EECS150 - Digital Design

Lecture 24 - Arithmetic Blocks, Part 2 + Shifters

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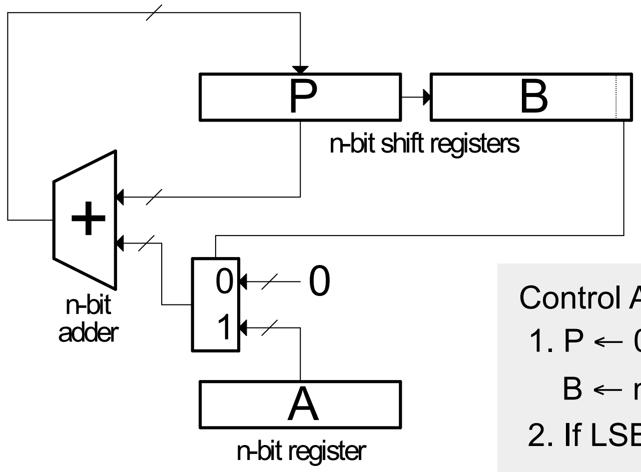
Multiplication

$$
\begin{array}{ccccccccl}
 & & & a_3 & a_2 & a_1 & a_0 & \leftarrow & \textit{Multiplicand} \\
 & & & b_3 & b_2 & b_1 & b_0 & \leftarrow & \textit{Multiplier} \\
\cline{4-7}
 & & X & a_3b_0 & a_2b_0 & a_1b_0 & a_0b_0 & \multirow{4}{*}{\Bigg\}} & \\
 & & a_3b_1 & a_2b_1 & a_1b_1 & a_0b_1 & & & \textit{Partial} \\
 & a_3b_2 & a_2b_2 & a_1b_2 & a_0b_2 & & & & \textit{products} \\
a_3b_3 & a_2b_3 & a_1b_3 & a_0b_3 & & & & & \\
\hline
 & & & \ldots & & a_1b_0+a_0b_1 & a_0b_0 & \leftarrow & \textit{Product}
\end{array}
$$

Many different circuits exist for multiplication.
Each one has a different balance between
speed (performance) and amount of logic (cost).

"Shift and Add" Multiplier



- Sums each partial product, one at a time.
- In binary, each partial product is shifted versions of A or 0.

Control Algorithm:
1. P ← 0, A ← multiplicand,
   B ← multiplier
2. If LSB of B==1 then add A to P
   else add 0
3. Shift [P][B] right 1
4. Repeat steps 2 and 3 n-1 times.
5. [P][B] has product.

- Cost α n, T = n clock cycles.
- What is the critical path for determining the min clock period?

"Shift and Add" Multiplier

Signed Multiplication:

Remember for 2's complement numbers MSB has negative weight:

$$X = \sum_{i=0}^{N-2} x_i 2^i - x_{n-1} 2^{n-1}$$

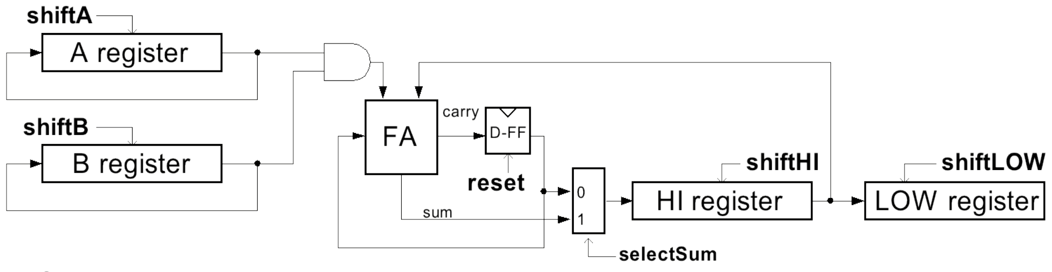
ex: $-6 = 11010_2 = 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 - 1 \cdot 2^4$

$= 0 + 2 + 0 + 8 - 16 = -6$

- Therefore for multiplication:
  - a) subtract final partial product
  - b) sign-extend partial products
- Modifications to shift & add circuit:
  - a) adder/subtractor
  - b) sign-extender on P shifter register

Bit-serial Multiplier

- Bit-serial multiplier (n^2 cycles, one bit of result per n cycles):



- Control Algorithm:

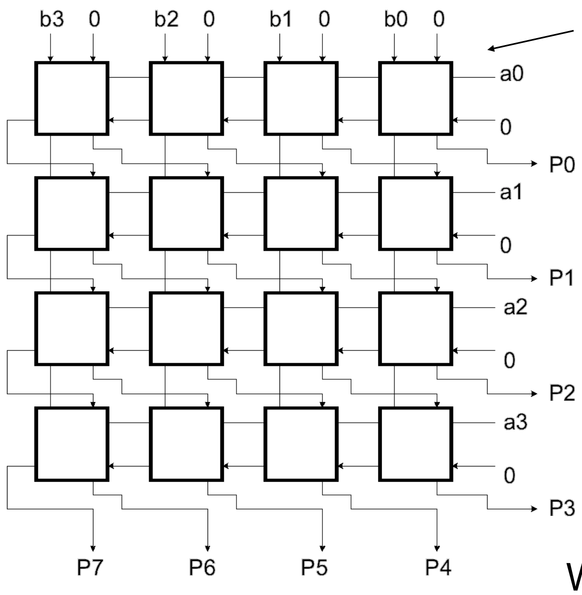
```
repeat n cycles {   // outer (i) loop
        repeat n cycles{    // inner (j) loop
                shiftA, selectSum, shiftHI
        }
        shiftB, shiftHI, shiftLOW, reset
}
```

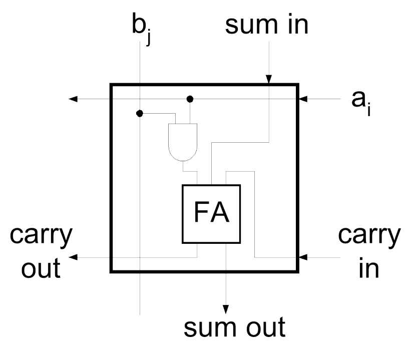
Note: The occurrence of a control signal x means x=1. The absence of x means x=0.

Array Multiplier

Single cycle multiply: Generates all n partial products simultaneously.

Each row: n-bit adder with AND gates





What is the critical path?

Carry-Save Addition

- Speeding up multiplication is a matter of speeding up the summing of the partial products.
- "Carry-save" addition can help.
- Carry-save addition passes (saves) the carries to the output, rather than propagating them.
- Example: sum three numbers, 3_{10} = 0011, 2_{10} = 0010, 3_{10} = 0011

$$\left.\begin{array}{rl} 3_{10} & 0011 \\ +\ 2_{10} & \underline{0010} \\ c & 0100 = 4_{10} \\ s & 0001 = 1_{10} \end{array}\right\}\text{carry-save add}$$

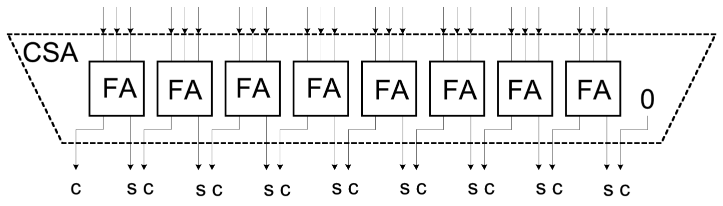
carry-save add

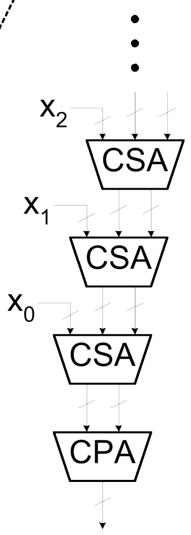
$$\begin{array}{rl} 3_{10} & \underline{0011} \\ c & 0010 = 2_{10} \\ s & \underline{0110} = 6_{10} \\ & 1000 = 8_{10} \end{array}$$

carry-propagate add

- In general, *carry-save* addition takes in 3 numbers and produces 2.
- Whereas, *carry-propagate* takes 2 and produces 1.
- With this technique, we can avoid carry propagation until final addition

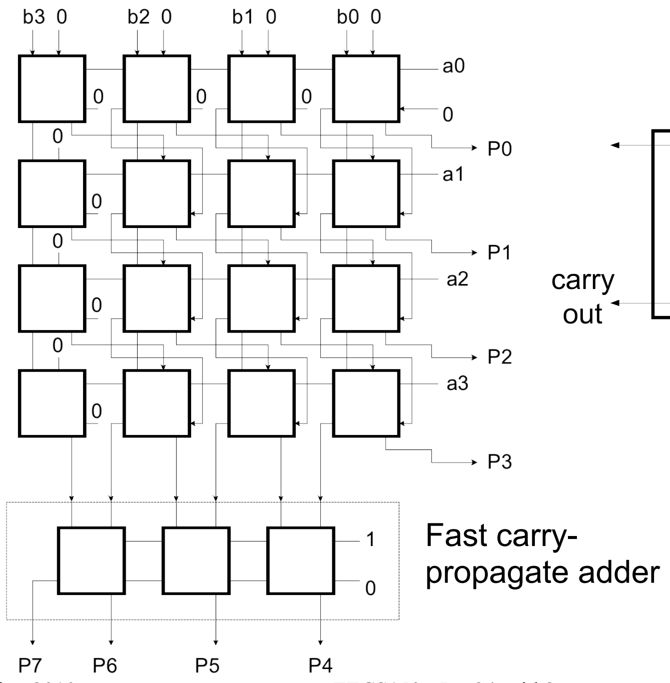
Carry-save Circuits

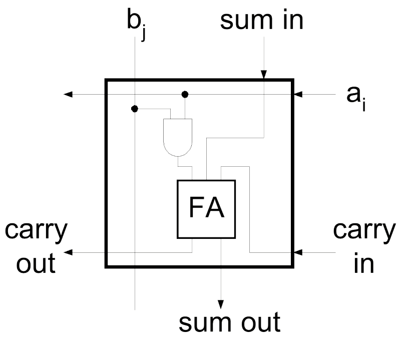




- When adding sets of numbers, carry-save can be used on all but the final sum.
- Standard adder (carry propagate) is used for final sum.
- Carry-save is fast (no carry propagation) and cheap (same cost as ripple adder)

Array Multiplier using Carry-save Addition

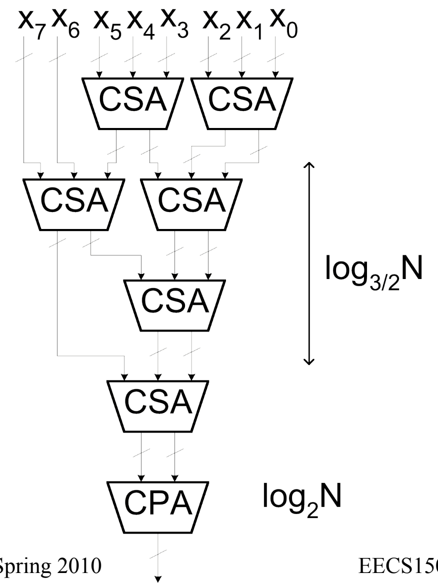




Carry-save Addition

CSA is associative and communitive. For example:

$$(((X_0 + X_1) + X_2) + X_3) = ((X_0 + X_1) + (X_2 + X_3))$$



- A balanced tree can be used to reduce the logic delay.
- This structure is the basis of the ***Wallace Tree Multiplier***.
- Partial products are summed with the CSA tree. Fast CPA (ex: CLA) is used for final sum.
- Multiplier delay α $\log_{3/2}N + \log_2 N$

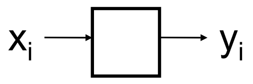
Constant Multiplication

- *Our discussion so far has assumed both the multiplicand (A) and the multiplier (B) can vary at runtime.*
- What if one of the two is a constant?

$$Y = C * X$$

- "Constant Coefficient" multiplication comes up often in signal processing and other hardware. Ex:

$$y_i = \alpha y_{i-1} + x_i$$

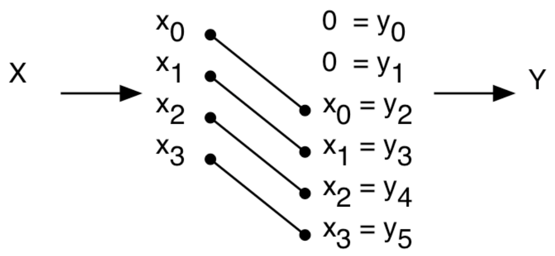


where α is an application dependent constant that is hard-wired into the circuit.

- How do we build and array style (combinational) multiplier that takes advantage of the constancy of one of the operands?

Multiplication by a Constant

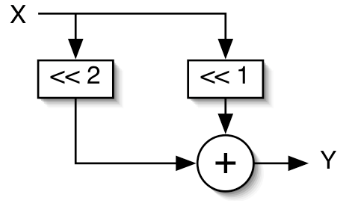
- If the constant C in C*X is a power of 2, then the multiplication is simply a shift of X.
- Ex: 4*X



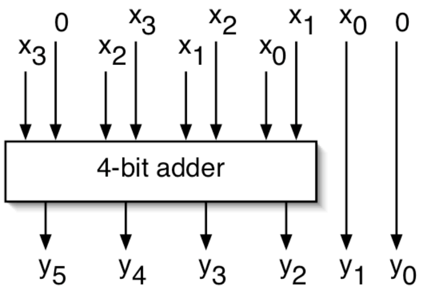
- What about division?
- What about multiplication by non- powers of 2?

Multiplication by a Constant

- In general, a combination of fixed shifts and addition:
  - Ex: 6*X = 0110 * X = $(2^2 + 2^1)$*X

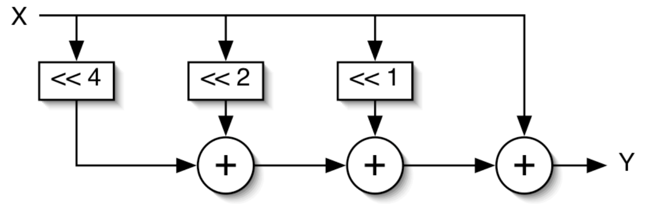


  - Details:


x_3 0 x_2 x_3 x_1 x_2 x_0 x_1 x_0 0
4-bit adder
y_5 y_4 y_3 y_2 y_1 y_0

Multiplication by a Constant

- Another example: C = 23_{10} = 010111



- *In general, the number of additions equals the number of 1's in the constant minus one.*
- Using carry-save adders (for all but one of these) helps reduce the delay and cost, but the number of adders is still the number of 1's in C minus 2.

- Is there a way to further reduce the number of adders (and thus the cost and delay)?

Multiplication using Subtraction

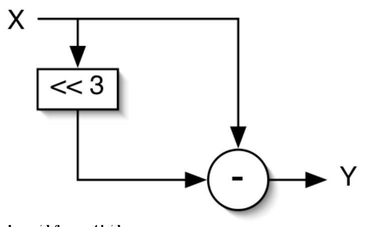
- *Subtraction is ~ the same cost and delay as addition.*
- Consider C*X where C is the constant value 15_{10} = 01111.

  C*X requires 3 additions.
- We can "recode" 15

  from 01111 = $(2^3 + 2^2 + 2^1 + 2^0)$

  to 1000$\bar{1}$ = $(2^4 - 2^0)$

  where $\bar{1}$ means negative weight.
- Therefore, 15*X can be implemented with only one subtractor.



Canonic Signed Digit Representation

- CSD represents numbers using 1, $\bar{1}$, & 0 with the least possible number of non-zero digits.
  - Strings of 2 or more non-zero digits are replaced.
  - Leads to a unique representation.
- To form CSD representation might take 2 passes:
  - First pass: replace all occurrences of 2 or more 1's:

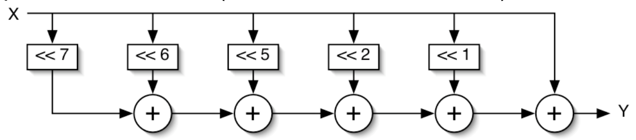
    01..10 by 10..$\bar{1}$0
  - Second pass: same as a above, plus replace 0$\bar{1}$10 by 00$\bar{1}$0
- Examples:

011101 = 29	0010111 = 23	0110110 = 54
100$\bar{1}$01 = 32 - 4 + 1	001100$\bar{1}$	10$\bar{1}$10$\bar{1}$0
	010$\bar{1}$00$\bar{1}$ = 32 - 8 - 1	100$\bar{1}$0$\bar{1}$0 = 64 - 8 - 2

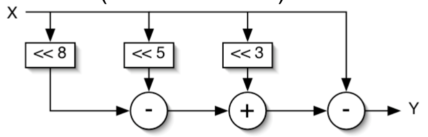
- Can we further simplify the multiplier circuits?

"Constant Coefficient Multiplication" (KCM)

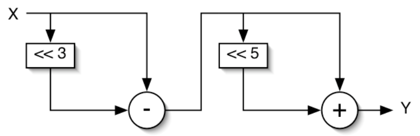
Binary multiplier: $Y = 231*X = (2^7 + 2^6 + 2^5 + 2^2 + 2^1+2^0)*X$



- CSD helps, but the multipliers are limited to shifts followed by adds.
  – CSD multiplier: $Y = 231*X = (2^8 - 2^5 + 2^3 - 2^0)*X$



- How about shift/add/shift/add …?
  – KCM multiplier: $Y = 231*X = 7*33*X = (2^3 - 2^0)*(2^5 + 2^0)*X$

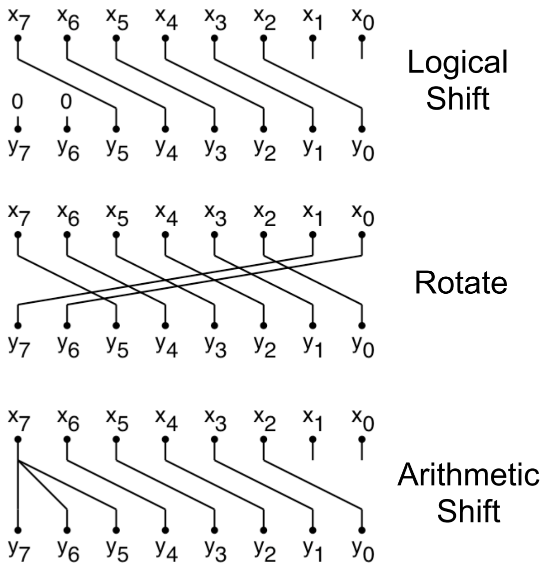


- No simple algorithm exists to determine the optimal KCM representation.
- Most use exhaustive search method.

Fixed Shifters / Rotators

- "fixed" shifters "hardwire" the shift amount into the circuit.
- Ex: verilog: X >> 2
  – (right shift X by 2 places)
- Fixed shift/rotator is nothing but wires!

  So what?

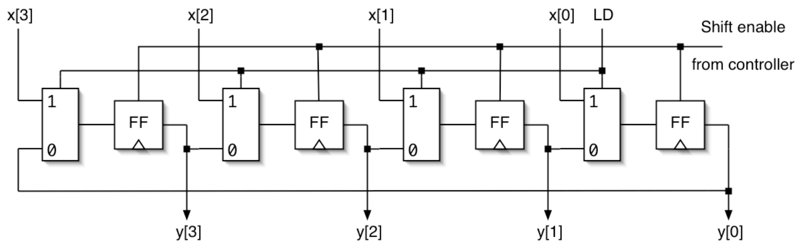


x_7 x_6 x_5 x_4 x_3 x_2 x_1 x_0
0 0
y_7 y_6 y_5 y_4 y_3 y_2 y_1 y_0
Logical Shift
Rotate
Arithmetic Shift

Variable Shifters / Rotators

- Example: X >> S, where S is unknown when we synthesize the circuit.
- Uses: shift instruction in processors (ARM includes a shift on every instruction), floating-point arithmetic, division/multiplication by powers of 2, etc.
- One way to build this is a simple shift-register:

  a) Load word, b) shift enable for S cycles, c) read word.

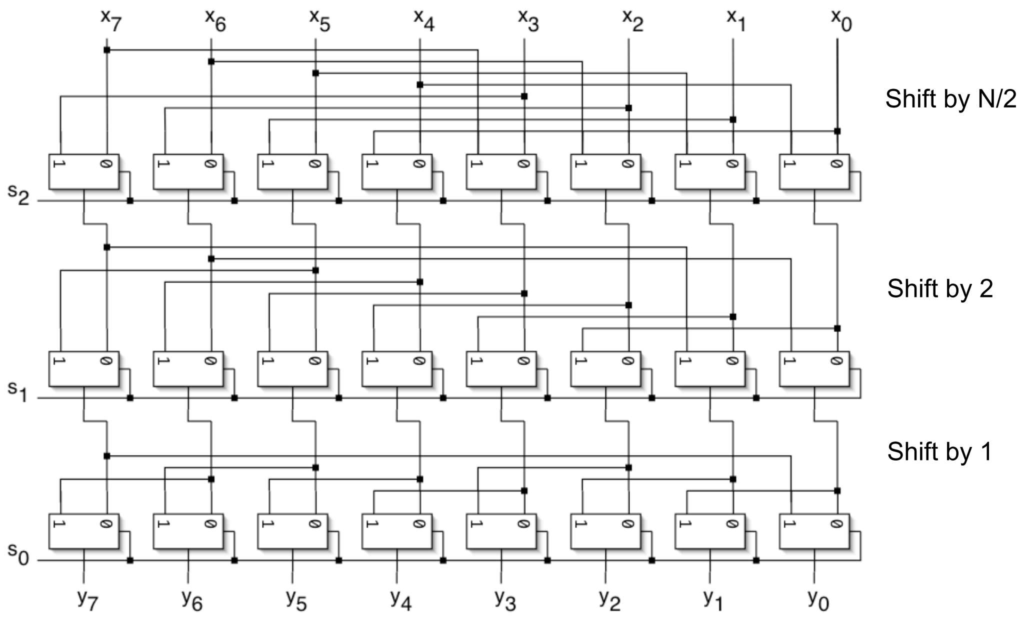


– Worst case delay O(N) , not good for processor design.

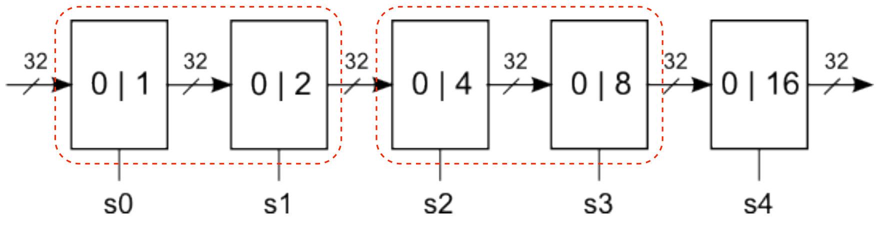
– Can we do it in O(logN) time and fit it in one cycle?

Log Shifter / Rotator

- Log(N) stages, each shifts (or not) by a power of 2 places,

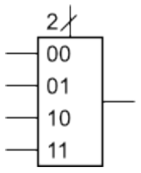

x_7 x_6 x_5 x_4 x_3 x_2 x_1 x_0
Shift by N/2
s_2
Shift by 2
s_1
Shift by 1
s_0
1
0
y_7 y_6 y_5 y_4 y_3 y_2 y_1 y_0

LUT Mapping of Log shifter

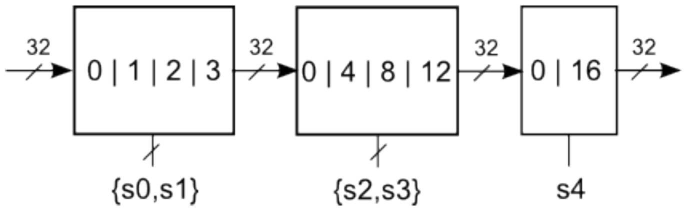


Efficient with 2to1 multiplexors, for instance, 3LUTs.

Virtex5 has 6LUTs. Naturally makes 4to1 muxes:



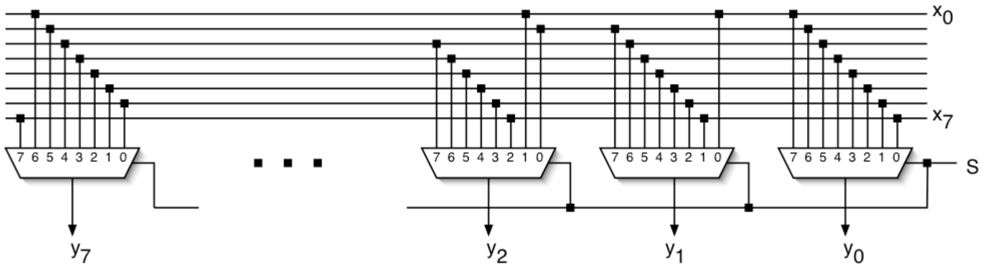
Reorganize shifter to use 4to1 muxes.



Final stage uses F7 mux

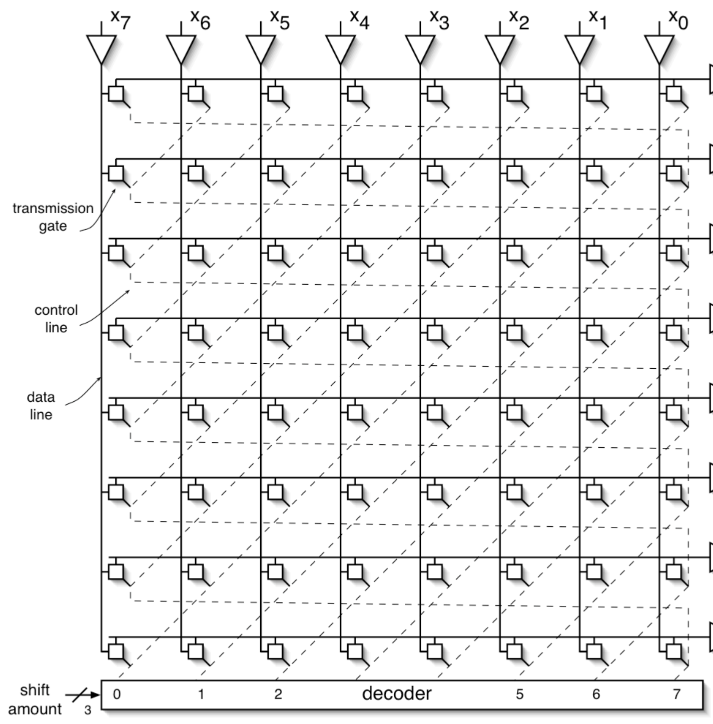
"Improved" Shifter / Rotator

- How about this approach? Could it lead to even less delay?



- What is the delay of these big muxes?
- Look a transistor-level implementation?

Barrel Shifter

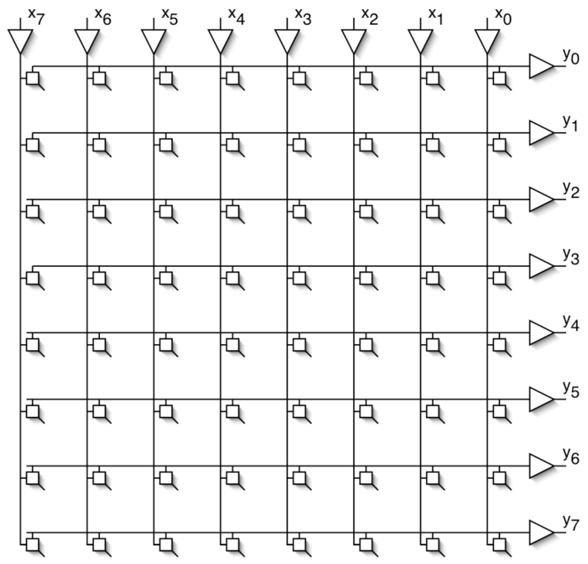


x_7 x_6 x_5 x_4 x_3 x_2 x_1 x_0
y_0 y_1 y_2 y_3 y_4 y_5 y_6 y_7
transmission gate
control line
data line
shift amount
3
0 1 2 decoder 5 6 7

Cost/delay?

- (don't forget the decoder)

Connection Matrix

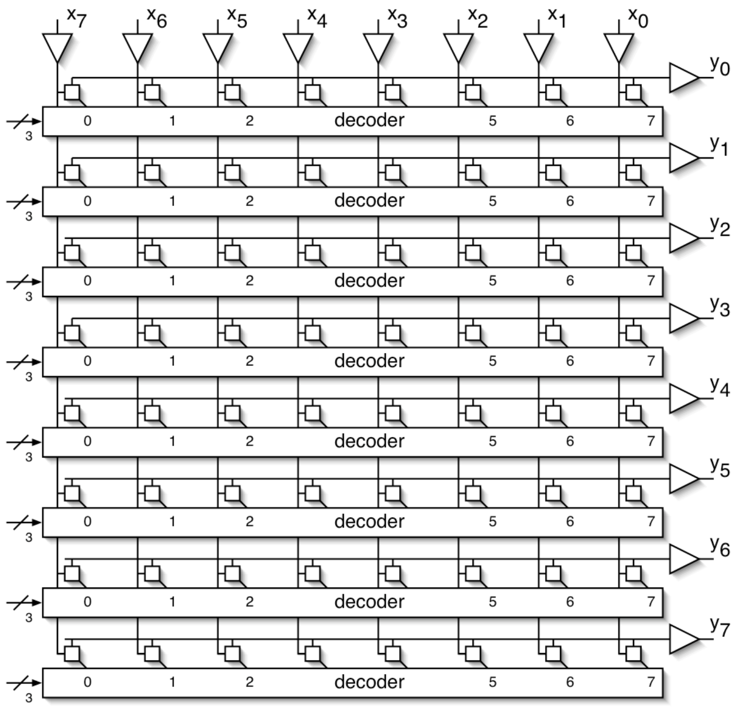


x_7 x_6 x_5 x_4 x_3 x_2 x_1 x_0
y_0 y_1 y_2 y_3 y_4 y_5 y_6 y_7

Generally useful structure:

- N^2 control points.
- What other interesting functions can it do?

Cross-bar Switch



- Nlog(N) control signals.
- Supports all interesting permutations
  - All one-to-one and one-to-many connections.
- Commonly used in communication hardware (switches, routers).