t_0)$ is the initial condition, meaning the voltage across the capacitor at time t_0 (when you started the experiment for instance).

In summary, voltage across a capacitor is the integral of the current flowing through it, and current through a capacitor is the derivative of the voltage across it. Of course there is a constant factor that is the capacitance. Note that the relation between the two quantity is a differential equation but (since we assume C time independent) it is a *LINEAR* relation. A circuit with resistor and capacitor is still a linear circuit.

1.3 Power and Energy in capacitors

We know the general expression for power absorbed by a circuit element:

$$p(t) = v(t)i(t)$$

note that this is instantaneous power. If we want to know the average power we have to integrate over time and den divide by the integration period.

For a capacitor we have:

$$p(t) = v(t)C\frac{dv(t)}{dt}$$

We can now compute the energy which is the integral of power over time:

$$e(t) = \int_{t_0}^{t} p(t)dt = C \int_{t_0}^{t} v(t) \frac{dv(t)}{dt} =$$
$$\frac{1}{2}Cv^2(t) + \frac{1}{2}Cv_0^2$$

1.4 Circuit analysis in presence of capacitors

Can we still analyze circuits if they use some capacitors? First of all we have to prove that we can still talk about lumped elements. This is indeed true and you can look at the appendix to understand why. We introduce a new symbol shown in figure 3 which represent a lumped capacitor which fixes a relation between current and voltage in the branch of the circuit it represents. We first note that in *steady state* the voltage across a capacitor is constant (by superposition), therefore the current through it is zero meaning that it behaves like an open circuit. Being in steady state means that each source has the same value (in time) since $t = -\infty$. Another way of looking at it is that we wait enough time for the circuit to stabilize (see appendix for some practical details.

Figure 3: Symbol indicating a lumped capacitor

If sources are time variant, then we cannot consider capacitors are open circuits anymore but we can still analyze it. We can apply the same methods that we have learned in this class. For instance we can use mesh-current analysis. The unknown in this method are the loop-currents. We first write the kVL to all meshes. Then we write each voltage across and element in terms of the current through it. If the element is a capacitor then we obtain a integral equation. We can still solve the system of integral equations (even if it is more difficult to do).

Using node-voltage analysis, we write KCL to all nodes and then substitute currents using node voltages. We obtain a system of differential equations.

Example 1 *Consider the series of a resistor and a capacitor like in figure 4 (we want to consider $v_c(t)$ as our output and, so we need to compute it). We set*

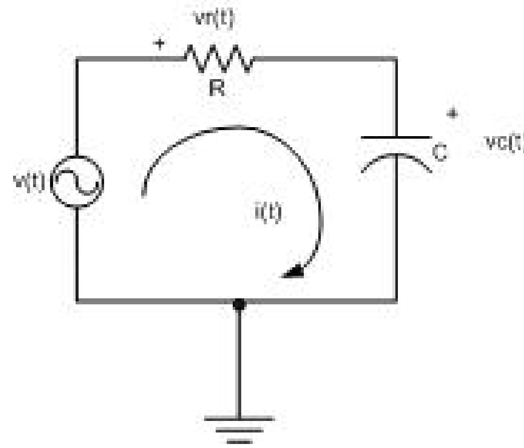

Figure 4: RC circuit

$v(t) = A\sin(2\pi f_0 t)$ *and we want to compute $v_c(t)$. We can write KVL to the only loop of this circuit:*

$$v(t) = v_c(t) + v_r(t) = Ri(t) + \frac{1}{C}\int_{t_0}^{t} i(t)dt + v(t_0)$$

we can take derivative of the whole equation:

$$\frac{dv(t)}{dt} = R\frac{di(t)}{dt} + \frac{1}{C}i(t)$$

this a linear differential equation with constant coefficients and the solution is very easy to computer (see appendix 1.5). The steady state solution for $v_c(t)$ is $v_c(t) = -\frac{1}{\sqrt{1+(RC2\pi f_0)^2}}\cos(2\pi f_0 t + \alpha)$ meaning that the output has the same shape as the input but amplitude and phase are different. This property is true for linear systems in general. If a system is linear then its response to a sinusoidal source is a sinusoid with a different amplitude and a different phase, but still a sinusoid and the reason will be more clear when we will talk about Fourier and Laplace transforms.

The output amplitude (the vpp of our signal) decreases by a factor which depends on the input frequency and actually inversely proportional to it. We are filtering high frequencies.

We define Cut-off frequency *that frequency for which the output is equal to* $1/\sqrt{2}$ *times the input. In our case it means* $f_0 = 1/(2\pi RC)$*. If we want to filter a signal up to* $5KHz$ *then we have to choose components in order to have* $5000 = 1/(2\pi RC)$*. There are of course many combination and the choice of* R *and* C *depends on many factors. Figure 5 shows the value of* $A = \frac{1}{\sqrt{1+(RC2\pi f_0)^2}}$ *as a function of frequency. It also shows the cut-off frequency.*

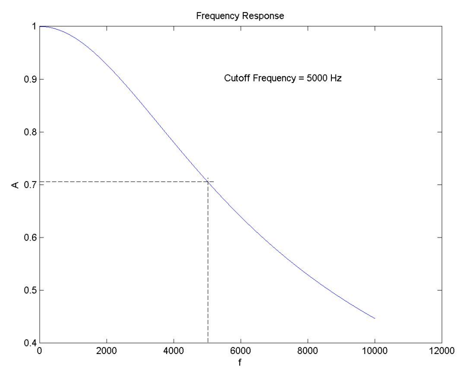

Figure 5: Frequency response of the RC circuit

Previous example shows a passive filter but we can of course build an active filter using operational amplifier.

1.5 Solution of the linear differential equation for the RC circuit

I suggest you to refresh your knowledge about differential equation even if this is a particular differential equation and very easy to solve. Given a differential equation we first find the homogeneous equation associated with it. The we find a basis for all the solution of such equation (the dimension of the basis being the order of the differential equation). Then we look for a particular solution of the original equation and finally we add them together.

For our circuit the homogeneous differential equation is:

$$\frac{di}{dt} + \frac{1}{RC}i = 0$$

We guess a solution for this equation $i(t) = e^{\alpha t}$ and we find α (substituting this in the previous equation):

$$\alpha e^{\alpha t} + \frac{1}{RC} e^{\alpha t} = 0 \Rightarrow$$
$$\alpha = -\frac{1}{RC}$$

A solution for the homogeneous equation (and actually a basis for all solutions) is $i(t) = e^{t/RC}$.

Now we look for a solution of the original equation. We look for a solution of type $\eta(t) = A\sin(2\pi f_0 t) + B\cos(2\pi f_0 t)$ (substituting in the original differential equation):

$$-A2\pi f_0 \cos(2\pi f_0 t) + B2\pi f_0 sin(2\pi f_0 t) + \frac{A}{RC}\sin(2\pi f_0 t) +$$
$$+\frac{B}{RC}\cos(2\pi f_0 t) = \frac{2\pi f_0}{R}\cos(2\pi f_0 t)$$

We get the following system of equations:

$$\frac{B}{RC} - A2\pi f_0 = \frac{2\pi f_0}{R}$$
$$B2\pi f_0 + \frac{A}{RC} = 0$$

Which gives:

$$A = -\frac{1}{(R(1 + 1/(RC2\pi f_0)^2)} = -\frac{1}{RD}$$
$$B = \frac{1}{R^2 C2\pi f_0 \left(1 + \frac{1}{1+(RC2\pi f_0)^2}\right)} = \frac{1}{R^2 C2\pi f_0 D}$$

Now we use the fact that $Asin(X)+Bcos(X) = \sqrt{A^2+B^2}\sin(X+tg^{-1}(-A/B))$. The equation can be rewritten as:

$$\eta(t) = \frac{1}{R\sqrt{1 + 1/(2\pi f_0 RC)^2)}}\sin(2\pi f_0 t + tg^{-1}(2\pi f_0 RC))$$

This equation is also the steady state current, meaning the current after that the exponential part of the circuit response is negligible. Since we know the relation between current and voltage across the capacitor, we can compute $v_c(t)$ as the integral of this expression which is:

$$v_c(t) = -\frac{1}{\sqrt{1 + (RC2\pi f_0)^2}} \cos(2\pi f_0 t + \alpha)$$

where $\alpha = tg^{-1}(2\pi f_0 RC)$.